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SPECTRAL TYPES OF SKEWED BERNOULLI SHIFT

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Abstract. For the transformation $T : x \mapsto kx \pmod 1$ for $k \geq 2$, it is proved that a real-valued function $f(x)$ of modulus 1 is not a multiplicative coboundary if the discontinuities $0 < x_1 < \cdots < x_n \leq 1$ of $f(x)$ are k-adic points and $x_1 \geq \frac{1}{k}$. It is also proved that the weakly mixing skew product transformations arising from Bernoulli shifts have Lebesgue spectrum.

1. Introduction

Given a measure-preserving transformation T on a probability space X and a measurable function $\phi : X \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, we define an isometry $V^{\phi,T}$ on $L^2(X)$ by $(V^{\phi,T} f)(x) = \phi(x)f(Tx)$. If T is invertible, then $V^{\phi,T}$ is unitary. When T is an irrational rotation, the properties of $V^{\phi,T}$ are investigated in [4], [5], [8], [10]. In [5] it is proved that, if ϕ is a step function with its discontinuities at rational points, $V^{\phi,T}$ has no eigenfunction. In this paper, we are interested in the case when T is a Bernoulli shift. We also investigate the skew product transformation associated with ϕ defined on $X \times \mathbb{T}$ by $T_\phi(x, z) = (Tx, \phi(x)z)$. For the spectral properties of T_ϕ, see [2], [7].

A function ϕ of modulus 1 is called a (multiplicative) *coboundary* if there exists $q(x)$ such that $\phi(x) = \overline{q(x)}q(Tx)$, $|q| = 1$ a.e. on X. Let $\mathcal{M} = \{h \in L^2(X) : Uh = h\}$. If T is ergodic and ϕ is real-valued, i.e. $\phi(x) = \exp(\pi i \chi_E(x))$ where χ_E is a characteristic function of E, then the dimension of $\mathcal{M}$ is 0 or 1. If dim $\mathcal{M} = 1$, then (i) $\phi(x)$ is a coboundary, and (ii) there exists q such that $q(x) = \exp(\pi i \chi_F(x))$ for some F, $\exp(\pi i \chi_E(x)) = q(x)q(Tx)$, $E = F \triangle T^{-1}F = F^c \triangle T^{-1}F^c$. Throughout the paper all the set equalities, set inclusions and function equalities are understood as being modulo measure zero sets and all the subsets are measurable unless otherwise stated.

Numbers of the form $\sum_{i=1}^{n} a_i\, k^{-i}$, $a_i \in \{0, ..., k-1\}$, are called *k-adic numbers* and denoted by $[a_1, a_2, ..., a_n]$. For $k = 2$ they are called dyadic numbers. We will regard 1 as a k-adic number. In [1] it is shown that if $0 \leq a < b \leq 1$ are dyadic numbers and if $\exp(\pi i \chi_{[a,b]})$ is a coboundary for $T : x \mapsto 2x \pmod 1$, then $a = \frac{1}{4}, b = \frac{3}{4}$.

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Let F be a Lebesgue measurable subset of $\mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$. For a point $x \in \mathbb{R}$ the *metric density* of F at x is defined to be

$$d_F(x) \equiv \lim_{r\to 0+} \frac{\lambda(F\cap(x-r,x+r))}{2r},$$

provided that this limit exists. The metric density of F equals 1 and 0 at a.e. point of F and F^c, respectively. If $(x-r,x+r)$ and $2r$ are replaced by $[x,x+r)$ and r, respectively, in the above limit, then we call the corresponding limit $d_F^+(x)$ *the right metric density* of F at x. Similarly the left metric density $d_F^-(x)$ is defined. Recall that for $f \in L^1(\mathbb{R})$, $x \in \mathbb{R}$ is called a Lebesgue point of f if

