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INVERTIBILITY IN INFINITE-DIMENSIONAL SPACES

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Abstract. An interesting result of Doyle and Hocking states that a topological n-manifold is invertible if and only if it is a homeomorphic image of the n-sphere S^n. We shall prove that the sphere of any infinite-dimensional normed space is invertible. We shall also discuss the invertibility of other infinite-dimensional objects as well as an infinite-dimensional version of the Doyle-Hocking theorem.

1. Introduction

The most interesting application of invertibility in finite-dimensional spaces is the Doyle-Hocking characterization of the n-sphere S^n.

Theorem 1 (Doyle and Hocking [8]). *A topological n-manifold is homeomorphic to S^n if and only if it is invertible.*

A (non-empty) topological space X is said to be *invertible* [9] if for each proper open subset U of X there is a homeomorphism T (called an *inverting homeomorphism*) of X onto X sending $X \setminus U$ into U. Recall that a subset U of X is *proper* if both U and its complement $X \setminus U$ are not empty. It is clear that invertibility is a topological property, *i.e.* preserved by homeomorphisms. In many cases, we may expect that a topological property which holds locally in an arbitrary proper open subset U of X holds indeed globally in all of X. For examples, we have

Proposition 2 ([9, 15, 10, 13, 16]). *Let U be a proper open subset of an invertible space X. If U has any of the following properties, then X also has the corresponding properties: (1) T_0, (2) T_1, (3) Hausdorff, (4) regular, (5) completely regular, (6) normal, (7) first countable, (8) second countable, (9) separable, (10) metrizable, (11) uniformizable, (12) compact, (13) pseudocompact, (14) extremally disconnected; unless X is a two point space, the list also includes: (15) T_1 and connected, and (16) T_1 and path connected.*

Recall that a topological space X is locally compact if every point x in X has a compact neighborhood U, *i.e.* x belongs to the interior of the compact subset U of X. Since locally compact invertible spaces must be compact, the intervals

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$(0,1)$, $[0,1)$ and $(0,1]$, and the n-space $\mathbb{R}^n$ $(n = 1, 2, \dots)$ cannot be invertible. By a simple connectedness argument, one can see that the compact interval $[0,1]$ is not invertible, either. On the other hand, all finite-dimensional spheres S^n $(n = 1, 2, \dots)$, the set $\mathbb{Q}$ of all rational points of the real line $\mathbb{R}$, and the Cantor set are all invertible. Moreover, it is easy to show that a topological space X is invertible if and only if for any proper closed subset F and proper open subset U of X there is a homeomorphism of X onto itself sending F into U. Consequently, one can see that many fractal figures are invertible along the line of reasoning in [9], in which together with several continua the universal one-dimensional plane curve is proved to be invertible. It seems to us that invertibility may be a useful tool in studying fractal geometry. Finally, an interesting presentation of the theory of function spaces of invertible spaces can be found in [18].

This paper is devoted to an infinite-dimensional version of Theorem 1. In particular, we shall show

Theorem 3. *The unit sphere of any normed space of finite or infinite dimension is invertible. Moreover, the inverting homeomorphisms T can be chosen to have period 2, i.e. $T \circ T$ is the identity map of the sphere.*

Conjecture 4. *All infinite-dimensional invertible topological Hilbert manifolds are homeomorphic to the unit sphere of the underlying Hilbert space.*

Recall that a topological space X is called a (topological) *manifold* modeled on a topological vector space E if there is an open cover of X each member of which is homeomorphic to E. The following result of Toruńczyk tells us that we may consider merely Hilbert manifolds (*i.e.* the case that the model space E is a Hilbert space).

Theorem 5 (Toruńczyk [19, 20])**.** *All infinite-dimensional Fréchet (i.e. complete metrizable locally convex) spaces are homeomorphic to Hilbert spaces.*

The invertibility of infinite-dimensional spheres and other convex objects will be verified in Section 2. Some approaches to solving Conjecture 4 will be presented in Section 3.

