

EVERY (λ^+, \varkappa^+) -REGULAR ULTRAFILTER IS (λ, \varkappa) -REGULAR

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ABSTRACT. We prove the following:

Theorem A. *If D is a (λ^+, \varkappa) -regular ultrafilter, then either*

- (a) *D is (λ, \varkappa) -regular, or*
- (b) *the cofinality of the linear order $\prod_D \langle \lambda, < \rangle$ is cf \varkappa , and D is (λ, \varkappa') -regular for all $\varkappa' < \varkappa$.*

Corollary B. *Suppose that \varkappa is singular, $\varkappa > \lambda$ and either λ is regular, or cf $\varkappa < cf \lambda$. Then every $(\lambda^{+\varkappa}, \varkappa)$ -regular ultrafilter is (λ, \varkappa) -regular.*

We also discuss some consequences and variations.

The notion of a (λ, \varkappa) -regular ultrafilter has been introduced by J. Keisler in [Kei]. An ultrafilter D is (λ, \varkappa) -regular iff there is a family of \varkappa members of D such that the intersection of any λ members of the family is empty.

In [Kei] Keisler proved some cardinality results about ultraproducts taken modulo such ultrafilters. Further results were proved in the 70's: for example, the following are theorems of ZFC:

- (a) Every (λ^+, λ^+) -regular ultrafilter is (λ, λ) -regular ([CC], [KP]).
- (b) If λ is singular, then every (λ^+, λ^+) -regular ultrafilter is (λ, λ^+) -regular [Ka]; moreover, it is either $(cf \lambda, cf \lambda)$ -regular or (λ', λ^+) -regular for some $\lambda' < \lambda$ ([CC], [KP]).
- (c) If $2^\varkappa = \varkappa^+$ and $2^{\varkappa^+} > \varkappa^{++}$, then every $(\varkappa^+, \varkappa^+)$ -regular ultrafilter is (\varkappa, \varkappa^+) -regular ([BK], [Ket]).

It was soon realized, however, that (ir)regular ultrafilters are connected with large cardinals, inner models, and combinatorial or reflection principles; this paved the way for significant applications to set theory, but seemed to dash any hope that other results besides (a)–(c) above can be proved in ZFC alone (see e.g. [KM] or [Lp1] for further references).

However, in [Lp1] (more than twenty years later) we proved some slight improvements of (b), as well as a “down from exponents” transfer result for (λ, λ) -regularity; we also suggested the possibility that further results are theorems of ZFC. It is actually so; in this paper we prove the following generalization of (a):

Theorem 1. *If $n < \omega$, then every $(\lambda^{+n}, \varkappa^{+n})$ -regular ultrafilter is (λ, \varkappa) -regular.*

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Theorem 1 solves a problem raised in [Lp1]. Actually, (the proof of) Theorem 1 has some further consequences:

Corollary 1. *Suppose that $\lambda \leq \varkappa$, λ is a regular cardinal, and D is a (λ^+, \varkappa) -regular ultrafilter. Then the following are equivalent:*

- (i) D is (λ, \varkappa) -regular;
- (ii) the cofinality of $\prod_D \langle \lambda, < \rangle$ is $> \varkappa$;
- (iii) the cofinality of $\prod_D \langle \lambda, < \rangle$ is different from $\text{cf } \varkappa$.

Corollary 2 ([Lp1]). *Suppose that $\text{cf } \lambda \neq \text{cf } \varkappa$. If D is (λ^+, \varkappa) -regular and not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, then D is (λ, \varkappa) -regular.*

Corollary 3. *If D is (λ^+, \varkappa) -regular and $(\text{cf } \lambda, \text{cf } \varkappa)$ -regular, then D is (λ, \varkappa) -regular.*

Corollary 4. *Suppose that D is (λ^+, \varkappa) -regular and (λ'^+, \varkappa') -regular. If $\text{cf } \lambda = \text{cf } \lambda'$ and $\text{cf } \varkappa \neq \text{cf } \varkappa'$, then D is either (λ, \varkappa) -regular or (λ', \varkappa') -regular.*