$$\lim_{r\to 0+} \frac{1}{2r} \int_{(x-r,x+r)} |f(y)-f(x)|\, d\lambda(y) = 0.$$

It is known that for $f \in L^1(\mathbb{R})$ almost every x is a Lebesgue point of f. If x is a Lebesgue point of χ_F, then $d_F(x) = d_F^+(x)$ (see [13]). The metric density of F at a specific point may not be defined, hence it is not a Lebesgue point of χ_F: For κ and $\eta, 0 \le \kappa \le \eta \le 1$, there exists $F \subset \mathbb{R}$ so that the upper and lower limits of $\lambda(F\cap(-\delta,\delta))/2\delta$ are η and κ, respectively, as $\delta \to 0$ (see [9]). Recall that for a point x a sequence $A_1, A_2, A_3, ...$ of measurable sets is said to *shrink to x nicely* if there is a constant $c > 0$ for which there is a sequence of positive numbers $r_1, r_2, r_3, ...$ with $\lim r_n = 0$ such that $A_n \subset (x-r_n, x+r_n)$ and $\lambda(A_n) \ge c \cdot r_n$. If a sequence $\{A_n\}_n$ shrinks to x nicely and x is a Lebesgue point of χ_F, then

$$d_F(x) = \lim_{n\to\infty} \frac{\lambda(F\cap A_n)}{\lambda(A_n)}.$$

(See p.140, [13].)

2. Weakly mixing skew product on Bernoulli shift

Suppose T is a measure-preserving transformation on a probability space (X,μ). For $\phi : X \to \mathbb{T}$, if the skew product transformation $T_\phi(x,z) = (Tx, \phi(x)\, z)$ on $X\times\mathbb{T}$ is weakly mixing, then we say that ϕ is weakly mixing. Let $C(T)$ be the centralizer for T, i.e. $C(T) = \{S : S\circ T = T\circ S\}$. We need the following facts.

Lemma 1. *Let $T : X \to X$ be Bernoulli, and let $\phi_1,\ \phi_2 : X \to \mathbb{T}$ be weakly mixing. Then there exists $S \in C(T)$ and a measurable function $f : X \to \mathbb{T}$ such that $\phi_1\circ S/\phi_2 = f\circ T/f$.*

For the proof see [14]. In this case we have a commutative diagram:

$$\begin{array}{ccc} L^2(X) & \xrightarrow{V^{\phi_1,T}} & L^2(X) \\ {\scriptstyle V^{f,S}}\big\downarrow & & \big\downarrow{\scriptstyle V^{f,S}} \\ L^2(X) & \xrightarrow{V^{\phi_2,T}} & L^2(X) \end{array}$$

Lemma 2. *Let $T : X \to X$ be Bernoulli, and let $\phi : X \to \mathbb{T}$ be weakly mixing. Then T_ϕ is also Bernoulli. Hence T and T_ϕ are isomorphic, since they are both Bernoulli with the same entropy.*

For the proof see [6] and [14]. As a corollary, $V^{\phi,T} : f(x) \mapsto \phi(x)f(Tx)$ in $L^2(X)$ has absolutely continuous spectrum since U_{T_ϕ} has countable Lebesgue spectrum on $L^2(X\times\mathbb{T})$ and since $V^{\phi,T}$ is spectrally equivalent to $U_{T_\phi}|_{H_1} : H_1 \to H_1$ where

$H_1 = \{f(x) \cdot z \in L^2(X \times \mathbb{T}) : f \in L^2(X)\}$ and U_{T_ϕ} is the unitary operator induced by T_ϕ.

Theorem 1. *Let T be a Bernoulli transformation on (X, μ). Define a unitary operator $V^{\phi,T} : f(x) \mapsto \phi(x)f(Tx)$ in $L^2(X)$. If ϕ is weakly mixing, then $V^{\phi,T}$ has countable Lebesgue spectrum.*

Proof. By Lemma 1 it suffices to show the existence of weakly mixing ϕ such that $V^{\phi,T}$ has Lebesgue spectrum.

First, we replace T on X by T_ψ on $X \times \mathbb{T}$ where $\psi : X \to \mathbb{T}$ is weakly mixing. By Lemma 2, T_ψ is isomorphic to T, so it is enough to prove that $\tilde{T} \equiv T_\psi$ has a weakly mixing skew product $\tilde{T}_{\tilde{\phi}}$ such that $V^{\tilde{\phi},\tilde{T}}$ has countable Lebesgue spectrum where $\tilde{X} = X \times \mathbb{T}$ and $\tilde{\phi} : \tilde{X} \to \mathbb{T}$.

