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BISHOP'S PROPERTY (β) AND ESSENTIAL SPECTRA OF QUASISIMILAR OPERATORS

LIN CHEN AND YAN ZIKUN

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Abstract. We analyze the notion of Bishop's property (β) to obtain some new concepts. We describe some conditions in terms of these concepts for an operator to have its essential spectrum (spectrum) contained in the essential spectrum (spectrum) of every operator quasisimilar to it. A subfamily of such operators is proved to be dense in $L(\mathbf{H})$.

1. Preliminaries and notations

The concept of quasisimilarity of linear operators was introduced by B. Sz-Nagy and C. Foias [1] in 1967. Quasisimilarity is an equivalent relation weaker than similarity. Similarity preserves the spectrum and essential spectrum of an operator, but this fails to be true for quasisimilarity. Suppose that $S \overset{q}{\sim} T$. What condition should be imposed on S and T to insure the equality relation $\sigma_e(S) = \sigma_e(T)$ ($\sigma(S) = \sigma(T)$)? A list of results have been announced along this line. We would like to recall that Yang [2] proved that two quasisimilar M-hyponormal operators have equal essential spectra and M. Putinar [3] proved that two densely similar tuples of operators having Bishop's property (β) ([4]) have equal essential spectra.

However, the foregoing works paid more attention to the equality of essential spectra and spectra than the inclusion relations among them, and the methods applied formerly to different families of operators were varied. We are now going to seek some general conditions for a bounded linear operator $S \in L(\mathbf{H})$ to have its essential spectrum (spectrum) contained in that of every operator quasisimilar to it. We call such an operator S a (Q) ((P)) operator, denoted as $S \in (Q)$ ($S \in (P)$). Of course, if both S and $T \in (Q)$ ((P)) and $S \overset{q}{\sim} T$, then S and T have equal essential spectra (spectra).

The results in [2], [3] motivate us to analyze Bishop's property (β). It is well known that the subdecomposability of an operator T is equivalent to having Bishop's property (β) [5]. In section 2, we "localize" the property (β) of an operator S to obtain the concepts $A(S), E_1(S), E_2(S), C_1(S), C_2(S)$ as defined below and discuss their mutual relations and the relations between them and the spectral structure of S. In section 3, we establish certain sufficient conditions for (Q) ((P), etc.) operators (Theorems 1,2). We then give a number of corollaries and

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examples to exhibit the applications of the above theorems. It is shown that the family (Q) ((P), etc.) contains many familiar operators. A subfamily (Q_2) of (Q) is proved (Theorem 3) to be norm-dense in $L(\mathbf{H})$.

Let $\mathbf{H}$ denote an infinite-dimensional separable complex Hilbert space, and let $L(\mathbf{H})$ denote the set of all bounded linear operators on $\mathbf{H}$. For $T \in L(\mathbf{H})$, $\sigma(T)$, $\rho(T)$, $\sigma_e(T)$ and $\sigma_{re}(T)$ denote the spectrum, resolvent set, essential spectrum and right essential spectrum of T respectively. Write

$\rho_e(T) = \mathbb{C} \setminus \sigma_e(T), \rho_{re}(T) = \mathbb{C} \setminus \sigma_{re}(T),\ \mathbb{C} =$ complex plane,
$\rho_{_D}(T) = \{\lambda \in \mathbb{C};\ R(T-\lambda)\ \text{is closed}\},\ \sigma_{_D}(T) = \mathbb{C} \setminus \rho_{_D}(T)$,
$\rho_r(T) = \{\lambda \in \mathbb{C};\ R(T-\lambda) = \mathbf{H}\}$,
$\psi(T) = \{\lambda \in \mathbb{C};\ T-\lambda \text{ is semi-Fredholm}\}$,
$\sigma_p(T) =$ the set of all eigenvalues of T,
$\nu(T) = \dim \mathrm{Ker} T, \quad \mu(T) = \dim R(T)^{\perp}$,
$H_{m,n}(T) = \{\lambda \in \mathbb{C};\ \nu(T-\lambda) = m,\ \mu(T-\lambda) = n\}\ (m, n = 0, 1, \cdots, \infty)$,
$\psi_{m,n}(T) = \rho_{_D}(T) \cap H_{m,n}(T)\ (m, n = 0, 1, \cdots, \infty), \quad \sigma_c(T) = \sigma(T) \cap H_{0,0}(T)$.

Suppose $\lambda \in \mathbb{C}$, $T \in L(\mathbf{H})$. λ is called a regular point of the operator T if $\|P_{\mathrm{Ker}(T-\mu)} - P_{\mathrm{Ker}(T-\lambda)}\| \to 0\ (\mu \to \lambda)$, where P_M denotes the orthogonal projection onto the subspace M of $\mathbf{H}$. $\tau^r(T)$ denotes the set of all regular points of T, $\tau^s(T) = \mathbb{C} \setminus \tau^r(T)$. For every set-valued function $B(\cdot) : L(\mathbf{H}) \to 2^{\mathbb{C}}$, write $B^r(T) = B(T) \cap \tau^r(T)$, $B^s(T) = B(T) \setminus \tau^r(T)$. By [6], $\psi^r(T)$ is an open set and $\psi^s(T)$ consists of isolated points only.