2. Main results

Recall that a convex subset of a topological vector space is called a convex body if it has non-empty interior. Since the unit ball of a normed space is a bounded convex body, Theorem 3 follows from the following seemingly more general

Theorem 6. *The (topological) boundary S of any bounded convex body V in any normed space N is invertible. Moreover, the inverting homeomorphisms can be chosen to have period 2.*

Proof. We may assume that N is a real normed space of dimension greater than 1. In fact, if the underlying field is complex, then we may consider the real normed space $N_{\mathbb{R}}$ instead. $N_{\mathbb{R}}$ is the vector space N over the real field $\mathbb{R}$ equipped with the norm $\|\cdot\|_{\mathbb{R}}$, where $\|x\|_{\mathbb{R}} = \|x\|$ for all x in N. It is plain that (N, V) and $(N_{\mathbb{R}}, V)$ are homeomorphic as topological pairs. The case that N is the one-dimensional line $\mathbb{R}$ is trivial. Moreover, we may assume that V is open and contains 0 since the boundary of any convex body coincides with the boundary of its interior.

Recall that in the proof of the invertibility of *finite*-dimensional spheres S^n, one utilizes the stereographic projection of $S^n \setminus \{\infty\}$ onto $\mathbb{R}^n$ and the inversions of $\mathbb{R}^n$

with respect to circles. To achieve an infinite-dimensional version of these type of arguments, the first task for us is to replace S with a homeomorphic image S_2 which looks "round" enough to have a stereographic projection onto a closed hyperplane of N. Then the inverting homeomorphisms will be obtained exactly the same way as in the finite-dimensional case.

Let r be the gauge functional of the open convex set V, namely,

$$r(x) = \inf\{\lambda > 0 : x \in \lambda V\}, \quad \forall x \in N.$$

r is a sublinear functional of N since V is convex. In other words, $r(x + y) \leq r(x) + r(y)$ and $r(\lambda x) = \lambda r(x)$ for all x, y in N and $\lambda \geq 0$.

Claim 1. There is a constant $\alpha > 1$ such that $\frac{1}{\alpha} U_N \subseteq V \subseteq \alpha U_N$; or equivalently,

$$\frac{1}{\alpha} r(x) \leq \|x\| \leq \alpha r(x), \quad \forall x \in N, \tag{1}$$

where $U_N = \{x \in N : \|x\| \leq 1\}$ is the closed unit ball of N.

In fact, the openness and boundedness of V establish the inclusions for some constant $\alpha > 1$. For the norm inequalities, we observe that, for any non-zero x in N, $x/\|x\| \in U_N \subseteq \alpha V$ implies that $r(x/\|x\|) \leq \alpha$ or $r(x) \leq \alpha\|x\|$. Similarly, since $x/r(x)$ belongs to the closure of $V \subseteq \alpha U_N$, we have $\|x/r(x)\| \leq \alpha$ or $\|x\| \leq \alpha r(x)$, as asserted.

As a consequence of Claim 1, the family $\{B_{r,1/n}(x) : n = 1, 2, \dots\}$ is a local base at each x in N in the norm topology, where $B_{r,1/n}(x) = \{y \in N : r(y - x) \leq 1/n\}$. It is easy to see that $S = \{x \in N : r(x) = 1\}$. Fix an arbitrary x_0 in S and let f be a continuous (real) linear functional of N supporting V at x_0, *i.e.* $f(x) \leq f(x_0) = 1, \forall x \in V$. Write

$$N = \mathbb{R}x_0 \oplus \mathrm{Ker} f$$

as a direct sum of the line $\mathbb{R}x_0$ in the direction of x_0 and the closed hyperplane $\mathrm{Ker} f = \{y \in X : f(y) = 0\}$ determined by f. For each x in N, write

$$x = f(x)x_0 + y_x$$

for some (unique) y_x in $\mathrm{Ker} f$. Define another sublinear functional r_2 of N by

$$r_2(x) = \sqrt{f(x)^2 + r(y_x)^2}, \quad \forall x \in N.$$

Claim 2. There are positive constants c and d such that $cr_2(x) \leq r(x) \leq dr_2(x)$, $\forall x \in N$.