Put $\tilde{\phi}(\tilde{x}) = \tilde{\phi}(x, z) = z$. Then $(V^{\tilde{\phi},\tilde{T}} F)(x, z) = zF(\tilde{T}(x, z))$ where $F \in L^2(\tilde{X})$. We use the following Fourier type argument: If $f \perp g$, then for any $n \in \mathbb{Z}$,

$$\begin{aligned}((V^{\tilde{\phi},\tilde{T}})^{(n)} f, g) &= \iint ((V^{\tilde{\phi},\tilde{T}})^{(n)} f(x, z))\overline{g(x, z)}\, d\mu\, dz \\ &= \iint \tilde{\phi}^{(n)}(x, z) f(\tilde{T}^n(x, z))\overline{g(x)}\, d\mu\, dz \\ &= \iint z^n \prod_{i=0}^{n-1} \psi^{(i)}(x) f(T^n x)\overline{g(x)}\, d\mu\, dz = 0.\end{aligned}$$

We used $\tilde{T}(x, z) = (Tx, \psi(x)z)$ and

$$\begin{aligned}\tilde{\phi}^{(n)}(x, z) &= \tilde{\phi}(x, z)\tilde{\phi}(\tilde{T}(x, z)) \cdots \tilde{\phi}(\tilde{T}^{n-1}(x, z)) \\ &= z \cdot \psi(x)z \cdot \psi^{(2)}(x)z \cdots \psi^{(n-1)}(x)z = z^n \prod_{i=0}^{n-1} \psi^{(i)}(x).\end{aligned}$$

Now take $f_n = f_n(x) \in L^2(\tilde{X})$ satisfying $f_n \perp f_m$ if $n \neq m$. Then the cyclic subspaces generated by f_n are mutually orthogonal. Similarly we have $((V^{\tilde{\phi},\tilde{T}})^{(n)} f, f) = 0$ for $n \neq 0$. □

3. A cocycle equation for one-sided Bernoulli shift

First we consider one-sided Bernoulli-$(\frac{1}{2}, \frac{1}{2})$ shift, which is measure-theoretically isomorphic to $T : x \mapsto 2x \pmod 1$ on $[0, 1)$ with Lebesgue measure λ. Note that $T^{-1}F \cap [0, r] = \frac{1}{2}F \cap [0, r]$ for $0 < r \leq \frac{1}{2}$, and $T^{-1}F \cap [r, 1] = (\frac{1}{2}F + \frac{1}{2}) \cap [r, 1)$ for $\frac{1}{2} \leq r < 1$.

Definition 1. For a fixed set F and a real $0 \leq t < 1$, define a continuous function $h_{F,t}(r)$ on $(0, 1 - t)$ by

$$h_{F,t}(r) \equiv h_t(r) = \frac{\lambda(F \cap [t, t + r])}{r}.$$

Note that $d_F^+(t) = \lim_{r \to 0+} h_{F,t}(r)$.

Lemma 3. *If $E \subset [\frac{1}{2}, 1)$ and $E = F \vartriangle T^{-1}F$ for some set F, then (i) $h_0(\frac{r}{2^n}) = h_0(r)$ for $n \in \mathbb{N}$, $0 < r \leq 1$; (ii) if $d_F^+(0)$ exists, then $d_F^+(0) = h_0(r) = 0$ or 1 for every r.*

Proof. (i) Take $0 < r \le 1$. Since $\lambda((F \triangle T^{-1}F) \cap [0, \frac{r}{2}]) = 0$, we have $F \cap [0, \frac{r}{2}] = T^{-1}F \cap [0, \frac{r}{2}]$. Hence $\lambda(F \cap [0, \frac{r}{2}]) = \lambda(T^{-1}F \cap [0, \frac{r}{2}]) = \lambda(\frac{1}{2}F \cap [0, \frac{r}{2}]) = \frac{1}{2}\lambda(F \cap [0, r])$ and $h_0(\frac{r}{2}) = h_0(r)$. Thus $h_0(\frac{r}{2^n}) = h_0(\frac{r}{2^{n-1}}) = \cdots = h_0(r)$.