Suppose $T, S \in L(\mathbf{H})$. T is called a dense (injective, quasiaffine) transform of S if there exists a dense-ranged (injective, dense-ranged and injective) operator $A \in L(\mathbf{H})$ such that $AT = SA$. We denote it as $T \xrightarrow{dr} S$ ($T \xrightarrow{inj} S$, $T \xrightarrow{q} S$). T is said to be densely (injectively, quasi-) similar to S if $T \xrightarrow{dr} S \xrightarrow{dr} T$ ($T \xrightarrow{inj} S \xrightarrow{inj} T$, $T \xrightarrow{q} S \xrightarrow{q} T$) and it is denoted as $T \overset{dr}{\sim} S$ ($T \overset{inj}{\sim} S$, $T \overset{q}{\sim} S$). Notice that, $\forall z \in \mathbb{C}$,

$T \xrightarrow{inj} S \Longrightarrow \nu(T-z) \le \nu(S-z)$,
$T \xrightarrow{dr} S \Longrightarrow \mu(T-z) \ge \mu(S-z)$,
$T \overset{q}{\sim} S \Longrightarrow \nu(T-z) = \nu(S-z),\ \mu(T-z) = \mu(S-z)$.
$(Q) = \{S \in L(\mathbf{H});\ \sigma_e(S) \subset \sigma_e(T),\ \forall T \overset{q}{\sim} S\}$,
$(P) = \{S \in L(\mathbf{H});\ \sigma(S) \subset \sigma(T),\ \forall T \overset{q}{\sim} S\}$,
$(P)_{\xrightarrow{inj}} = \{S \in L(\mathbf{H});\ \sigma(S) \subset \sigma(T),\ S \xrightarrow{inj} T)\}$,
$(P)_{\xleftarrow{dr}} = \{S \in L(\mathbf{H});\ \sigma(S) \subset \sigma(T),\ \forall T \xrightarrow{dr} S\}$,
$(Q)_{\overset{dr}{\sim}} = \{S \in L(H);\ \sigma_e(S) \subset \sigma_e(T),\ \forall T \overset{dr}{\sim} S\}$.

2. Localization of Bishop's property (β)

Bishop's property (β) was first introduced in [4]. It can be defined as follows. $\mathcal{O}(U, \mathbf{H})$ denotes the Fréchet space of all $\mathbf{H}$-valued analytic functions on the open set $U \subset \mathbb{C}$ with the topology defined by uniform convergence on every compact subset of U.

Definition 1. $T \in L(\mathbf{H})$ is said to have Bishop's property (β) (denoted by $T \in (\beta)$) if the mapping $\alpha_{T,U} :\ \mathcal{O}(U, \mathbf{H}) \to \mathcal{O}(U, \mathbf{H}),\ f \mapsto (T-z)f$ is injective and has closed range on every open subset U of $\mathbb{C}$.

We are now going to "localize" Bishop's property (β) (with respect to $\lambda \in \mathbb{C}$).

Definition 2. Suppose $T \in L(\mathbf{H}),\ \lambda \in \mathbb{C}$. Then

$\lambda \in A(T) \overset{d}{\Longleftrightarrow} \exists\delta > 0$ such that for $U = O(\lambda, \delta'),\ 0 < \delta' < \delta$, $\alpha_{T,U}$ is injective,

$\lambda \in E_1(T) \overset{d}{\Longleftrightarrow} \exists\delta > 0$ such that for $F = O(\lambda, \delta')^-,\ 0 < \delta' < \delta$, α_{T,F^c} has closed range,

$\lambda \in E_2(T) \overset{d}{\Longleftrightarrow} \exists\delta > 0$ such that for $U = O(\lambda, \delta'),\ 0 < \delta' < \delta$, $\alpha_{T,U}$ has closed range,

$\lambda \in C_1(T) \overset{d}{\Longleftrightarrow} \exists\delta > 0$ such that for $F = O(\lambda, \delta')^-,\ 0 < \delta' < \delta$, the spectral manifold $H_T(F) = \{x \in \mathbf{H};\ \exists f \in \mathcal{O}(F^c, \mathbf{H}),\ x = (T - z)f(z)\ (z \in F^c)\}$ is closed (without assuming that T has the single-valued extension property, cf. [7]),

$\lambda \in C_2(T) \overset{d}{\Longleftrightarrow} \exists\delta > 0$ such that for $U = O(\lambda, \delta'),\ 0 < \delta' < \delta$, $H_T(U^c)$ is closed.

Definition 3. Suppose that $T \in L(\mathbf{H})$. Then

$T \in (A)\ ((E_1),\ (E_2),\ (C_1),\ (C_2)) \overset{d}{\Longleftrightarrow} A(T)\ (E_1(T),\ E_2(T),\ C_1(T),\ C_2(T)) = \mathbb{C}$,

$T \in (E) \overset{d}{\Longleftrightarrow} \forall$ open set $U \subset \mathbb{C}$, $\alpha_{T,U}$ has closed range,

$T \in (C) \overset{d}{\Longleftrightarrow} \forall$ closed set $F \subset \mathbb{C}$, $H_T(F)$ is closed.

With $(E_1) \cap (E_2)$ abbreviated as (E_1E_2), etc., it is clear that $(\beta) = (AE),\ (E) \subset (E_1E_2),\ (C) \subset (C_1C_2)$.

