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# ON A SOBOLEV INEQUALITY WITH REMAINDER TERMS

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ABSTRACT. In this note we consider the Sobolev inequality

$$|| \triangle \phi||_2 \geq S_2||\phi||_{\frac{2N}{N-4}},\ N > 4,\ \phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N),$$

where $S_2$ is the best Sobolev constant and $\mathcal{D}_0^{2,2}(\mathbb{R}^N)$ is the space obtained by taking the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm $|| \triangle \phi||_2$. We prove here a refined version of this inequality,

$$|| \triangle \phi||_2^2 - S_2^2||\phi||^2_{\frac{2N}{N-4}} \geq \alpha d^2(\phi, M_2),\ N > 4,\ \phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N),$$

where $\alpha$ is a positive constant, the distance is taken in the Sobolev space $\mathcal{D}_0^{2,2}(\mathbb{R}^N)$, and $M_2$ is the set of solutions which attain the Sobolev equality. This generalizes a result of Bianchi and Egnell (*A note on the Sobolev inequality*, J. Funct. Anal. 100 (1991), 18-24), which was posed by Brezis and Lieb (*Sobolev inequalities with remainder terms*, J. Funct. Anal. 62 (1985), 73-86). regarding the classical Sobolev inequality

$$|| \bigtriangledown \phi||_2 \geq S_1||\phi||_{\frac{2N}{N-2}}, \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^N).$$

A key ingredient in our proof is the analysis of eigenvalues of the fourth order equation

$$\triangle^2 v - \mu S_2^{p+1} U^{\frac{8}{N-4}} v = 0, v \in \mathcal{D}_0^{2,2}(\mathbb{R}^N),$$

where $p = \frac{N+4}{N-4}$ and $U$ is the unique radial function in $M_2$ with $\|\Delta U\|_2 = 1$. We will show that the eigenvalues $\mu$ of the above equation are discrete:

$$\mu_1 = 1, \mu_2 = \mu_3 = \cdots = \mu_{N+2} = p < \mu_{N+3} \leq \cdots$$

and the corresponding eigenfunction spaces are

$$V_1 = \{U\}, V_p = \{\frac{\partial U}{\partial y_j}, j = 1, \cdots, N, x \cdot \bigtriangledown U + \frac{N-4}{2} U\}, \cdots .$$

## 1. INTRODUCTION

In this note we consider the Sobolev inequality

$$|| \triangle \phi||_2 - S_2||\phi||_{\frac{2N}{N-4}} \geq 0,\ N > 4,\ \phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N), \tag{1.1}$$

where $\mathcal{D}_0^{2,2}(\mathbb{R}^N)$ is the completion of the space of smooth functions with compact support under the norm $|| \triangle \phi||_2$, and $S_2$ is the best Sobolev constant. We assume $N > 4$ throughout this paper, and we denote by $|| \cdot ||_p$ the $L^p$ norm in $\mathbb{R}^N$.

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In [2], Brezis and Lieb asked if the following refined classical Sobolev inequality holds:

$$(1.2)\qquad ||\bigtriangledown\phi||_2^2 - S_1^2||\phi||^2_{\frac{2N}{N-2}} \geq \alpha d^2(\phi, M_1),\ N>2,\ \phi\in\mathcal{D}_0^{1,2}(\mathbb{R}^N),$$

where $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(R^N)$ under the norm of $||\bigtriangledown\phi||_2$ when $N\geq 3$, $\alpha>0$ and $M_1$ is the $(N+2)$-dimensional manifold which consists of all solutions attaining the classical Sobolev inequality. In [1], Bianchi and Egnell gave an affirmative answer to this question.

Some related results were proved by Brezis and Nirenberg in [3], among many other things. They showed in [3] that if $q<\frac{N}{N-2}$, then there is a positive number $A$ such that

$$(1.3)\qquad ||\bigtriangledown\phi||_2^2 - S_1^2||\phi||^2_{\frac{2N}{N-2}} \geq A||\phi||_q^2, N>2,\ \phi\in\mathcal{D}_0^{1,2}(\Omega),$$

where $A$ only depends on $N, q$ and the bounded domain $\Omega$. Moreover, the result (1.3) is proved to be sharp in the sense that it fails for $q=\frac{N}{N-2}$.