(ii) Since $h_0(\frac{r}{2^n}) = h_0(r)$, for all $n \in \mathbb{N}$ and $0 \le r < 1$ by (i),

$$d_F^+(0) = \lim_{s \to 0+} \frac{\lambda(F \cap [0, s])}{s} = \lim_{n \to \infty} h_0(\frac{r}{2^n}) = h_0(r).$$

Assume that $d_F^+(0) = \alpha$, $0 < \alpha < 1$. Since for every $0 \le r < 1$, there exists sufficiently small $\delta(r) > 0$ such that $0 \le r + \epsilon < 1$ for all $0 < \epsilon < \delta(r)$, i.e.,

$$\frac{\lambda(F \cap [0, r + \epsilon])}{r + \epsilon} = \alpha,$$

$$\begin{aligned} \lambda(F \cap [r, r + \epsilon]) &= \lambda(F \cap [0, r + \epsilon]) - \lambda(F \cap [0, r]) \\ &= \alpha(r + \epsilon) - \alpha r = \alpha\epsilon. \end{aligned}$$

Hence $\lambda(F \cap [r, r + \epsilon])/\epsilon = \alpha$, so r has right metric density α for all $0 \le r < 1$. Since $0 < \alpha < 1$, we arrive at a contradiction. □

Hence we investigate the existence of $d_F^+(0)$ in Lemmas 4 and 5.

Lemma 4. *Let $E = F \triangle T^{-1}F$. If $E \subset [\frac{1}{2}, 1)$ is a finite union of intervals with dyadic endpoints, then there exists an r_0 such that for $t = [c_1, ..., c_l]$, $h_t(\frac{r}{2^n}) = h_t(r)$ and $h_t \equiv h_0$ or $h_t \equiv 1 - h_0$ for $n \in \mathbb{N}$ and $0 < r \le \frac{r_0}{2^l}$.*

Proof. Let $E = \bigcup_{i=1}^m [a_i, b_i]$ with $a_i = [a_{i,1}, ..., a_{i,p_i}]$ and $b_i = [b_{i,1}, ..., b_{i,q_i}]$ for $i = 1, ..., m$. Put $r_0 = \frac{1}{2^k}$ where $k = \max\{p_1, ..., p_m, q_1, ..., q_m\}$.

Step I. We consider the case for $l = 1$. By Lemma 3 we have $h_0(r) = h_0(\frac{r}{2^n})$ for $0 < r \le r_0$ and $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = 0$ or $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = \frac{r_0}{2}$.

Case 1. Assume $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = 0$. Since $E = F \triangle T^{-1}F$, $\lambda(F \cap [\frac{1}{2}, \frac{1}{2} + r]) = \lambda(T^{-1}F \cap [\frac{1}{2}, \frac{1}{2} + r]) = \lambda(T^{-1}F \cap [0, r]) = \frac{1}{2}\lambda(F \cap [0, 2r])$. Thus $h_{\frac{1}{2}}(r) = h_0(2r) = h_0(r)$, and $h_{\frac{1}{2}}(\frac{r}{2^n}) = h_0(\frac{r}{2^n}) = h_0(r) = h_{\frac{1}{2}}(r)$ for $0 < r \le \frac{r_0}{2}$.

Case 2. Assume $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = \frac{r_0}{2}$. Then

$$\begin{aligned} \lambda\left(F \cap \left[\frac{1}{2}, \frac{1}{2} + r\right]\right) &= r - \lambda\left(T^{-1}F \cap \left[\frac{1}{2}, \frac{1}{2} + r\right]\right) \\ &= r - \lambda(T^{-1}F \cap [0, r]) = r - \frac{1}{2}\lambda(F \cap [0, 2r]). \end{aligned}$$

Thus $h_{\frac{1}{2}}(r) = 1 - h_0(2r) = 1 - h_0(r)$ and $h_{\frac{1}{2}}(\frac{r}{2^n}) = 1 - h_0(\frac{r}{2^n}) = 1 - h_0(r) = h_{\frac{1}{2}}(r)$ for $n \in \mathbb{N}$ and $0 < r \le \frac{r_0}{2}$. Hence $h_{\frac{1}{2}}(\frac{r}{2^n}) = h_{\frac{1}{2}}(r) = h_0(r)$ or $h_{\frac{1}{2}}(\frac{r}{2^n}) = 1 - h_0(r)$.