We notice also that

$T \in (A) \Longleftrightarrow T$ has the single-valued extension property,

$T \in (AC) \Longleftrightarrow T$ satisfies Dunford's condition C ([7], [8]).

Proposition 1. *Suppose* $T \in L(\mathbf{H})$. *Then* $\rho(T) \subset A(T) \cap E_1(T) \cap E_2(T) \cap C_1(T) \cap C_2(T)$.

Proof. Suppose $\lambda \in \rho(T)$. We only need to show that $\lambda \in E_1(T)$, the rest is obvious from the above definitions.

Take any $\delta > 0$ such that $O(\lambda, \delta)^- \subset \rho(T)$. Suppose that $F = O(\lambda, \delta')^-,\ 0 < \delta' < \delta$, and $f_n,\ g_n \in \mathcal{O}(F^c, \mathbf{H}),\ (T - z)f_n = g_n,\ g_n \to g \in \mathcal{O}(F^c, \mathbf{H})$.

Take a Cauchy domain $U' \supset \sigma(T),\ \overline{U'} \subset F^c$. $f_n(z) = (T - z)^{-1} g_n(z)\ (z \in F^c \setminus \sigma(T))$, where $g_n(z)$ converges uniformly on $\partial U'$. It follows that $f_n(z)$ converges uniformly on $\partial U'$. By the principle of maximum modulus, $f_n(z)$ converges uniformly on $\overline{U'}$ and hence on every compact subset of F^c. Hence $f_n \to f \in \mathcal{O}(F^c, \mathbf{H}),\ (T - z)f(z) = g(z)\ (z \in F^c)$ and the conclusion $\lambda \in E_1(T)$ follows. □

Proposition 2. *Suppose* $T \in L(\mathbf{H})$. *Then* (1) $E_2(T) \subset C_2(T)$, (2) $E_1(T) \subset C_1(T) \subset A(T)$.

Notice that $E_2(T) \subset C_2(T),\ E_1(T) \subset C_1(T)$ is obvious. The proof of $C_1(T) \subset A(T)$ is a simple modification of that of $(C) \subset (A)$ in [7, Th.2] and we omit it.

Proposition 3. *Suppose that* $\mathbf{H} = H_1 \oplus H_2,\ T_i \in L(H_i),\ i = 1, 2,\ T = T_1 \oplus T_2$. *Then* $A(T) = A(T_1) \cap A(T_2),\ E_j(T) = E_j(T_1) \cap E_j(T_2),\ C_j(T) = C_j(T_1) \cap C_j(T_2),\ j = 1, 2$.

This is obvious from the definitions.

Proposition 4. $(\beta) = (E) = (E_1E_2) = (AE_2) = (CE) = (CE_2) \subset (C) = (C_2C_1) = (AC) = (AC_2) \subset (A)$.

Proof. (1) $(E) \subset (C)$. Obvious from the definitions.

(2) $(AE_2) \subset (E)$. Suppose that $T \in (AE_2)$, U open, $f_n,\ g_n,\ g_0 \in \mathcal{O}(U, \mathbf{H}),\ g_n \to g_0,\ (T-z)f_n(z) = g_n(z)\ (z \in U)$. Let $\lambda \in U$. By the definition of (E_2), there exists $\delta > 0,\ V = O(\lambda, \delta) \subset U,\ f_V \in \mathcal{O}(V, \mathbf{H}),\ (T-z)f_V(z) = g_0(z)\ (z \in V)$. Since $T \in (A)$, there is a unique $f \in \mathcal{O}(U, \mathbf{H}),\ f(z) = f_V(z)\ (z \in V)$ and hence $(T-z)f(z) = g_0(z)\ (z \in U)$ and $T \in (E)$.

(3) $(AC_2) \subset (C)$. The argument is similar to (2).

(4) Other relations follow from (1),(2),(3) and Proposition 2. □

Proposition 5. *Suppose $T \in L(H),\ \lambda \in \mathbb{C}$.*

(1) If $\lambda \in \rho_D(T) \setminus \psi^s_{\infty,\infty}(T)$, then $\lambda \in A(T) \Longrightarrow \lambda \notin \sigma_p(T)^0 \Longleftrightarrow \exists m,\ 0 \le m \le \infty$, and $\delta > 0$ such that $O'(\lambda, \delta) = O(\lambda, \delta) \setminus \{\lambda\} \subset \psi_{0,m}(T)$;

(2) $A(T) \cap \rho_r(T) \subset \rho(T)$.

Proof. (1) Suppose that $\lambda \in \rho_D(T) \setminus \psi^s_{\infty,\infty}(T)$. The statement that $\lambda \notin \sigma_p(T)^0 \Longleftrightarrow \exists 0 \le\ m \le \infty,\ \delta > 0,\ O'(\lambda, \delta) \subset \psi_{0,m}(T)$ is obvious from the structure of the spectrum of $T \in L(\mathbf{H})$.

Now suppose that $\lambda \in \sigma_p(T)^0 \cap \rho_D(T) \setminus \psi^s_{\infty,\infty}(T) = (\sigma_p(T)^0 \cap \rho^r_D(T)) \cup (\sigma_p(T)^0 \cap \psi^s(T))$. It is to be shown that $\lambda \notin A(T)$. We assume that $\lambda = 0$.