By the norm inequalities (1), we have

$$|f(x)| \leq \|f\|\|x\| \leq \alpha\|f\|r(x)$$

and

$$\begin{aligned} r(y_x) = r(x - f(x)x_0) &\leq \alpha\|x - f(x)x_0\| \leq \alpha(\|x\| + |f(x)|\|x_0\|) \\ &\leq \alpha^2(1 + \|f\|\|x_0\|)r(x) \end{aligned}$$

for all x in N. Consequently,

$$r_2(x)^2 \leq (\alpha^2\|f\|^2 + \alpha^4(1 + \|f\|\|x_0\|)^2)r(x)^2, \quad \forall x \in N.$$

On the other hand,

$$\begin{aligned} r(x) &\leq r(f(x)x_0) + r(y_x) \leq \alpha|f(x)|\|x_0\| + r(y_x) \leq \alpha^2|f(x)| + r(y_x) \\ &\leq \alpha^2(|f(x)| + r(y_x)), \end{aligned}$$

and hence

$$r(x) \leq \sqrt{2}\alpha^2 r_2(x),$$

for all x in N.

It follows from Claims 1 and 2 that the family $\{B_{r_2,1/n}(x) : n = 1, 2, \dots\}$ forms a local base at each x in N in the norm topology. As a result, we have proved

Claim 3. A sequence (x_n) converges to x in N if and only if $r_2(x_n - x) \longrightarrow 0$ as $n \longrightarrow \infty$.

Note also that r and r_2 coincide on $\mathrm{Ker}f$. Let

$$S_2 = \{x \in N : r_2(x) = 1\}.$$

It is easy to see that $h(x) = x/r_2(x)$ defines a homeomorphism of S onto S_2. As invertibility is a topological property, it suffices to show that S_2 is invertible.

Observe that $f(x) < 1$ whenever $x = f(x)x_0 + y_x \in S_2 \setminus \{x_0\}$ since in this case $r_2(x) = \sqrt{f(x)^2 + r(y_x)^2} = 1$. This enables us to define a stereographic projection $P : S_2 \setminus \{x_0\} \longrightarrow \mathrm{Ker}f$ by

$$P(x) = \frac{y_x}{1 - f(x)} = \frac{x - f(x)x_0}{1 - f(x)}. \tag{2}$$

Claim 4. P is a homeomorphism.

First, we note that for each $x = f(x)x_0 + y_x$ in $S_2 \setminus \{x_0\}$ with y_x in $\mathrm{Ker}f$,

$$P(x) - x_0 = \frac{x - f(x)x_0}{1 - f(x)} - x_0 = \frac{x - x_0}{1 - f(x)}$$

by (2). Therefore,

$$x = f(x)x_0 + (1 - f(x))P(x), \quad \forall x \in S_2 \setminus \{x_0\}. \tag{3}$$

Thus, $f(x)^2 + r((1 - f(x))P(x))^2 = r_2(x)^2 = 1$. Since $f(x) < 1$, we have

$$r((1 - f(x))P(x)) = (1 - f(x))r(P(x)).$$

So $(1 - f(x))r(P(x))^2 = 1 + f(x)$, and thus

$$f(x) = \frac{r(P(x))^2 - 1}{r(P(x))^2 + 1}, \quad \forall x \in S_2 \setminus \{x_0\}. \tag{4}$$

Now, suppose x, x' in $S_2 \setminus \{x_0\}$ are such that $P(x) = P(x')$. Then we have $f(x) = f(x')$ by (4), and consequently, $x = x'$ by (3). In other words, P is one-to-one. P is also onto. In fact, for any y in $\mathrm{Ker}f$, we have

$$P^{-1}(y) = \frac{(r(y)^2 - 1)x_0 + 2y}{r(y)^2 + 1}$$

by (3) and (4) again. The continuity of P and P^{-1} follows from that of f and r, respectively.

Claim 5. S_2 is invertible and the inverting homeomorphisms can be chosen to have period 2.