Step II. By induction we assume that if $s = [s_1, ..., s_{l-1}]$, then $h_s(\frac{r}{2^n}) = h_s(r)$ for all $0 < r \le \frac{r_0}{2^{l-1}}$ and $h_s = h_0$ or $1 - h_0$.

Let $t = [c_1, ..., c_l]$ and $s = [c_2, ..., c_l]$; then $t = [0, c_2, ..., c_l]$ or $t = [1, c_2, ..., c_l]$. If $t = [0, c_2, ..., c_l]$, then $t = \frac{1}{2}s$, and if $t = [1, c_2, ..., c_l]$, then $t = \frac{1}{2}s + \frac{1}{2}$. Note that $\lambda(E \cap [t, t + \frac{r_0}{2^l}]) = 0$ or $\lambda(E \cap [t, t + \frac{r_0}{2^l}]) = \frac{r_0}{2^l}$.

Case 1. Assume $\lambda(E \cap [t, t + \frac{r_0}{2^l}]) = 0$. Then $\lambda(E \cap [t, t + r]) = 0$ for $0 < r \le \frac{r_0}{2^l}$. Since $E = F \triangle T^{-1}F$,

$$\lambda(F \cap [t, t + r]) = \lambda(T^{-1}F \cap [t, t + r]) = \frac{1}{2}\lambda(F \cap [s, s + 2r]).$$

Thus

$$h_t(r) = h_s(2r) = h_s(r) = h_0(r) \quad \text{or} \quad h_t(r) = 1 - h_0(r)$$

and

$$h_t(\frac{r}{2^n}) = h_s(\frac{r}{2^n}) = h_s(r) = h_t(r)$$

for $0 < r \leq \frac{r_0}{2^l}$.

Case 2. If $\lambda(E \cap [t, t + \frac{r_0}{2^l}]) = \frac{r_0}{2^l}$, then $\lambda(E \cap [t, t+r]) = r$ for $0 < r \leq \frac{r_0}{2^l}$. Since

$$\begin{aligned}\lambda(F \cap [t, t+r]) &= r - \lambda(T^{-1}F \cap [t, t+r]) \\ &= r - \frac{1}{2}\lambda(F \cap [s, s+2r]),\end{aligned}$$

we have

$$h_t(r) = 1 - h_s(2r) = 1 - h_s(r) = h_0(r) \quad \text{or} \quad h_t(r) = 1 - h_0(r)$$

and

$$h_t(\frac{r}{2^n}) = 1 - h_s(\frac{r}{2^n}) = 1 - h_s(r) = h_t(r).$$

Hence for $t = [c_1, ..., c_l]$ we have

$$h_t(\frac{r}{2^n}) = h_t(r) = h_0(r) \quad \text{or} \quad h_t(\frac{r}{2^n}) = 1 - h_0(r).$$

□

Remark. By Lemma 4 we know that $d_F^+(0)$ exists if and only if $d_F^+(t)$ exists for every dyadic point t and if it exists, then $d_F^+(t) = d_F^+(0)$ or $d_F^+(t) = 1 - d_F^+(0)$.

Lemma 5. *Let $E = F \triangle T^{-1}F$. If $E \subset [\frac{1}{2}, 1)$ is a finite union of intervals with dyadic endpoints, then every dyadic point has the right metric density for F.*

Proof. By Lemma 4 it suffices to show that the point 0 has the right metric density for F. Hence we assume that $\lim_{r \to 0+} h_0(r)$ does not exist. Let r_0 be the same as in Lemma 4. We see that for $t = [c_1, ..., c_l]$, $h_t(\frac{r}{2^n}) = h_t(r) = h_0(r)$ or $1 - h_0(r)$ for $0 < r \leq \frac{r_0}{2^l}$ by Lemma 5.