Suppose that $\lambda \in \sigma_p(T)^0 \cap \rho^r_D(T)$. Let P be the orthogonal projection onto $\mathrm{Ker}T$, $T' \in L(\mathbf{H}),\ T'|_{R(T)^\perp} = 0,\ T'|_{R(T)}$ is the inverse of $T|_{\mathrm{Ker}T^\perp}$. It is clear that $T'T = I - P,\ PT' = 0$. Let $P(z)$ denote $(I - zT')^{-1}P\ (|z| < \delta_1)$. A simple calculation shows that $P(z)^2 = P(z)$ is a projection, $R(P(z)) = \mathrm{Ker}(I - P(z)) \supset \mathrm{Ker}(T - z)$. Making use of the regularity of λ (i.e. $P_{\mathrm{Ker}(T-z)} \to P_{\mathrm{Ker}T} = P\ (z \to 0)$) we can verify the existence of $\delta',\ 0 < \delta' < \delta_1$, such that $\mathrm{Ker}(T - z) = R(P(z))\ (|z| < \delta')$.

Take any $x \in \mathrm{Ker}T = R(P),\ x \ne 0$, and $f(z) = P(z)x = (I - zT')^{-1}Px$. Then $f \in \mathcal{O}(U, \mathbf{H}),\ U = O(0, \delta'),\ f(0) = P(0)x = Px = x \ne 0$. But $(T - z)f(z) = (T - z)P(z)x = 0\ (z \in U)$. Hence $\lambda = 0 \notin A(T)$.

Suppose $0 \in \sigma_p(T)^0 \cap \psi^s(T)$. By [6, Th.3.3], there exists $H_i \in \mathrm{Lat}\,T,\ T_i = T|_{H_i},\ i = 1, 2,\ \mathbf{H} = H_1 \oplus H_2,\ T = T_1 \oplus T_2,\ 0 \in \psi^r(T_1) \cap \sigma_p(T_1)^0,\ \sigma(T_2) = \{0\}$. Hence $0 \notin A(T_1)$ and Proposition 3 implies that $0 \notin A(T)$.

(2) Suppose that $\lambda \in A(T) \cap \rho_r(T) \cap \sigma(T)$. Then there exists $\delta > 0,\ O(\lambda, \delta) \subset \sigma_p(T)^0$. This contradicts (1) and hence $A(T) \cap \rho_r(T) \subset \rho(T)$. □

3. Conditions for (Q) and (P) operators

Proposition 6. *Suppose that $S,\ T \in L(\mathbf{H}),\ T \overset{dr}{\sim} S$. Then $\rho_{re}(T) \cap E_2(S) \subset \rho_D(S)$.*

Proof. (1) Suppose first that $\lambda \in \rho^r_{re}(T) \cap E_2(S)$. We may assume that $\lambda = 0$.

By [6], there exists a neighborhood U_1 of 0 such that

$$\text{(i)} \qquad \mu(T - z) = \mu(T) = m < \infty\ (z \in U_1).$$

Analogous to the proof of Proposition 5,(1) , we may define $Q,\ T',\ Q(z) \in L(\mathbf{H})$ such that

$$\text{(ii)} \qquad \begin{aligned} &Q = \text{orthogonal projection onto } R(T)^\perp, \\ &T'Q = 0,\ TT' = I - Q,\ \|zT'\| < 1\ (z \in U_2 = O(0, \delta_2)), \\ &Q(z) = Q(I - zT')^{-1}, \\ &Q(z)H = QH = R(T)^\perp \quad (z \in U_2), \end{aligned}$$

and show by a simple calculation that

$$Q(z)^2 = Q(z),\ I - Q(z) = (T-z)T'(I-zT')^{-1},$$
$$(I-Q(z))\mathbf{H} \subset (T-z)\mathbf{H}\ (z \in U_2).$$

Suppose that $z \in U_3 = U_1 \cap U_2$. It follows from (i), (ii) that

$$\text{codim}(I-Q(z))\mathbf{H} = \text{dim}Q(z)\mathbf{H} = \mu(T-z) = \text{codim}(T-z)\mathbf{H} < \infty.$$

But $\text{codim}(I-Q(z))\mathbf{H} = \text{codim}(T-z)\mathbf{H} + \dim((T-z)\mathbf{H} \ominus (I-Q(z))\mathbf{H})$, hence

(iii) $$(T-z)\mathbf{H} = (I-Q(z))\mathbf{H}\ (z \in U_3).$$

(2) By hypothesis, there exist $A,\ B \in L(\mathbf{H}),\ A,\ B$ having dense ranges such that $AT = SA,\ BS = TB$.

For every $u \in \mathbf{H},\ z \in U_3$, we have

$$u = Q(z)u + (I-Q(z))u = Q(z)u + (T-z)f_u(z)$$

where $f_u(z) = T'(I-zT')^{-1}u,\ f_u \in \mathcal{O}(U_3, \mathbf{H})$. This implies that

(iv) $$\begin{aligned} Au &= AQ(z)u + A(T-z)f_u(z) \\ &= AQ(z)u + (S-z)Af_u(z) \quad (z \in U_3,\ u \in \mathbf{H}). \end{aligned}$$

(3) For every $x \in \mathbf{H}$, there exists $u_n \in \mathbf{H}$ $(n = 1, 2, \cdots)$, $Au_n \to x$ $(n \to \infty)$. We are going to show that $AQ(z)u_n$ converges too.