The following refinement of (1.3) was also proved in [2]:

$$(1.4)\qquad ||\bigtriangledown\phi||_2^2 - S_1^2||\phi||^2_{\frac{2N}{N-2}} \geq A||\phi||^2_{\frac{N}{N-2},w}, N>2,\ \phi\in\mathcal{D}_0^{1,2}(\Omega),$$

where $||\cdot||_{q,w}$ is the weak $L^q$ norm. An alternate proof of (1.4) was given in [1].

The result of Brezis and Nirenberg [3] was generalized to $1<p<N$ and $q<\frac{N(p-1)}{N-p}$ by Egnell, Pacella and Tricarico in [5]. They proved in [5] that there is a positive number $A$ depending only $N, q, p$ and the bounded domain $\Omega$ such that

$$(1.5)\qquad ||\bigtriangledown\phi||_p^p - S_p^p||\phi||^p_{p*} \geq A||\phi||_q^p, N>2,\ \phi\in\mathcal{D}_0^{1,2}(\Omega),$$

where $p*=\frac{Np}{N-p}$ and $S_p$ is the best Sobolev constant in the Sobolev embedding theorem with the critical exponent $p*$. Furthermore, (1.5) fails for $q=\frac{N(p-1)}{N-p}$.

The purpose of this paper is to generalize the result of [1] and prove the following refined inequality of (1.1):

$$(1.6)\qquad ||\bigtriangleup\phi||_2^2 - S_2^2||\phi||^2_{\frac{2N}{N-4}} \geq \alpha d^2(\phi, M_2),$$

where $M_2=\{\phi\in\mathcal{D}_0^{2,2}(\mathbb{R}^N): ||\bigtriangleup\phi||_2 = S_2||\phi||_{\frac{2N}{N-4}}\}$.

By Theorem 2.1 of [4], $M_2$ is an $(N+2)$-dimensional manifold and consists of functions of the form

$$(1.7)\qquad \phi(x) = cU_{\lambda,y}(x) = c\lambda U(\lambda^{\frac{2}{N-4}}(x-y)),$$

where $c\in\mathbb{R}, \lambda\in\mathbb{R}_+, U(x)=k_0(1+|x|^2)^{-\frac{N-4}{2}}$ and $k_0$ is chosen so that $||\bigtriangleup U||_2=1$. The best constant is

$$S_2 = \pi^2(N+2)N(N-2)(N-4)\left(\Gamma(N/2)/\Gamma(N)\right)^{4/N}.$$

Hence

$$M_2=\{cU_{\lambda,y}: c\in\mathbb{R}, \lambda\in\mathbb{R}_+, y\in\mathbb{R}^N\}.$$

(In fact, C.-S. Lin recently showed (see Theorem 1.4 in [7]) that any solution of the equation

$$(1.8)\qquad (-\Delta)^2u = u^{\frac{N+4}{N-4}}, u>0 \text{ in } \mathbb{R}^N\ (N\geq 5)$$

is symmetric around some point and is of the form (1.7).)

We define the distance between this manifold and a function $\phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N)$ as

$$d(\phi, M_2) = \inf_{u \in M_2} || \triangle (\phi - u)||_2 = \inf_{c,\lambda,y} || \triangle (\phi - cU_{\lambda,y})||_2.$$

Note that $d(c\lambda\phi(\lambda^{\frac{2}{N-4}}(\cdot - y)), M_2) = cd(\phi, M_2)$.

We shall prove

**Theorem 1.9.** *There is a positive constant $\alpha$ depending only on the dimension $N$ such that*

$$|| \triangle \phi||_2^2 - S_2^2||\phi||^2_{\frac{2N}{N-4}} \geq \alpha d^2(\phi, M_2).$$

*Furthermore, this result is sharp in the sense that it is false if the left hand side is replaced with $d(\phi, M_2)^\beta|| \triangle \phi||_2^{2-\beta}$, where $\beta < 2$.*

We refer the reader to [1], [2], [5], [6], [8] and [9] for more information on Sobolev inequalities of such type.