The following improves [BK, Theorem 1.3 and Corollaries 1.8 and 2.5]:

Proposition. *Suppose that D is a (λ^+, \varkappa) -regular ultrafilter, and that the cofinality of the linear order $\prod_D \langle \lambda, < \rangle$ is different from $\text{cf } \varkappa$. Then D is (λ, \varkappa) -regular.*

Notice that, trivially, the cofinality of $\prod_D \langle \lambda, < \rangle$ equals the cofinality of $\prod_D \langle \text{cf } \lambda, < \rangle$.

In order to discuss some further results, let us introduce some notation and definitions (see [CK] for basics about models and ultrafilters).

$S_\mu(X)$ denotes the set of all subsets of X of cardinality $< \mu$. If X is a well-ordered set and α is an ordinal, $S_\alpha(X)$ is the set of all subsets of X which have order type $< \alpha$ (notice that the two definitions are consistent).

Throughout this paper, we shall use the following reformulation of (λ, \varkappa) -regularity. As Benda and Ketonen [BK] noticed, it makes sense even when λ is allowed to be an ordinal: the next corollary deals with such a situation. See [Lp1] for further comments.

An ultrafilter D is (β, \varkappa) -regular iff in the ultrapower $\prod_D \langle S_\beta(\varkappa), \subseteq, \{\alpha\}_{\alpha \in \varkappa} \rangle$ there is an element x such that $\{\alpha\} \subseteq x$, for every $\alpha \in \varkappa$ (equivalently, for \varkappa -many α 's).

Corollary 5. *Let D be an ultrafilter, and let \varkappa be a cardinal. Let γ be the least ordinal such that D is (γ, \varkappa) -regular. Then either:*

- (i) γ is a limit ordinal and D is $(\text{cf } \gamma, \text{cf } \gamma)$ -regular, or
- (ii) $\gamma = \delta + 1$, δ is a limit ordinal and the cofinality of $\prod_D \langle \delta, < \rangle$ is $\text{cf } \varkappa$. Hence either D is $(\text{cf } \delta, \text{cf } \delta)$ -regular and $\text{cf } \varkappa > \text{cf } \delta$, or $\text{cf } \varkappa = \text{cf } \delta$ and D is not $(\text{cf } \delta, \text{cf } \delta)$ -regular.

Theorem 2. *There is a finite expansion \mathbf{A}^+ of the model $\langle S_{\lambda+n}(\mu^{+n}), V, \subseteq, \{\alpha\}_{\alpha \in \mu^{+n}} \rangle$ (where $V(a)$ iff $a \in S_\lambda(\mu)$) such that whenever $\mathbf{B} \equiv \mathbf{A}^+$ and there is $b \in B$ such that $|\{\alpha \in \mu^{+n} \mid \mathbf{B} \models \{\alpha\} \subseteq b\}| = \mu^{+n}$, then there is $b' \in B$ such that $V(b')$ and $|\{\alpha \in \mu \mid \mathbf{B} \models \{\alpha\} \subseteq b'\}| = \mu$.*

The statement of Theorem 2 reads $\text{alm}(\lambda^{+n}, \mu^{+n}) \Rightarrow \text{alm}(\lambda, \mu)$, in the notation of [Lp2]. There we also discuss the interest of such results. Notice that, in view of the reformulation of (λ, μ) -regularity we have given, Theorem 2 is stronger than Theorem 1.

Finally, let us mention that (λ, \varkappa) -regular ultrafilters have found other applications both in general topology and in the model theory for abstract logics (see [Lp1], [Lp2] for further references).

A notice to the reader wanting to consult the literature on regular ultrafilters: results like (a)–(c) above are sometimes stated in equivalent forms in terms of the closely related notions of λ -decomposability, λ -descending incompleteness and *uniformity*. We shall not need those notions in the present paper, but, for the convenience of the reader, we give below a “table of correspondence”.