Take a Lebesgue point ξ of χ_F, with $d_F(\xi) = 1$ and put $r_n = \frac{r_0}{2^n}$. For every n take ξ_n in the set of dyadic numbers $\{[c_1, ..., c_m] : m \leq n\}$ so that the sequence ξ_n converges to ξ and $[\xi_n, \xi_n + r_n] \subset (\xi - \frac{1}{2^{n-1}}, \xi + \frac{1}{2^{n-1}})$. Since

$$\frac{\lambda([\xi_n, \xi_n + r_n])}{2^{2-n}} = \frac{r_0}{4},$$

the subsets $[\xi_n, \xi_n + r_n]$ shrink to ξ nicely. For a fixed r_0, there exists $\epsilon > 0$ such that $\epsilon < h_0(r_0) < 1 - \epsilon$. If not, the metric density at the point 0 exists. Hence for all n,

$$\frac{\lambda(F \cap [\xi_n, \xi_n + r_n])}{r_n} = h_{\xi_n}(r_n) = h_0(r_n)$$

or

$$1 - h_0(r_n) < 1 - \epsilon.$$

Since $d_F(\xi) = 1$, we have a contradiction. □

Lemma 6. *Let $E = F \triangle T^{-1}F$. If E is a finite union of intervals with dyadic endpoints and there exists a dyadic number $p > 0$ such that $E \supset [0, p)$, then there is an r_0 satisfying the following:*

(i) $h_0(\frac{r}{2^{2n}}) = h_0(r)$ and $h_0(\frac{r}{2^{2n-1}}) = 1 - h_0(r)$ for all $n \in \mathbb{N}$ and $0 < r \le r_0$.

(ii) For $t = [c_1, ..., c_l]$, $h_t(\frac{r}{2^{2n}}) = h_t(r)$, $h_t(\frac{r}{2^{2n-1}}) = 1 - h_t(r)$ and $h_t \equiv h_0$ or $h_t \equiv 1 - h_0$ for $n \in \mathbb{N}$ and $0 < r \le \frac{r_0}{2^l}$.

(iii) There is no measurable set F such that $E = F \triangle T^{-1}F$.

Proof. Let $E = \bigcup_{i=1}^m [a_i, b_i]$ with $a_i = [a_{i,1}, ..., a_{i,p_i}]$ and $b_i = [b_{i,1}, ..., b_{i,q_i}]$ for $i = 1, ..., m$. Put $r_0 = \frac{1}{2^k}$ where $k = \max\{p_1, ..., p_m, q_1, ..., q_m\}$.

(i) Take r, $0 < r \le r_0$. Since $\lambda((F \triangle T^{-1}F) \cap [0, \frac{r}{2}]) = \frac{r}{2}$, we have $\lambda(F \cap [0, \frac{r}{2}]) = \frac{r}{2} - \lambda(T^{-1}F \cap [0, \frac{r}{2}]) = \frac{r}{2} - \lambda(\frac{1}{2}F \cap [0, \frac{r}{2}]) = \frac{r}{2} - \frac{1}{2}\lambda(F \cap [0, r])$ and $h_0(\frac{r}{2}) = 1 - h_0(r)$. Hence $h_0(\frac{r}{2^{2n}}) = 1 - h_0(\frac{r}{2^{2n-1}}) = \cdots = h_0(r)$ and $h_0(\frac{r}{2^{2n-1}}) = 1 - h_0(\frac{r}{2^{2n-2}}) = \cdots = 1 - h_0(r)$.

(ii) By (i), $h_0(\frac{r}{2^{2n}}) = h_0(r)$ and $h_0(\frac{r}{2^{2n-1}}) = 1 - h_0(r)$ for $n \in \mathbb{N}$ and $0 < r \le r_0$, and $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = 0$ or $\lambda(E \cap [\frac{1}{2}, \frac{1}{2} + \frac{r_0}{2}]) = \frac{r_0}{2}$. Now proceed as in Lemma 4.

(iii) Take ξ, r_0, r_n and ξ_n as in Lemma 5. Then

$$d_F(\xi) = \lim_{n \to \infty} h_{\xi_n}(r_n) = 1 - h_0(r_0) \text{ or } h_0(r_0)$$

by the relations (i, ii). If $d_F(\xi) = 1 - h_0(r_0)$, then $h_0(r_0) = 0$ and $E \ne F \triangle T^{-1}F$ for this F. □

Remark. For $T : x \mapsto kx \pmod 1$, we can use the same argument: Put $E = \bigcup_{i=1}^m [a_i, b_i]$ with k-adic endpoints $\frac{1}{k} \le a_i < b_i \le 1$ for $i = 1, ..., m$. If $E = F \triangle T^{-1}F$ for some set F, then

(i) $h_0(\frac{r}{k^n}) = h_0(r)$ for all $n \in \mathbb{N}$ and for all $0 < r \le 1$.