Let U be a proper open subset in S_2. Choose an a in $U \setminus \{x_0\}$. There exists a $\delta > 0$ such that the closure of $B_{r_2,\delta}(a) \cap S_2 = \{x \in S_2 : r_2(x - a) < \delta\}$ is contained in $U \setminus \{x_0\}$. Let $b = P(a)$. Since P is an open map, there exists a $\delta' > 0$ such that $B_{r_2,\delta'}(b) \cap \mathrm{Ker} f = \{y \in \mathrm{Ker} f : r_2(y-b) < \delta'\} \subseteq P(B_{r_2,\delta}(a) \cap S_2)$. Define the inversion $h_{b,\delta'}$ from $\mathrm{Ker} f \setminus \{b\}$ onto itself by the condition that

$$r_2(h_{b,\delta'}(x) - b) r_2(x-b) = {\delta'}^2. \tag{5}$$

In other words,

$$h_{b,\delta'}(x) = b + \frac{{\delta'}^2}{r_2(x-b)^2}(x-b), \quad \forall x \in \mathrm{Ker} f \setminus \{b\}.$$

Clearly, $h_{b,\delta'} = {h_{b,\delta'}}^{-1}$ is continuous and maps $\{y \in \mathrm{Ker} f : r_2(y-b) > \delta'\}$ onto $B_{r_2,\delta'}(b) \cap \mathrm{Ker} f = \{y \in \mathrm{Ker} f : r_2(y-b) < \delta'\}$. Define $T : S_2 \longrightarrow S_2$ by

$$Tx = \begin{cases} P^{-1} h_{b,\delta'} P(x) & \text{if } x \neq a, x_0; \\ x_0 & \text{if } x = a; \\ a & \text{if } x = x_0. \end{cases}$$

It is plain that T is one-to-one, onto and $T = T^{-1}$. To ensure that T is a homeomorphism, we need only to check the continuity of T at x_0 and at a.

Suppose a sequence $x_n = f(x_n)x_0 + y_{x_n}$ in $S_2 \setminus \{x_0\}$ approaches x_0. In particular, $1 = r_2(x_n)^2 = f(x_n)^2 + r(y_{x_n})^2$. By (2), we have

$$r_2(P(x_n))^2 = \frac{r(y_{x_n})^2}{(1-f(x_n))^2} = \frac{1-f(x_n)^2}{(1-f(x_n))^2} = \frac{1+f(x_n)}{1-f(x_n)} \longrightarrow +\infty,$$

since $f(x_n) \longrightarrow f(x_0) = 1$. It then follows from $r_2(P(x_n) - b) \geq r_2(P(x_n)) - r_2(b) \longrightarrow +\infty$ that $r_2(h_{b,\delta'}P(x_n) - b) = \frac{{\delta'}^2}{r_2(P(x_n)-b)} \longrightarrow 0$ by (5). Hence, $Tx_n = P^{-1}h_{b,\delta'}P(x_n) \longrightarrow P^{-1}(b) = a$ by the continuity of P^{-1}. We have thus proved the continuity of T at x_0. Similarly, suppose a sequence (x_n) in $S_2 \setminus \{x_0\}$ approaches a. Then it follows that $P(x_n) \longrightarrow P(a) = b$. By (5), we have

$$r_2(h_{b,\delta'}P(x_n) - b) = \frac{{\delta'}^2}{r_2(P(x_n)-b)} \longrightarrow +\infty. \tag{6}$$

Since

$$Tx_n = f(Tx_n)x_0 + (1 - f(Tx_n))PTx_n \tag{7}$$

by (3), we have

$$1 = r_2(Tx_n)^2 = f(Tx_n)^2 + (1-f(Tx_n))^2 r(PTx_n)^2. \tag{8}$$

Hence, (6) implies that

$$\sqrt{\frac{1+f(Tx_n)}{1-f(Tx_n)}} = r(PTx_n) = r(h_{b,\delta'}P(x_n)) \geq r(h_{b,\delta'}P(x_n) - b) - r(-b) \longrightarrow +\infty.$$

Consequently, $f(Tx_n) \longrightarrow 1$ since f is bounded on the norm bounded set S_2. It then follows from (7) and (8) that

$$\begin{aligned} r_2(Tx_n - x_0)^2 &= (f(Tx_n) - 1)^2 + (1 - f(Tx_n))^2 r(PTx_n)^2 \\ &= (f(Tx_n) - 1)^2 + 1 - f(Tx_n)^2 \longrightarrow 0. \end{aligned}$$

Hence, $Tx_n \longrightarrow x_0$. The continuity of T at a is thus verified.