For every cardinal λ and every ultrafilter D , the following hold:

- (i) D is λ -descendingly incomplete iff it is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular.
- (ii) If D is λ -decomposable, then D is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular.
- (iii) If D is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, then D is (λ, λ) -regular.
- (iv) D is λ -decomposable iff there is a D' uniform on λ and $D' \leq D$ in the Rudin Keisler (pre)-order.

Moreover, if λ is regular, then,

- (v) if D is (λ, λ) -regular, then D is λ -decomposable.

By the above statements, for λ a regular cardinal, the notions of (λ, λ) -regularity, λ -descending incompleteness and λ -decomposability are all equivalent. In particular, the notion of (λ, \varkappa) -regularity encompasses all the above notions, except for λ -decomposability when λ is singular.

Moreover, it is trivial that if $D' \leq D$ in the Rudin Keisler order, and D' is (λ, \varkappa) -regular, then D is (λ, \varkappa) -regular, also.

These facts make it possible, for example, to reformulate (a) together with the second statement in (b) as: every uniform ultrafilter over λ^+ is either λ -descendingly incomplete or (λ', λ^+) -regular for some $\lambda' < \lambda$.

We now prove the stated results. The statement in the title of the paper is a consequence of Theorem A in the abstract (when \varkappa is a successor cardinal). Theorem 1 follows now from a simple induction. Corollary 4 is immediate from the proposition.

Proof of the Proposition. For every $x \in S_{\lambda^+}(\varkappa)$, let $G(x, \beta)$ ($\beta < \lambda$) be a sequence of subsets of x such that:

- (i) if $\beta \leq \beta' < \lambda$, then $G(x, \beta) \subseteq G(x, \beta')$;
- (ii) $|G(x, \beta)| \leq |\beta|$, for every $\beta < \lambda$;
- (iii) $\bigcup_{\beta < \lambda} G(x, \beta) = x$.

Consider the model $\mathbf{A} = \langle S_{\lambda^+}(\varkappa), \subseteq, U, <, G, \{\alpha\}_{\alpha \in \varkappa} \rangle$, where $\langle U, < \rangle = \langle \lambda, < \rangle$ and G is the above function from $A \times U$ to A .

Thus, by (iii) above, for every $\alpha \in \varkappa$, \mathbf{A} satisfies

$$(*) \quad \forall x(\{\alpha\} \subseteq x \Rightarrow \exists w(U(w) \wedge \{\alpha\} \subseteq G(x, w))).$$

Let D be a (λ^+, \varkappa) -regular ultrafilter; let us work in $\mathbf{B} = \prod_D \mathbf{A}$; and let y witness the (λ^+, \varkappa) -regularity of D . By $(*)$, and since y witnesses that D is (λ^+, \varkappa) -regular, for every $\alpha \in \varkappa$, there is a w_α in $\mathbf{B}|_U = \prod_D \lambda$ such that, in \mathbf{B} , $\{\alpha\} \subseteq G(y, w_\alpha)$.

For every $w \in \mathbf{B}|_U$, let $X_w = \{\alpha \mid \mathbf{B} \models \{\alpha\} \subseteq G(y, w)\}$. Notice that:

- (I) The union of all the X_w 's ($w \in \mathbf{B}|_U$) is \varkappa , by the above argument.
- (II) If $w \leq w' \in \mathbf{B}|_U$, then $X_w \subseteq X_{w'}$, because of clause (i) (and elementarity).

It is enough to show that there is $w \in \mathbf{B}|_U$ such that $|X_w| = \varkappa$. If this happens, then $G(y, w)$ witnesses the (λ, \varkappa) -regularity of D (because of (ii), $G(y, w)$ is in $\prod_D S_\lambda(\varkappa)$).

Suppose by contradiction that

(III) $|X_w| < \varkappa$, for all $w \in \mathbf{B}|_U$.