A key ingredient in our proof is the analysis of eigenvalues of the following fourth order equation:

$$\triangle^2 v - \mu S_2^{p+1} U^{\frac{8}{N-4}} v = 0, v \in \mathcal{D}_0^{2,2}(\mathbb{R}^N), \tag{1.10}$$

where $p = \frac{N+4}{N-4}$. We will show that the eigenvalues $\mu$ of (1.10) are discrete and

$$\mu_1 = 1, \mu_2 = \mu_3 = \cdots = \mu_{N+2} = p < \mu_{N+3} \leq \cdots. \tag{1.11}$$

The corresponding eigenfunction spaces are

$$V_1 = \{U\}, V_p = \{\frac{\partial U}{\partial y_j}, j = 1, \cdots, N, x \cdot \bigtriangledown U + \frac{N-4}{2}U\}. \tag{1.12}$$

This result is interesting and may have applications in analyzing blow up problems involving the biharmonic operator. We prove this in Section 2. In Section 3, we prove Theorem 1.9.

## 2. An eigenvalue problem

In this section, we solve the eigenvalue problem

$$\triangle^2 v - \mu S_2^{p+1} U^{\frac{8}{N-4}} v = 0, v \in \mathcal{D}_0^{2,2}(\mathbb{R}^N), \tag{2.1}$$

where $U = k_0(1 + |x|^2)^{-\frac{N-4}{2}}$. Note that $U$ satisfies

$$\triangle^2 U - S_2^{p+1} U^p = 0, U(0) = \max_{z \in \mathbb{R}^N} U(z)$$

and $p = \frac{N+4}{N-4}$.

Note also that (2.1) is a fourth order eigenvalue problem. We will show the following

**Theorem 2.2.** *The eigenvalues $\mu$ of* (2.1) *are discrete and*

$$\mu_1 = 1, \mu_2 = \mu_3 = \cdots = \mu_{N+2} = p < \mu_{N+3} \leq \cdots. \tag{2.3}$$

*The corresponding eigenfunction spaces are*

$$V_1 = \{U\}, V_p = \{\frac{\partial U}{\partial y_j}, j = 1, \cdots, N, x \cdot \bigtriangledown U + \frac{N-4}{2}U\}. \tag{2.4}$$

*Proof.* We decompose the fourth order equation (2.1) into a system of two second order equations.

We first note that the eigenvalues of $\triangle_{S^{N-1}}$ are given by

$$\lambda_1 = 0, \lambda_2 = \cdots = \lambda_{N+1} = N-1, \lambda_{N+1} < \lambda_{N+2}. \tag{2.5}$$

Let $\Psi_i(\theta)$ be the corresponding eigenfunctions, i.e., $\triangle_{S^{N-1}}\Phi_i = -\lambda_i \Phi_i$ for each $i$. Set

$$\phi_i(r) = \int_{S^{N-1}} v(x)\Psi_i(\theta)d\theta$$

and

$$w_i(r) = -\int_{S^{N-1}} \triangle v(x)\Psi_i(\theta)d\theta.$$

Then we obtain a system of two equations

$$\triangle\, \phi_i - \frac{\lambda_i}{r^2}\phi_i + w_i = 0, \tag{2.6}$$

$$\triangle\, w_i - \frac{\lambda_i}{r^2}w_i + \mu S_2^{p+1}U^{p-1}\phi_i = 0. \tag{2.7}$$

We now show that

$$\phi_i = 0, \text{ if } i > N+2, \mu \leq p. \tag{2.8}$$

We will prove it by contradiction. Suppose now that $\mu \leq p$ and $\phi_i \not\equiv 0$, $i > N+2$. Without loss of generality, we assume that

$$\phi_i < 0 \text{ in } (0, r_1), \phi_i(r_1) = 0. \tag{2.9}$$

Since $i > N+2$, $\phi_i(0) = w_i(0) = \phi_i'(0) = w_i'(0) = 0$.

We now compare $(\phi_i, w_i)$ with a new pair of functions.

Observe that $U$ satisfies $\triangle^2 U - S_2^{p+1}U^p = 0$, and let $U_1 = -\triangle U$. Then both $U$ and $U_1$ are radial and monotone decreasing functions. We observe that for any radial function $f(r)$ we have

$$(\triangle f)' = (f''(r) + \frac{N-1}{r}f'(r))' = f'''(r) + \frac{N-1}{r}f''(r) - \frac{N-1}{r^2}f'(r).$$