(v) $$\begin{aligned} Q(z)BAu &= Q(z)BAQ(z)u + Q(z)(T-z)BAf_u(z) \\ &= Q(z)BAQ(z)u \ \ \text{(by (iii))}\ (z \in U_3, u \in \mathbf{H}). \end{aligned}$$

Let $\phi_z = Q(z)BA|_{Q\mathbf{H}}$ $(z \in U_3)$. Since $\text{dim}Q\mathbf{H} < \infty$,

$$\phi_z Q\mathbf{H} = \overline{\phi_z Q\mathbf{H}} \ = \ \overline{Q(z)BA\mathbf{H}} \supset Q(z)\overline{BA\mathbf{H}} = Q(z)\mathbf{H} = Q\mathbf{H},$$

i.e. ϕ_z is surjective and has an inverse ψ_z continuous with respect to z on U_3. It then follows that there exists $U = O(0, \delta) \subset U_3$ and $M_1 > 0$ such that

$$\sup_{z \in U} \|Q(z)\| \le M_1, \quad \sup_{z \in U} \|\psi_z\| \le M_1.$$

By (v), we have, $\forall u \in \mathbf{H}$,

$$Q(z)u = \psi_z \phi_z Q(z)u = \psi_z Q(z)BAu,$$

$$\|Q(z)u\| \le \|\psi_z\| \cdot \|Q(z)\| \cdot \|B\| \cdot \|Au\| \le M\|Au\| \quad (z \in U)$$

where $M = M_1^2\|B\|$. Hence $Q(z)u_n$ converges uniformly on U when $n \to \infty$ and $AQ(z)u_n \to g(z)$ uniformly on U, $g \in \mathcal{O}(U, \mathbf{H}),\ g(z) \in AQ\mathbf{H}$.

(4) (iv) implies that $(S-z)Af_{u_n}(z) \to x - g(z) = h(z)$ uniformly on U, $h \in \mathcal{O}(U, \mathbf{H})$.

Since $\lambda = 0 \in E_2(S)$, there exists $\delta' > 0$ such that $V = O(0, \delta') \subset U$ and $(S-z)\mathcal{O}(V, \mathbf{H})$ is closed. It follows that $h \in (S-z)\mathcal{O}(V, \mathbf{H})$, i.e. there exists $f \in \mathcal{O}(V, \mathbf{H})$ such that

$$h(z) = (S-z)f(z),$$
$$x = g(z) + (S-z)f(z),$$
$$x = g(0) + Sf(0), \quad g(0) \in AQ\mathbf{H}.$$

Since $x \in \mathbf{H}$ is arbitrary, $\mathbf{H}/S\mathbf{H}$ is isomorphic to a subspace of the finite-dimensional space $AQ\mathbf{H}$, $\dim \mathbf{H}/S\mathbf{H} < \infty$. This implies that $S\mathbf{H}$ is closed (cf. [9, Chap.4]), i.e. $\lambda = 0 \in \rho_{_D}(s)$.

(5) Suppose $\lambda = 0 \in \rho^s_{re}(T)$. By [6, Th3.3], there exist subspaces H_1, H_2 of $H, H = H_1 + H_2, \dim H_2 < \infty, \sigma(T|_{H_2}) = \{0\}, 0 \in \rho^r_{re}(T|_{H_1})$. Applying the above reasoning to T and $T|_{H_1}$, we can derive that, for every $x \in H$, there exist $x_1 \in \overline{AH_1}, x_2 \in AH_2, g(0) \in \overline{AQH_1}, f(0) \in H$ such that

$$x = x_1 + x_2 = g(0) + Sf(0) + x_2.$$

Since $\dim(\overline{AQH_1} + AH_2) < \infty$, $\dim H/SH < \infty$, SH is closed, $\lambda = 0 \in \rho_{_D}(S)$. $\square$

Proposition 7. *Suppose that $S,\ T \in L(\mathbf{H}),\ T \xrightarrow{dr} S$. Then $\rho(T) \cap C_2(S) \subset \rho_r(S) \subset \rho_{_D}(S)$.*

Proof. Suppose that $A,\ S,\ T \in L(\mathbf{H}),\ R(A)$ is dense, $AT = SA,\ \lambda = 0 \in \rho(T) \cap C_2(S)$.

$\forall x \in \mathbf{H}$, there exists $u_n \in \mathbf{H},\ Au_n \to x\ (n \to \infty)$. Since $0 \in \rho(T)$, there exists $\delta > 0$ such that $U = O(0,\delta) \subset \rho(T)$ and $f_n = (T-z)^{-1}u_n \in \mathcal{O}(U, \mathbf{H}),\ u_n = (T-z)f_n(z),\ Au_n = A(T-z)f_n(z) = (S-z)Af_n(z)\ (z \in U)$, hence $Au_n \in H_S(U^c)$. It follows from $0 \in C_2(S)$ that $H_S(U^c)$ is closed for sufficiently small U and hence $x = \lim Au_n \in H_S(U^c)$, i.e. there exists $f \in \mathcal{O}(U, \mathbf{H}),\ x = (S-z)f(z)\ (z \in U)$. Therefore S is surjective and $\lambda = 0 \in \rho_r(S)$. $\square$