Easy cofinality arguments then show that (I)–(III) imply that $\text{cf}\langle \mathbf{B}|_U, < \rangle = \text{cf } \varkappa$, contradicting the hypothesis. \square

Proof of Theorem A. If (a) fails, then the proposition implies that $\text{cf} \prod_D \langle \lambda, < \rangle$ is $\text{cf } \varkappa$.

If $\varkappa' < \varkappa$, then D is trivially (λ^+, \varkappa') -regular. If $\text{cf } \varkappa' \neq \text{cf } \varkappa$, then the proposition (applied to (λ^+, \varkappa') -regularity) implies that D is (λ, \varkappa') -regular.

But this is enough, since for every $\varkappa'' < \varkappa$ there is \varkappa' such that $\text{cf } \varkappa' \neq \text{cf } \varkappa$ and $\varkappa'' \leq \varkappa' < \varkappa$. \square

Proof of Corollary 1. (i) \Rightarrow (ii) is standard (e.g. [BK, p. 233]).

(ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i) is immediate from the proposition. \square

Remark. The hypothesis λ regular is necessary in Corollary 1. If μ is strongly compact, then there exists a μ -complete $(\mu, \mu^{+\omega+\omega})$ -regular ultrafilter D . If we take $\lambda = \mu^{+\omega}$ and $\varkappa = \mu^{+\omega+\omega}$, then condition (i) holds, but (ii) and (iii) fail ($\text{cf} \prod_D \langle \lambda, < \rangle = \text{cf} \prod_D \langle \omega, < \rangle = \omega$, since D is ω_1 -complete).

We wonder whether some generalization of Corollary 1 is possible when λ is singular.

Notice that Corollary 1 and Theorem A imply that if λ is regular, and D is (λ^+, \varkappa) -regular, then $\text{cf} \prod_D \langle \lambda, < \rangle \geq \varkappa$.

Proof of Corollary B. An induction on n shows that it is enough to prove the particular case when $n = 1$. So assume that \varkappa is singular, $\varkappa > \lambda$ and D is (λ^+, \varkappa) -regular.

Case 1: λ is regular. By Theorem A, D is (λ, \varkappa') -regular for all $\varkappa' < \varkappa$. By, e.g., Corollary 1, the cofinality of $\prod_D \langle \lambda, < \rangle$ is $> \varkappa'$, for all $\varkappa' < \varkappa$. Since \varkappa is singular, the cofinality of $\prod_D \langle \lambda, < \rangle$ must be $> \varkappa$; in particular, it is $\neq \text{cf } \varkappa$. Hence, case (b) in Theorem A cannot occur.

Case 2: $\text{cf } \varkappa < \text{cf } \lambda$. Then case (b) in Theorem A cannot occur, since the cofinality of $\prod_D \langle \lambda, < \rangle$ is $\text{cf } \lambda$ if D is not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, and is $> \text{cf } \lambda$ otherwise. Hence, we are in case (a) and we are done. \square

Proof of Corollary 2. D is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular iff the cofinality of $\prod_D \langle \text{cf } \lambda, < \rangle$ is different from $\text{cf } \lambda$ (e.g., by Corollary 1). So it is enough to apply the proposition. \square

Proof of Corollary 3. If $\varkappa \leq \lambda$, the result is trivial. The case $\text{cf } \varkappa < \text{cf } \lambda$ is covered by Corollary B. Otherwise, by Corollary 1, the cofinality of $\prod_D \langle \text{cf } \lambda, < \rangle$ is $> \varkappa$. Now apply the proposition. \square

Proof of Theorem 2. It is enough to prove the case $n = 1$.

The proofs of the proposition and of Theorem A have been devised to work as a proof for Theorem 2.

Add to the model \mathbf{A} the relations and functions necessary in order to carry over the proof of the proposition in the two cases $\varkappa = \mu^+$ and $\varkappa = \mu$. Notice that $U = \lambda$ is the same in both cases.