Then $(U', U_1')$ satisfies

$$\triangle\, U' - \frac{N-1}{r^2}U' + U_1' = 0. \tag{2.10}$$

We note

$$\triangle^2 U = -\triangle U_1 = -\left(U_1'' + \frac{N-1}{r}U_1'\right);$$

thus

$$U_1'' + \frac{N-1}{r}U_1' + S_2^{p+1}U^p = 0.$$

Differentiating both sides of the above identity with respect to $r$, we get

$$U_1''' + \frac{N-1}{r}U_1'' - \frac{N-1}{r^2}U_1' + pS_2^{p+1}U^{p-1}U' = 0.$$

This is equivalent to

$$\triangle U_1' - \frac{N-1}{r^2}U_1' + pS_2^{p+1}U^{p-1}U' = 0. \tag{2.11}$$

Multiplying (2.7) by $U'$ and integrating over $B_r$ we get

(2.12)

$$\begin{aligned} 0 &= \int_{B_r} \mu S_2^{p+1} U^{p-1} \phi_i U' + \int_{B_r} (\triangle w_i - \frac{\lambda_i}{r^2}\phi_i) U' \\ &= (\mu - p)\int_{B_r} S_2^{p+1} U^{p-1}\phi_i U' + p\int_{B_r} S_2^{p+1} U^{p-1}\phi_i U' + \int_{B_r} (\triangle w_i - \frac{\lambda_i}{r^2}\phi_i) U' \\ &= (\mu - p)\int_{B_r} S_2^{p+1} U^{p-1}\phi_i U' - \int_{B_r} \left[\triangle U_1' - \frac{N-1}{r^2} U_1'\right]\phi_i \\ &\quad + \int_{B_r} \left[\triangle w_i - \frac{\lambda_i}{r^2} w_i\right] U'. \end{aligned}$$

The second term in (2.12) can be seen as follows:

$$\begin{aligned} & -\int_{B_r} \left[\triangle U_1' - \frac{N-1}{r^2} U_1'\right]\phi_i \\ = & -\int_{B_r} div(\triangledown U_1')\phi_i + \int_{B_r} \frac{N-1}{r^2} U_1'\phi_i \\ = & -\int_{\partial B_r} (\triangledown U_1' \cdot \nu)\phi_i + \int_{B_r} (\triangledown U_1') \cdot \phi_i + \int_{B_r} \frac{N-1}{r^2} U_1'\phi_i \\ = & -\int_{\partial B_r} \frac{\partial U_1'}{\partial \nu}\phi_i + \int_{\partial B_r} U_1' \frac{\partial \phi_i}{\partial \nu} - \int_{B_r} U_1' \triangle \phi_i + \int_{B_r} \frac{N-1}{r^2} U_1' \phi_i. \end{aligned}$$

By using equation (2.6), the second term of (2.12) becomes

$$\int_{B_r} \frac{N-1-\lambda_i}{r^2} U_1'\phi_i + \int_{\partial B_r} \left(U_1' \frac{\partial \phi_i}{\partial \nu} - \phi_i \frac{\partial U_1'}{\partial \nu}\right) + \int_{B_r} U_1' w_i.$$

The third term of (2.12) is

$$\int_{B_r} \left[\triangle w_i - \frac{\lambda_i}{r^2} w_i\right] U' = \int_{\partial B_r} \frac{\partial w_i}{\partial \nu} U' - \int_{\partial B_r} w_i \frac{\partial U'}{\partial \nu} + \int_{B_r} w_i \triangle U' - \int_{B_r} \frac{\lambda_i}{r^2} w_i U'.$$

Note that by (2.10) we have $\triangle U' = \frac{N-1}{r^2} U' - U_1'$; thus the third term of (2.12) is equal to

$$\int_{B_r} \frac{N-1-\lambda_i}{r^2} U' w_i + \int_{\partial B_r} \left(U' \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial U'}{\partial \nu}\right) - \int_{B_r} w_i U_1'.$$