Proposition 8. *Suppose that $S,\ T \in L(\mathbf{H}),\ S \xrightarrow{inj} T,\ \lambda \in C_1(S)$ and $H_S(\overline{O(\lambda,\delta)}) \neq \{0\}\ (\delta > 0)$. Then $\lambda \in \sigma(T)$.*

Proof. Suppose $S,\ T,\ \lambda$ satisfy the above hypothesis. By the definition of $C_1(S)$, there exists $\delta' > 0$ such that $M_\delta = H_S(\overline{O(\lambda,\delta)})$ is closed for every $\delta,\ 0 < \delta < \delta'$, and hence $M_\delta \neq \{0\},\ M_\delta \in \mathrm{Lat}S$. By [10, Theorem 2.5], it follows that $\sigma(S|_{M_S}) \cap \sigma(T) \neq \emptyset$. But $\sigma(S|_{M_\delta}) \subset \overline{O(\lambda,\delta)}$ ([7, Prop. 3]) and δ may be arbitrarily small, hence $\lambda \in \sigma(T)$. $\square$

Lemma 1 ([11, Th.1])**.** *If $S,\ T \in L(\mathbf{H}),\ S \overset{q}{\sim} T$, then every connected component of $\sigma_e(S)$ intersects $\sigma_e(T)$.*

Define

$G_1(S) = \{\lambda \in \sigma_{_D}(S) : \exists \delta > 0, 0 < m+n < \infty, \nu(S-z) = m, \mu(S-z) = n, z \in O(\lambda,\delta)\}$,

$G_0(S) = \{\lambda \in \sigma_{_D}(S) : \exists \delta > 0, \mu(S-z) = \nu(S-z) = 0, z \in O(\lambda,\delta)\}$,

$F_1(S) = \{\lambda \in \sigma_{_D}(S) : \exists \delta > 0, 0 \le m < \infty, 0 \le n < \infty, \mu(S-\lambda) = m+n, \mu(S-z) = n, z \in O'(\lambda,\delta)\}$,

$F_0(S) = \sigma_{_D}(S) \cap \{z \in \mathbb{C} : \mu(S-z) = 0\}^0$,

$J(S) = \sigma(S) \cap \{z \in \mathbb{C} : \nu(S-z) = 0\}^0$.

Theorem 1. *Suppose that $S \in L(H)$.*

(1) $G_0(S) \subset C_2(S) \Longrightarrow S \in (P)$,

(2) $G_0(S) \subset C_2(S), G_1(S) \subset E_2(S) \Longrightarrow S \in (Q)$,

(3) $S \in (A), F_0(S) \subset C_2(S) \Longrightarrow S \in (P)_{\overset{dr}{\leftarrow}}$,

(4) $S \in (A), F_1(S) \subset E_2(S) \Longrightarrow S \in (Q)_{\overset{dr}{\sim}}$,

(5) $H_S(\overline{O(\mu,\delta)}) \neq \{0\}(\delta > 0, \mu \in \sigma(S)), J(S) \subset C_1(S) \Longrightarrow S \in (P)_{\xrightarrow{inj}}$.

Proof. (1) Suppose that $T \in L(H), T \overset{q}{\sim} S, G_0(S) \subset C_2(S)$.

$$\rho(T) \cap \sigma(S) = \rho(T) \cap G_0(S) \subset \rho(T) \cap C_2(S) \cap \sigma_{_D}(S) \subset \rho_{_D}(S) \cap \sigma_{_D}(S) = \emptyset$$

by Proposition 7, hence $\sigma(S) \subset \sigma(T), S \in (P)$.

(2) Suppose that $T \in L(H), T \overset{q}{\sim} S, G_0(S) \subset C_2(S), G_1(S) \subset E_2(S)$,

$$\rho(T) \cap \sigma_e(S) \subset \rho(T) \cap \sigma(S) = \emptyset \qquad \text{by (1)},$$

$$\rho_e^r(T) \cap \sigma(T) \cap \sigma_e(S) \subset \rho_e(T) \cap G_1(S) \subset \rho_e(T) \cap E_2(S) \cap \sigma_{_D}(S)$$
$$\subset \rho_{_D}(S) \cap \sigma_{_D}(S) = \emptyset \qquad \text{by Proposition 6}$$

so that $\rho_e^r(T) \cap \sigma_e(S) = \emptyset$.

If $\mu \in \rho_e^s(T) \cap \sigma_e(S)$, then $O(\mu, \delta) \setminus \{\mu\} \subset \rho_e^r(T) \subset \rho_e(S)$ for some $\delta > 0$ and $\{\mu\}$ would be a connected component of $\sigma_e(S)$ contained in $\rho_e(T)$, a contradiction with Lemma 1. It follows that $\rho_e^s(T) \cap \sigma_e(S) = \emptyset$, too.

It follows that $\sigma_e(S) \subset \sigma_e(T), S \in (Q)$.

(3) Suppose that $T \in L(H), T \xrightarrow{dr} S, S \in (A), F_0(S) \subset C_2(S)$ and $\lambda \in \rho(T)$. Then $\mu(S - z) = \mu(T - z) = 0$ for $z \in$ some neighbourhood of λ.

If $\lambda \in \sigma_{_D}(S)$, then $\lambda \in F_0(S) \cap \rho(T) \subset \rho(T) \cap C_2(S) \subset \rho_{_D}(S)$ (by Proposition 7), a contradiction. Hence $\lambda \in \rho_{_D}(S)$ and it follows from $\lambda \in A(S)$ and Proposition 5,(1) that $\lambda \in \rho(S), S \in (P)_{\overset{dr}{\leftarrow}}$.