Finally, we show that $T(S_2 \setminus U) \subseteq U$. If $x_0 \in S_2 \setminus U$, then $Tx_0 = a \in U$. If $x \neq x_0$ and $x \in S_2 \setminus U$, then x does not belong to the closure of $B_{r_2,\delta}(a) \cap S_2$. This implies $P(x)$ does not belong to the closure of $B_{r_2,\delta'}(b) \cap \mathrm{Ker} f$. In other words, $P(x) \in \{y \in \mathrm{Ker} f : r_2(y-b) > \delta'\}$, and thus $h_{b,\delta'}P(x) \in B_{r_2,\delta'}(b) \cap \mathrm{Ker} f \subseteq P(B_{r_2,\delta}(a) \cap S_2)$. Consequently, $Tx = P^{-1}h_{b,\delta'}P(x) \in B_{r_2,\delta}(a) \cap S_2 \subseteq U$. Hence, $T(S_2 \setminus U) \subseteq U$, as asserted.

Since S is homeomorphic to S_2, we conclude that S is invertible. Moreover, the inverting homeomorphisms of S can be chosen to have period 2 as we can do so for the inverting homeomorphisms T of S_2. □

In fact, Theorem 3 also implies Theorem 6 by quoting a deep result of Bessaga and Klee. Recall that the *characteristic cone* of a convex body V in a topological linear space X is the set $\mathrm{cc}V = \{y \in X : \text{there is an } x \text{ in } X \text{ with } x + \lambda y \in V, \forall \lambda > 0\}$. If $\mathrm{cc}V$ is a linear subspace of X of codimension m $(0 \leq m \leq \infty)$, then we say that V has type m. V has type ∞ also if $\mathrm{cc}V$ is not a linear subspace of X. In the following, we write $(X, V) \simeq (Y, U)$ to indicate the existence of a relative homeomorphism from a topological space X onto a topological space Y which sends the topological subspace V of X onto the topological subspace U of Y.

Theorem 7 (Bessaga and Klee [2], see also [3, p. 110])**.** *Let V_1 and V_2 be closed convex bodies in a topological linear space X. Then $(X, V_1) \simeq (X, V_2)$ if and only if V_1 and V_2 have the same type. In this case, the topological boundaries of V_1 and V_2 are also homeomorphic.*

It is evident that all closed bounded convex bodies in a normed space N have the same type, *i.e.* the dimension of N. Therefore, Theorems 3 and 6 imply each other. In fact, much more can be said with the help of Theorem 7.

Corollary 8. *Every infinite-dimensional normed space N is invertible.*

Proof. Let $N_1 = N \times \mathbb{R}$ be the normed space direct product of N and the real line $\mathbb{R}$. Then $N = \{x \in N_1 : f(x) = 0\}$ for some continuous linear functional f of N_1. Since the closed half-space $\{x \in N_1 : f(x) \leq 0\}$ and the closed unit ball of N_1 have the same type $(= \infty)$, N is homeomorphic to the unit sphere of N_1 by Theorem 7. Consequently, N is invertible. □

Remark 9. The invertibility of infinite-dimensional *complete* normed spaces should not be surprising. Unlike the finite dimensional case, every infinite-dimensional Banach space E is homeomorphic to its unit sphere S [14, 3]. A key ingredient of the proof is the topological equivalence $L \simeq L \times \mathbb{R}$ for every infinite-dimensional Banach space L. The assertion will follow from this since S is homeomorphic to an (infinite-dimensional) closed hyperplane L of E which is in turn homeomorphic to $L \times \mathbb{R} \simeq E$ (see [3, p. 190]). One even has that every infinite-dimensional Hilbert space is real analytically isomorphic to its unit sphere [7]. However, this equivalence between spaces and their unit spheres may not extend to non-complete spaces. In fact, for every infinite-dimensional Banach space E there is a dense linear subspace L of E such that L is not homeomorphic to $L \times \mathbb{R}$ [17]. Consequently, the unit sphere of $L \times \mathbb{R}$, which is homeomorphic to L as in the proof of Corollary 8, is not homeomorphic to the whole space $L \times \mathbb{R}$.