Therefore

$$\begin{aligned} 0 &= (\mu - p)\int_{B_r} S_2^{p+1} U^{p-1}\phi_i U' + \int_{B_r} \frac{N-1-\lambda_i}{r^2} U_1'\phi_i + \int_{\partial B_r} \left(U_1' \frac{\partial \phi_i}{\partial \nu} - \phi_i \frac{\partial U_1'}{\partial \nu}\right) \\ &\quad + \int_{B_r} \frac{N-1-\lambda_i}{r^2} U' w_i + \int_{\partial B_r} \left(U' \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial U'}{\partial \nu}\right) \\ &= (\mu - p)\int_{B_r} S_2^{p+1} U^{p-1}\phi_i U' + \int_{B_r} \frac{N-1-\lambda_i}{r^2} (U_1'\phi_i + U' w_i) \\ &\quad + \int_{\partial B_r} \left(U_1' \frac{\partial \phi_i}{\partial \nu} - \phi_i \frac{\partial U_1'}{\partial \nu}\right) + \int_{\partial B_r} \left(U' \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial U'}{\partial \nu}\right) \\ &= I_1(r) + I_2(r) + I_3(r) + I_4(r), \end{aligned}$$

where $I_i(r), i = 1, 2, 3, 4$, are defined at the last equality.

We now choose appropriate $r$ and estimate each term. Recall that $\phi_i < 0$ in $(0, r_1)$, $\phi_i(r_1) = 0$ (we can take $r_1 = \infty$ if $\phi_i < 0$ in $(0, \infty)$).

We first deduce from equations (2.6) and (2.7) some useful facts:

1) $U'(r) < 0, U_1'(r) < 0$ if $r \neq 0$.

2) $\triangle w_i - \frac{\lambda_i}{r^2} w_i = -\mu S_2^{p+1} U^{p-1} \phi_i$. Thus $w_i$ cannot have a positive local maximum in $(0, r_1)$.

3) There exists some $r^* \in (0, r_1)$ such that $w_i(r^*) < 0$. This is because $\phi_i$ attains at least one local minimum in $(0, r_1)$ and by using equation (2.7).

4) By 2) and 3), $w_i(r) < 0$ for $r \in (0, r_2)$ for some $r_2 > 0$ and $w_i(r_2) = 0$.

If $r_1 = r_2$, then we take $r = r_1 = r_2$. In this case, $I_1(r) \leq 0$ since $\mu \leq p$, $\phi_i, U' < 0$; $I_2(r) \leq 0$ since $\lambda_i \geq N - 1$; $I_3(r) \leq 0$ since $\frac{\partial \phi_i}{\partial \nu} \geq 0$; $I_4(r) \leq 0$ since $\frac{\partial w_i}{\partial \nu} \geq 0$.

Therefore, we have $I_1(r) = I_2(r) = I_3(r) = I_4(r) = 0$. This is a contradiction since $I_3(r) < 0$ and $I_4(r) < 0$.

The rest of the cases can be discussed in the following:

Case 1: $r_2 < r_1$.

In this case, we take $r = r_2$. Then $I_1(r_2) \leq 0$, $I_2(r_2) \leq 0$, $I_4(r_2) \leq 0$. We only need to know if $I_3(r_2) \leq 0$. To this end, we consider the function

$$w = r^{N-1} \phi_i' U_1' - r^{N-1} U_1'' \phi_i.$$

By an easy calculation we get

$$w' = r^{N-1} \left[ \phi_i'' + \frac{N-1}{r} \phi_i' \right] U_1' - r^{N-1} \left[ U_1''' + \frac{N-1}{r} U_1'' \right] \phi_i.$$

By using equations (2.6) and (2.11) we can easily see that in $(r_2, r_1)$

$$w' = (\lambda_i - (N-1)) r^{N-3} \phi_i U_1' - r^{N-1} w_i U_1' + p S_2^{p+1} U^{p-1} U' r^{N-1} \phi_i > 0,$$

since $w_i > 0$ in $(r_2, r_1)$ by fact 2), $U_1' < 0$, $\phi_i < 0$ and $\lambda_i > N - 1$.

Thus

$$w(r_2) = r_2^{N-1} \phi_i'(r_2) U_1'(r_2) - r_2^{N-1} U''(r_2) \phi_i(r_2) < w(r_1) < 0$$

and

$$I_3(r_2) = \int_{\partial B_{r_2}} r^{1-N} w(r) < 0.$$

This is again a contradiction.

Case 2: $r_1 < r_2$.

In this case,

$$\triangle \phi_i - \frac{\lambda_i}{r^2} \phi_i = -w_i > 0, \text{ for } r \in (0, r_2).$$

Hence $\phi_i > 0$ for $r \in (r_1, r_2)$.