(4) Suppose that $T \in L(H), T \overset{dr}{\sim} S, S \in (A), F_1(S) \subset E_2(S)$ and $\lambda \in \rho_e(T)$. Then there exist $\delta > 0$ and integers m, n such that $0 \le m, n < \infty, \mu(S - \lambda) = \mu(T - \lambda) = m + n, \mu(S - z) = \mu(T - z) = n, z \in O'(\lambda, \delta)$. Notice that $\lambda \in A(S)$.

If $\lambda \in \sigma_{_D}(S)$, then $\lambda \in \rho_e(T) \cap F_1(S) \subset \rho_e(T) \cap E_2(S) \subset \rho_{_D}(S)$ by Proposition 6, a contradiction, and hence $\lambda \in \rho_{_D}(S)$. It follows from Proposition 5,(1) that $\lambda \in \rho_e(S)$, $\rho_e(T) \subset \rho_e(S), S \in (Q)_{\overset{dr}{\sim}}$.

(5) Suppose $T \in L(H), S \xrightarrow{inj} T$ and the hypothesis is true. Then

$$\rho(T) \cap \sigma(S) = \rho(T) \cap J(S) \subset \rho(T) \cap C_1(S) \cap \sigma(S) \subset \rho(T) \cap \sigma(T) = \emptyset$$

by Proposition 8, and hence $\sigma(S) \subset \sigma(T), S \in (P)_{\overset{inj}{\longrightarrow}}$. □

The conditions of Theorem 1 are not necessary (see Example 1 below). We may weaken these conditions. The following theorem presents an example.

Theorem 2. *If* $S \in L(H)$ *and*

$$G_1(S) \subset \bigcup_{R \overset{q}{\sim} S} (\sigma_e(R) \cap (E_2(R) \cup E_2(R^*)^*)),$$

$$G_0(S) \subset \bigcup_{R \overset{q}{\sim} S} (\sigma(R) \cap (C_2(R) \cup C_2(R^*)^*)),$$

then $S \in (Q)$ (M^* *denotes* $\{z \in \mathbb{C} : \overline{z} \in M\}$ *for* $M \subset \mathbb{C}$).

Proof. Suppose that $T \in L(H), T \overset{q}{\sim} S$. Then we have

$$\rho(T) \cap \sigma(S) = \rho(T) \cap G_0(S) \subset [\bigcup_{R \overset{q}{\sim} T} (\rho(T) \cap \sigma(R) \cap C_2(R))]$$
$$\cup [\bigcup_{R^* \overset{q}{\sim} T^*} (\rho(T^*) \cap \sigma(R^*) \cap C_2(R^*))]^*.$$

Now, for every $R \overset{q}{\sim} T$,

$$\begin{aligned}\rho(T) \cap C_2(R) \cap \sigma(R) &\subset \rho(T) \cap \rho_{_D}(R) \cap \sigma(R) \qquad \text{(by Proposition 7)}\\ &\subset \rho(R) \cap \sigma(R) = \emptyset,\end{aligned}$$

hence $\rho(T) \cap \sigma(S) = \emptyset$. Similarly $\rho_e^r(T) \cap \sigma(T) \cap \sigma_e(S) = \emptyset$.

We see then that $\rho_e^r(T) \subset \rho_e(S), \rho_e(T) \subset \rho_e(S)$ and $S \in (Q)$. □

Corollary 1. *If $S \in L(H), S$ or $S^* \in (E_2)$, then $S \in (Q)$.*

Corollary 2. *If S or $S^* \in (C_2)$, then $S \in (P)$; if $S \in (C)$, then $S \in (P)_{\underset{\leftarrow}{dr}}$.*

Corollary 3. *If S is subdecomposable, then $S \in (Q)_{\underset{\sim}{dr}}, S \in (P)_{\underset{\leftarrow}{dr}}$.*

The operators mentioned in Corollaries 1 and 3 include many familiar operators such as decomposable, spectral, subscalar, M-hyponormal, semihyponormal operators, operators with totally-disconnected spectrum, Riesz operators, etc.

The family (Q) has, of course, a much larger population.

Example 1. Let S_1 be the operator given in [1] which is quasisimilar to the simple bilateral shift U and whose spectrum is equal to the closed unit disk $\overline{D}$. Let S_2 be a normal operator with $\sigma(S_2) = \sigma_c(S_2) = \overline{D},\ S = S_1 \oplus S_2,\ T = U \oplus S_2$. We obviously have $T \overset{q}{\sim} S$ and

$\sigma(S_1) = \sigma_e(S_1) = \sigma_c(S_1) = \overline{D}$,

$E_2(S_1) \cup E_2(S_1^*) \cup C_2(S_1) \cup C_2(S_1^*) \subset \mathbb{C} \setminus D$ (by Proposition 6),

$E_2(S) \cup E_2(S^*) \cup C_2(S) \cup C_2(S^*) \subset \mathbb{C} \setminus D$ (by Proposition 3).

Hence $S,\ S^*$ are not (β), (E_2) or (C_2) operators and fail to satisfy the conditions of Theorem 1, but $G_1(S) = \emptyset,\ G_0(S) \subset \sigma(T) \cap C_2(T) = \overline{D}$ so that $S \in (Q)$ by Theorem 2.