Corollary 10. *An infinite-dimensional metrizable locally convex space X is invertible whenever X is complete or σ-compact.*

Proof. X is homeomorphic to a Hilbert space if X is complete by Theorem 5, or to a pre-Hilbert space if X is σ-compact by a result of Bessaga and Dobrowolski [1]. In both cases, X is invertible. □

Corollary 11. *Every non-empty open convex subset of an invertible topological vector space is invertible. Every closed convex body in an infinite-dimensional Fréchet space or an algebraically $\aleph_0$-dimensional normed space is invertible.*

Proof. We may assume that $0 \in V$. If V is an open convex subset of a topological vector space X, then the map $h(x) = \frac{x}{1-r(x)}$ is a homeomorphism of V onto X, where r is the gauge functional of V (see [3, p. 114]). Similarly, V is homeomorphic to the whole space if V is a closed convex body in either an infinite-dimensional Fréchet space (see [3, p. 190]) or an algebraically $\aleph_0$-dimensional normed space [5]. In all three cases, V is invertible. □

Recall that a subset A of a topological vector space is said to be infinite-dimensional if the vector subspace spanned by A is of infinite dimension. The first example of an invertible infinite-dimensional compact set is the Hilbert cube $[0,1]^\omega$ given in [9]. $[0,1]^\omega$ is the product space of countably infinitely many copies of the compact interval $[0,1]$, and can be embedded into the separable Hilbert space ℓ_2 as the set $\{(x_n) : |x_n| \le 1/n\}$. In fact, it was proved in [9] that the product space of arbitrary infinitely many copies of $[0,1]$ is invertible. In a similar manner, one can show that the product space of arbitrary infinitely many copies of the real line $\mathbb{R}$ is also invertible. This turns out to give another proof of the invertibility of infinite-dimensional *separable* Fréchet spaces, which are known to be homeomorphic to the countable product of lines $\mathbb{R}$ by the Kadec-Anderson Theorem (see [3, p. 189]).

Corollary 12. *Let A be an infinite-dimensional separable closed convex set in a Fréchet space. A is invertible if and only if A is either compact or not locally compact.*

Proof. If A is compact, then A is homeomorphic to the Hilbert cube (see [3, p. 100]). If A is not locally compact, then A is homeomorphic to ℓ_2 [6]. Therefore, A is invertible in both cases. Finally, we note that locally compact invertible space must be compact. Consequently, if A is locally compact but not compact, then A cannot be invertible. □

3. Conjectures

We do not know too much about the invertibility of the boundary of a closed convex set except for *bounded* convex bodies (Theorem 6). The following result of Klee might give us some hints.

Proposition 13 (Klee [14])**.** *Suppose C is a closed convex body in an infinite-dimensional reflexive Banach space E. Then the boundary of C is homeomorphic to E or to $E \times S^n$ for some finite n.*

Concerning Conjecture 4, we collect some results of Henderson which might be useful.

Theorem 14 (Henderson [11, 12])**.** *Let H be a separable Hilbert space. Every separable metric H-manifold M can be embedded as an open subset U of H such that* *the boundary of U and the closure of U are homeomorphic to U, and its complement $H \setminus U$ is homeomorphic to H.*

In the proof of Theorem 1, Doyle and Hocking [8] utilized a high-dimensional Jordan Curve Theorem [4]. In attacking Conjecture 4, we also found that an infinite-dimensional version of Jordan Curve Theorem is needed. We state it as

Conjecture 15. *Let V be a connected open subset of an infinite-dimensional Hilbert space H. If the boundary of V is homeomorphic to the unit sphere of H, then V is homeomorphic to the open unit ball of H.*

We would like to say a few words to explain why Conjecture 15 is an infinite-dimensional extension of the Jordan Curve Theorem. Suppose V is a connected open subset of the plane $\mathbb{R}^2$, and the boundary of V is homeomorphic to the unit circle S^1. Under the usual embedding of $\mathbb{R}^2$ into the unit sphere S^2, we may consider the boundary of V as a homeomorphic image of S^1 into S^2. By the Jordan Curve Theorem, this image divides S^2 into two components each of which is homeomorphic to the open unit ball of $\mathbb{R}^2$. By connectedness, V is homeomorphic to one of them. This is also an essential part of Doyle and Hocking's arguments in proving Theorem 1 in [8].

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