Now we take $r = r_1$. Then $I_1(r_1) \leq 0$, $I_2(r_1) \leq 0$, $I_3(r_1) \leq 0$. Thus we only need to show $I_4(r_1) \leq 0$. Consider

$$h = r^{N-1} w_i' U' - r^{N-1} U'' w_i.$$

We note that

$$\begin{aligned} h' &= r^{N-1} \left[ w_i'' + \frac{N-1}{r} w_i' \right] U' - r^{N-1} \left[ U''' + \frac{N-1}{r} U'' \right] w_i \\ &= r^{N-1} (\triangle w_i) U' - r^{N-1} (\triangle U') w_i. \end{aligned}$$

By using equations (2.7) and (2.10) we thus have

$$h' = (\lambda_i - (N-1))\, w_i U' r^{N-3} - pS_2^{p+1} U^{p-1} \phi_i U' r^{N-1} + r^{N-3}(N-1) U_1' w_i > 0$$

for $r \in (r_1, r_2)$ in $\lambda_i > N-1$. Hence $h(r_1) < h(r_2) = r_2^{N-1} w_i'(r_2) U'(r_2) < 0$. Therefore we have

$$I_4(r_1) = \int_{B_{r_1}} r^{1-N} h(r) < 0.$$

In conclusion, we have that if $\mu \le p$, then $\phi_i \equiv 0$ for $i > N+2$. Hence, when $\mu \le p$, the eigenfunction space has at most $(N+2)$ dimensions. On the other hand, when $\mu = 1$, $v = U$ is a solution; when $\mu = p$, $\frac{\partial U}{\partial y_j}$ $(1 \le j \le N)$ and $\frac{\partial U_{\lambda,0}}{\partial \lambda}|_{\lambda=1} = x \cdot \bigtriangledown U + \frac{N-4}{2} U$ are eigenfunctions.

Note that $U$, $\frac{\partial U}{\partial y_j}$ $(1 \le j \le N)$, and $\frac{\partial U_{\lambda,0}}{\partial \lambda} = x \cdot \bigtriangledown U + \frac{N-4}{2} U$ are linearly independent. Thus (2.2) has the following solutions:

$$\mu = 1, v = \alpha U,$$

$$\mu = p, v \in span\ \{\frac{\partial U}{\partial y_j}, \frac{\partial U_\lambda}{\partial \lambda}|_{\lambda=1}\}.$$

□

Consider the operator

$$L_{\lambda,y} = \frac{1}{S_2^{p+1}} U_{\lambda,y}^{1-p} \triangle^2, \text{ on } L^2(U_{\lambda,y}^{p-1}).$$

Since the imbedding

$$\mathcal{D}_0^{2,2}(\mathbb{R}^N) \to L^2(U_{\lambda,y}^{p-1})$$

is compact, the spectrum is discrete.

Consider

$$L_{\lambda,y} v - \mu v = 0. \tag{2.13}$$

Then we have

$$\triangle^2 v - \mu S_2^{p+1} U_{\lambda,y}^{p-1} v = 0. \tag{2.14}$$

By a simple scaling argument, we have that $\mu$ does not depend on $\lambda, y$. Moreover, by Theorem 2.1 we have

**Lemma 2.15.** *Let $\mu_i$, $i = 1, \cdots$, denote the eigenvalues of* (2.13) *in increasing order. Then $\mu_1 = 1$ is simple with eigenfunction $U_{\lambda,y}$ and $\mu_2 = ... = \mu_{N+2} = p$ with the corresponding $(N+1)$-dimensional eigenfunction space spanned by $(\partial_\lambda U_{\lambda,y}, \bigtriangledown_y U_{\lambda,y})$. Furthermore, eigenvalues do not depend on $\lambda, y$, and $\mu_{N+3} > \mu_2$.*

## 3. Proof of the theorem

The main ingredient in the proof of the theorem is contained in the lemma below, where the behavior near $M_2$ is studied.