(ii) If $d_F^+(0)$ exists, then $d_F^+(0) = h_0(r) = 0$ or 1.

The existence of $d_F^+(0)$ is proved similarly as in Lemmas 4 and 5. As in Lemma 6, if there is a k-adic number $p > 0$ such that $E \supset [0, p)$, then there is no measurable set F such that $E = F \triangle T^{-1}F$.

Proposition 1. *Let T be defined by $T : x \mapsto kx \pmod 1$ on $[0, 1)$ and let E be a finite union of intervals with k-adic endpoints. If there exists an integer l, $0 \le l < k$, such that $E^c \supset [\frac{l}{k}, \frac{l+1}{k}]$, then $\pm\exp(\pi i \chi_E)$ is not a coboundary.*

Proof. For the sake of notational simplicity, we consider the case $k = 2$. The other case is proved by exactly the same argument.

Assume $E \subset [\frac{1}{2}, 1]$ and $\exp(\pi i \chi_E)$ is a coboundary. Then by Lemmas 4 and 5 $d_F^+(0)$ exists, and $d_F^+(0) = 0$ or 1 by Lemma 3. If $d_F^+(0) = 1$, then $h_0(1) = d_F^+(0) = 1$, and

$$h_0(1) = \lambda(F \cap [0, 1]) = 1.$$

Thus $F = X$. If $d_F^+(0) = 0$, then $h_0(1) = d_F^+(0) = 0$, and

$$h_0(1) = \lambda(F \cap [0, 1]) = 0.$$

Thus $F^c = X$. So there is no measurable subset F such that $E = F \triangle T^{-1}F$. Since $-\exp(\pi i \chi_E) = \exp(\pi i \chi_{E^c})$ and $E^c \supset [0, \frac{1}{2})$, $-\exp(\pi i \chi_E)$ is not a coboundary.

Assume that $E \subset [0, \frac{1}{2}]$. We know that T is measure-theoretically isomorphic to itself by $\psi : x \mapsto x + \frac{1}{2} \pmod 1$. Since $\psi(E) \subset [\frac{1}{2}, 1]$ and coboundaries are sent to coboundaries by an isomorphism, the conclusion follows. □

Corollary 1. *For the transformation T defined by $x \mapsto kx \pmod 1$ for $k \geq 2$, a real-valued function $f(x)$ of modulus one is not a coboundary if the discontinuities $0 < x_1 < \cdots < x_n \leq 1$ of $f(x)$ are k-adic points and $x_1 \geq \frac{1}{k}$.*

Proof. Note that there exists a measurable set E such that $f(x) = \pm\exp(\pi i \chi_E)$ and E is a finite union of k-adic endpoints with $E^c \supset [0, \frac{1}{k}]$. □

4. A cocycle equation for two-sided Bernoulli shift

Let $(Y, \mathcal{C}, \mu)$ be a probability space, $f \in L^1(Y, \mathcal{C}, \mu)$ and $\mathcal{B} \subset \mathcal{C}$ a sub-σ-algebra. Define a complex measure $\nu(B) = \int_B f\,d\mu$, $B \in \mathcal{B}$. By the Radon-Nikodym Theorem, there is a function $g \in L^1(Y, \mathcal{B}, \mu)$ such that $\nu(B) = \int_B g\,d\mu$. We use the notation $E(f|\mathcal{B})$ for g, and call it *the conditional expectation of* f with respect to $\mathcal{B}$. Also let S be a transformation defined on Y. Then we say that $\mathcal{B}$ is an *exhaustive σ-algebra* if $S^{-1}\mathcal{B} \subset \mathcal{B}$ and $S^n\mathcal{B} \uparrow \mathcal{C}$. The Martingale Theorem states that $E(f|S^n\mathcal{B})$ converges to f a.e. and in $L^1(Y, \mathcal{C}, \mu)$ for $f \in L^1(Y, \mathcal{C}, \mu)$.