Let \mathbf{B} be as in the statement of Theorem 2. The cofinality of $\langle \mathbf{B}|_U, < \rangle$ is different from $\text{cf } \varkappa$ for at least one of the choices $\varkappa = \mu^+$ and $\varkappa = \mu$. Now it is enough to apply the arguments in the proof of the proposition, for the appropriate choice of \varkappa . \square

Of course, a generalization of Theorem A can be given along the lines of the above proof. We can obtain the separation in cases (a) and (b); we can also obtain that, in case (b), the cofinality of $\langle U, < \rangle$ becomes $\text{cf } \varkappa$, in \mathbf{B} (but notice that, when \varkappa is singular, we need $\text{cf } \varkappa$ symbols in the expansion \mathbf{A}^+). Corollaries 2 and 4, too, can be generalized.

We can also generalize Corollaries 1, 3 and B in this fashion, but some technicalities are needed in the proof: we have to add Skolem functions to \mathbf{A}^+ , and consider the substructure of \mathbf{B} generated by b (seemingly, \varkappa^+ symbols are needed in \mathbf{A}^+).

Proof of Corollary 5. (i) Let γ be limit. For every $x \in S_\gamma(\varkappa)$ let $F(x)$ be the order type of x . Consider an appropriate expansion of $S_\gamma(\varkappa)$, and take its ultrapower under D . If y witnesses (γ, \varkappa) -regularity, then $\beta < F(y)$, for every $\beta < \gamma$, since, otherwise, D would be $(\beta + 1, \varkappa)$ -regular. This easily implies that D is $(\text{cf } \gamma, \text{cf } \gamma)$ -regular.

Essentially, this is the argument used in [Lp2, Corollary 1.4].

(ii) If $\gamma = \delta + 1$, then it is easy to show that δ is limit.

A variation on the proof of the proposition shows that the cofinality of $\prod_D \langle \delta, < \rangle$ is $\text{cf } \varkappa$. Just consider U to be $\langle \delta, < \rangle$ and for $\beta < \delta$ let $G(x, \beta)$ be the initial subset of x of order type β . Now observe that the case corresponding to (a) in Theorem A cannot occur since γ is the least ordinal for which D is (γ, \varkappa) -regular.

The last remark follows from the fact that the cofinality of $\prod_D \langle \delta, < \rangle$ is equal to the cofinality of $\prod_D \langle \text{cf } \delta, < \rangle$. \square

REFERENCES

- [BK] M. Benda and J. Ketonen, *Regularity of ultrafilters*, Israel J. Math. **17** (1974), 231–240. MR **53**:132
- [CC] G. V. Cudnovskii and D. V. Cudnovskii, *Regular and descending incomplete ultrafilters* (English Translation), Soviet Math. Dokl. **12** (1971), 901–905.
- [CK] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland, Amsterdam, 1977. MR **58**:27177
- [Ka] A. Kanamori, *Weakly normal filters and irregular ultrafilters*, Trans Amer. Math. Soc. **220** (1976), 393–399. MR **58**:240
- [KM] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in Set Theory*, in Higher Set Theory, Lecture Notes in Mathematics, 669, Springer-Verlag, Berlin, 1978. MR **80b**:03083
- [Kei] J. Keisler, *On cardinalities of ultraproducts*, Bull. Amer. Math. Soc. **70** (1964), 644–647.
- [Ket] J. Ketonen, *Nonregular ultrafilters and large cardinals*, Trans. Amer. Math. Soc. **224** (1976), 61–73. MR **54**:7260
- [KP] K. Kunen and K. L. Prikry, *On descendingly incomplete ultrafilters*, J. Symbolic Logic **36** (1971), 650–652. MR **46**:1585
- [Lp1] P. Lipparini, *More on regular ultrafilters in ZFC*, revised for J. Symbol. Logic.
- [Lp2] P. Lipparini, *Ultrafilter translations, I: (λ, λ) -compactness of logics with a cardinality quantifier*, Arch. Math. Logic **35** (1996), 63–87. MR **96k**:03087

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