**Lemma 3.1.** *There is a constant $\alpha$, depending only on the dimension, such that*

$$||\triangle \phi||_2^2 - S_2^2 ||\phi||_{\frac{2N}{N-4}}^2 \ge \alpha d^2(\phi, M_2) + o(d(\phi, M_2)^2),$$

*for all $\phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N)$ with $d(\phi, M_2) < ||\triangle\phi||_2$.*

*Proof.* Since $M_2$ is an $(N+2)$-dimensional manifold embedded in $\mathcal{D}_0^{2,2}(\mathbb{R}^N)$,

$$(c,\lambda,y) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^N \to cU_{\lambda,y} \in \mathcal{D}_0^{2,2}(\mathbb{R}^N).$$

Take $\phi \in \mathcal{D}_0^{2,2}(\mathbb{R}^N)$ with

$$\begin{aligned} d &= d(\phi, M_2) \\ &= \inf_{c,\lambda,y} ||\triangle(\phi - cU_{\lambda,y})||_2^2 \\ &= \inf_{c,\lambda,y} \left( ||\triangle\phi||_2^2 + c^2 - 2c\int_{\mathbb{R}^N} \triangle\phi\cdot\triangle U_{\lambda,y} \right) \\ &< ||\triangle\phi||_2^2. \end{aligned}$$

It is easy to see that the infimum above is attained at a point $(c_0,\lambda_0,y_0)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^N$ with $c_0\neq 0$. Since $M_2\backslash\{0\}$ is a smooth manifold and the tangential space at $(c_0,\lambda_0,y_0)$ is given by

$$TM_{c_0U_{\lambda_0,y_0}} = span\ \{U_{\lambda_0,y_0}, \partial_\lambda U_{\lambda_0,y_0}, \bigtriangledown_y U_{\lambda_0,y_0}\},$$

we must have that $(\phi - c_0U_{\lambda_0,y_0})$ is perpendicular to $TM_{c_0U_{\lambda_0,y_0}}$. By Lemma 2.15, we necessarily have

$$\int_{\mathbb{R}^N} |\triangle(\phi - c_0U_{\lambda_0,y_0})|^2 \geq \mu_{N+3}S_2^{p+1}\int_{\mathbb{R}^N} U_{\lambda_0,y_0}^{p-1}(\phi - c_0U_{\lambda_0,y_0})^2.$$

Let $\phi = c_0U_{\lambda_0,y_0} + dv$, $||\triangle v||_2 = 1$. Then

$$||\triangle\phi||_2^2 = d^2 + c_0^2||\triangle U_{\lambda_0,y_0}||_2^2 = d^2 + c_0^2,$$

$$\begin{aligned} \int_{\mathbb{R}^N}\phi^{p+1} &= \int_{\mathbb{R}^N}(c_0U_{\lambda_0,y_0} + dv)^{p+1} \\ &= \int_{\mathbb{R}^N} c_0^{p+1}U_{\lambda_0,y_0}^{p+1} + d(p+1)c_0^p\int_{\mathbb{R}^N}U_{\lambda_0,y_0}^p v \\ &\quad + d^2\frac{p(p+1)}{2}|c_0|^{p-1}\int_{\mathbb{R}^N}U_{\lambda_0,y_0}^{p-1}v^2 + o(d^2). \end{aligned}$$

Since $v$ is perpendicular to $U_{\lambda_0,y_0}$, then $\int_{\mathbb{R}^N}U_{\lambda_0,y_0}^p v dx = 0$, i.e.,

$$\int_{\mathbb{R}^N}\triangle v\cdot\triangle U_{\lambda_0,y_0} = 0 = \int_{\mathbb{R}^N} v\triangle^2 U_{\lambda_0,y_0} = S_2^{p+1}\int_{\mathbb{R}^N} vU_{\lambda_0,y_0}^p.$$

Thus

$$\int_{\mathbb{R}^N}\phi^{p+1} \leq c_0^{p+1}S_2^{-(p+1)} + d^2|c_0|^{p-1}\frac{p(p+1)}{2\mu_{N+3}S_2^{p+1}} + o(d^2).$$

Therefore

$$\begin{aligned} \left(\int_{\mathbb{R}^N}\phi^{p+1}\right)^{\frac{2}{p+1}} &\leq \left(c_0^{p+1}S_2^{-(p+1)} + d^2|c_0|^{p-1}\frac{p(p+1)}{2\mu_{N+3}S_2^{p+1}} + o(d^2)\right)^{\frac{2}{p+1}} \\ &= c_0^2S_2^{-2}\left(1 + d^2c_0^{-2}\frac{p(p+1)}{2\mu_{N+3}} + o(d^2)\right)^{\frac{2}{p+1}} \\ &= c_0^2S_2^{-2}\left(1 + \frac{2}{p+1}d^2c_0^{-2}\frac{p(p+1)}{2\mu_{N+3}} + o(d^2)\right) \\ &= c_0^2S_2^{-2} + \frac{d^2p}{\mu_{N+3}}S_2^{-2} + o(d^2). \end{aligned}$$