Proposition 2. *Let S be a transformation on $(Y, \mathcal{C}, \mu)$, let $\mathcal{B}$ be an exhaustive σ-algebra $\mathcal{B} \subset \mathcal{C}$, and let $f : Y \to \mathbb{T}$ be a $\mathcal{B}$-measurable map to the circle group $\mathbb{T}$. If $q : Y \to \mathbb{T}$ is a $\mathcal{C}$-measurable solution to the equation $f \cdot q = q \circ S$, then q is $\mathcal{B}$-measurable.*

Proof. We follow the idea in [11]. Applying the conditional expectation operator $E(\cdot|\mathcal{B})$ to the equation

$$(*) \qquad f \cdot q = q \circ S$$

then $f \cdot E(q|\mathcal{B}) = E(q \circ S|\mathcal{B})$ or $f \cdot E(q|\mathcal{B}) = E(q|S\mathcal{B}) \circ S$. Multiplying this with $(*)$ inverted we have

$$\overline{q(y)} \cdot E(q|\mathcal{B})(y) = \overline{q(Sy)} \cdot E(q|S\mathcal{B}) \circ S(y) \qquad \text{a.e.}$$

so that

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\,d\mu = \int_Y \overline{q} \cdot E(q|S\mathcal{B})\,d\mu.$$

By exactly the same argument, using $S^n\mathcal{B}$ in place of $\mathcal{B}$, we have

$$\int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\,d\mu = \int_Y \overline{q} \cdot E(q|S^{n+1}\mathcal{B})\,d\mu$$

so that

$$\int_Y \overline{q} \cdot E(q|\mathcal{B})\,d\mu = \int_Y \overline{q} \cdot E(q|S^n\mathcal{B})\,d\mu.$$

Taking limits, we get $\int_Y \overline{q} \cdot E(q|\mathcal{B})\,d\mu = \int_Y |q|^2\,d\mu$.

Thus $E(q|\mathcal{B}) = q$ a.e., and q is $\mathcal{B}$-measurable. □

Definition 2. Let k be fixed and $B = \bigcup_{i=1}^m B_i$ where

$$B_i = \{y|(y_0, ..., y_{n_i}) = (a_{i,0}, ..., a_{i,n_i})\}.$$

We call B an *elementary set* generated by $B_1, ..., B_m$. If there exists an integer l, $0 \leq l < k$, such that $a_{i,0} \neq l$ for all $0 \leq i \leq m$, then we say that $B_1, ..., B_m$ have *the same missing initial digit*.

Theorem 2. *For the two-sided Bernoulli-$(\frac{1}{k}, ..., \frac{1}{k})$ shift with $k \geq 2$, a function $\pm\exp(\pi i \chi_B)$ is not a coboundary if B is an elementary set generated by a finite union of blocks with the same missing initial digit.*

Proof. Without loss of generality, we assume that B is a cylinder set generated by a block. Let S be a shift on the Bernoulli-$(\frac{1}{k}, ..., \frac{1}{k})$ shift space $(Y, \mathcal{C}, \mu)$. Let $B = \{y|(y_0, ..., y_n) = (a_0, ..., a_n)\}$ be a cylinder set generated by a block $(a_0, ..., a_n)$ of finite length.

Let $\mathcal{B} \subset \mathcal{C}$ be a sub-σ-algebra generated by the cylinder sets

$$B = \{y|(y_0, ..., y_m) = (b_0, ..., b_m)\}.$$

Then $\mathcal{B}$ is an exhaustive σ-algebra of $\mathcal{C}$. If $f \cdot q = q \circ S$, then q is $\mathcal{B}$-measurable by Proposition 2. Since $(Y, \mathcal{B}, \mu_{|\mathcal{B}}, S)$ is measure-theoretically isomorphic to $(X, \mathcal{A}, \lambda, T)$ where $X = [0, 1)$, $\mathcal{A}$ is a Borel σ-algebra, λ is Lebesgue measure, and T is defined by $x \mapsto kx \pmod 1$, it follows that $f(y)$ is a coboundary if and only if $\exp(\pi i \chi_{[a,b]}(x))$ is a coboundary where $a = [a_0, ..., a_n]$, $b = [a_0, ..., a_n + 1]$. Hence $f(y)$ is not a coboundary by Proposition 1.

For the finite union of blocks with the same missing initial digit, the proof is similar. □

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