Example 2. (1) If S is decomposable, then $S \in (P)_{\underset{\longrightarrow}{inj}}$.

(2) If S is decomposable, $T \in (C),\ S \overset{q}{\to} T$, then $\sigma(S) = \sigma(T)$.

Proof. (1) If S is decomposable, then $H_S(\overline{O(\mu, \delta)}) \neq \{0\}$ $(\mu \in \sigma(S),\ \delta > 0)$, $S \in (\beta) \subset (C_1)$. Hence $S \in (P)_{\underset{\longrightarrow}{inj}}$ (Theorem 1,(5)).

(2) $S \in (P)_{\underset{\longrightarrow}{inj}}$ by (1); $T \in (P)_{\underset{\longleftarrow}{dr}}$ by Corollary 2; hence $S \overset{q}{\to} T \Longrightarrow \sigma(S) = \sigma(T)$. □

(Q) operators may be divided into two subfamilies:

$S \in (Q_1) \overset{d}{\Longleftrightarrow} S \in (Q)$ and there exists $T \overset{q}{\sim} S,\ \sigma_e(T) \neq \sigma_e(S)$,

$S \in (Q_2) \overset{d}{\Longleftrightarrow} S \in (Q)$ and $\sigma_e(T) = \sigma_e(S)$ $(\forall T \overset{q}{\sim} S)$.

Example 3. If $S \in L(\mathbf{H})$ is an algebraic operator, $T \overset{q}{\sim} S$, then T is also an algebraic operator and $S,\ T \in (Q)$ (since their spectra are finite), $\sigma_e(T) = \sigma_e(S)$. It follows that algebraic operators are (Q_2) operators.

In [12], D. Herrero proved in fact that (Q_2) and $(Q)^c = L(\mathbf{H}) \setminus (Q)$ are both dense in $L(\mathbf{H})$. We are now going to show that (Q_1) is also dense in $L(\mathbf{H})$.

Lemma 2. *$\{T \in L(\mathbf{H});\ G_1(T) \cup\ G_0(T) = \emptyset\}$ is dense in $L(\mathbf{H})$.*

Proof. [13, Theorem 2.5] told us that simple model operators are dense in $L(\mathbf{H})$. A simple model operator T is defined to be an operator similar to an elementary operator. It follows from the spectral structure of elementary operators that $G_1(T) \cup G_0(T) = \emptyset$. □

Lemma 3 ([12, Lemma 2])**.** *If* $T,\ Q \in L(\mathbf{H}),\ \varepsilon > 0,\ \sigma(Q) \subset O(0, \varepsilon/5)$*, then there exist* $T_\varepsilon \in L(\mathbf{H}),\ C \in L(\mathbf{H})$*, such that* $\|T - T_\varepsilon\| < \varepsilon$*,* T_ε *is similar to* $(\lambda + Q) \oplus C$*, where* $\lambda \in \mathbb{C},\ \sigma(\lambda + Q)$ *lies in the unbounded component of* $\rho(C)$*.*

Theorem 3. (Q_1) *is dense in* $L(\mathbf{H})$*.*

Proof. Suppose that $T \in L(\mathbf{H}),\ \varepsilon > 0$ and U denotes the simple bilateral shift. By Lemma 3, there exist $T_\varepsilon \in L(\mathbf{H})$ and $X \in L(\mathbf{H})$, X invertible such that $\|T - T_\varepsilon\| < \varepsilon/2,\ X^{-1}T_\varepsilon X = (\lambda + (\varepsilon/11)U) \oplus C$; here $\lambda + (\varepsilon/11)U \in L(H_1),\ C \in L(H_2),\ \mathbf{H} = H_1 \oplus H_2,\ \sigma(\lambda + (\varepsilon/11)U) \subset$ the unbounded component of $\rho(C)$.

By Lemma 2, there exists $T' \in L(H_2)$ such that $G_1(T') \cup G_0(T') = \emptyset,\ \|T' - C\| < (\varepsilon/2)(\|X^{-1}\|\|X\|)^{-1}$. We can still require that $\sigma(\lambda + (\varepsilon/11)U) \subset$ the unbounded component of $\rho(T')$.

Let $T'' = X((\lambda + (\varepsilon/11)U) \oplus T')X^{-1}$. Since $G_1(T'') = \emptyset,\ G_0(T'') = \lambda + (\varepsilon/11)\partial D \subset C_2(T'')$, it follows that $T'' \in (Q)$ (Theorem 1,(2)) and we have

$$\|T'' - T\| \le \|T - T_\varepsilon\| + \|T_\varepsilon - T'\| < \frac{\varepsilon}{2} + \|C - T'\|\|X\|\|X^{-1}\| < \varepsilon.$$

Let S_1 denote the operator quasisimilar to U in Example 1. Then

$$S = (\lambda + (\varepsilon/11)S_1) \oplus T' \overset{q}{\sim} T'',$$
$$\sigma_e(S) \ne \sigma_e(T'') \text{ (since } \sigma_e(S_1) \ne \sigma_e(U)),$$

and hence $T'' \in (Q_1)$. □

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Department of Mathematics, Fujian Normal University, Fuzhou, 350007, The People's Republic of China

E-mail address: `xhyan@fjtu.edu.cn`