Thus

$$\begin{aligned}||\triangle\phi||_2^2 &- S_2^2||\phi||^2_{\frac{2N}{N-4}}\\ &\geq d^2 + c_0^2 - S_2^2\left(c_0^2 S_2^{-2} + \frac{d^2 p S_2^{-2}}{\mu_{N+3}} + o(d^2)\right)\\ &\geq d^2(1 - \frac{p}{\mu_{N+3}}) + o(d^2).\end{aligned}$$

This is true when $d$ is small. Thus the lemma holds with $\alpha = (1 - \frac{p}{\mu_{N+3}})$. To see that this is the best possible result we can argue as follows:

Take $\phi = U + dv$, where $v$ is the $(N+3)$th eigenfunction of $L_{1,0}$ and $d$ is a small positive number. We then have $d(\phi, M_2) = d$ if $d$ is small and the closest point on $M$ is $U$. □

*Proof of Theorem 1.9.* The fact that the result is sharp follows from the last part of the proof of the lemma above. Assume that the theorem were not true, then we could find a sequence $\{\phi_m\}$ such that

$$\frac{||\triangle\phi_m||_2^2 - S_2^2||\phi_m||^2_{\frac{2N}{N-4}}}{d(\phi_m, M_2)^2} \to 0, \text{ as } m \to \infty.$$

By homogeneity we can assume that $||\triangle\phi_m||_2 = 1$, and after selecting a subsequence we can assume that $d(\phi_m, M_2) \to L \in [0,1]$. Note that $d(\phi_m, M_2) \leq ||\triangle\phi_m||_2 = 1$. If $L = 0$, then we have a contradiction by Lemma 3.1 above. The other possibility is that $L > 0$. In this case we must have

$$||\triangle\phi_m||_2^2 - S_2^2||\phi_m||^2_{\frac{2N}{N-4}} \to 0, ||\triangle\phi_m||_2^2 = 1.$$

By P. L. Lions' concentration and compactness principle (see Corollary 1.2 of Section I.4 in Part I of P. L. Lions [6]) we have two sequences of numbers $\lambda_m, y_m$ so that

$$\lambda_m\phi_m(\lambda_m^{\frac{4}{N-4}}(x - y_m)) \to +U \text{ (or } -U) \in \mathcal{D}_0^{2,2}(\mathbb{R}^N) \text{ as } m \to \infty.$$

This implies

$$d(\phi_m, M_2) = d\left(\lambda_m\phi_m\left(\lambda_m^{\frac{4}{N-4}}(x - y_m)\right), M_2\right) \to 0 \text{ as } m \to \infty.$$

This is a contradiction.

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## References

[1] G. Bianchi and H. Egnell, A note on the Sobolev inequality, *J. Funct. Anal.* 100 (1991), 18-24. MR **92i:**46033

[2] H. Brezis and E. Lieb, Sobolev inequalities with remainder terms, *J. Funct. Anal.* 62 (1985), 73-86. MR **86i:**46033

[3] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* 36 (1983), 437-477. MR **84h:**46053

[4] D.E. Edmunds, D. Fortunato and E. Janelli, Critical exponents, critical dimensions, and the biharmonic operator, *Arch. Rational Mech. Anal.* 112 (1990), 269-289. MR **91k:**35191

[5] H. Egnell, F. Pacella and M. Tricarico, Some remarks on Sobolev inequalities, *Nonlinear Anal. T.M.A.* 13 (1989), 671-681. MR **90h:**46061

[6] P. L. Lions, The concentration-compactness principle in the calculus of variations: the limiting case (parts 1 and 2), *Riv. Mat. Iberoamericana* 1 (1985), 145-201, 45-121. MR **87c:**49007; MR **87j:**49012

[7] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in $R^n$, Comment. Math. Helv. **73** (1998), 206–231. MR **99c:**35062

[8] G. Talenti, Best constant in Sobolev inequalities, *Ann. Mat. Pura Appl.* 110 (1976), 353-372. MR **57:**3846

[9] X. Wang, Sharp constant in a Sobolev inequality, *Nonlinear Analysis: TMA*, 20 (1993), 261-268. MR **94g:**35035

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