

THE BEST POSSIBILITY OF THE GRAND FURUTA INEQUALITY

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ABSTRACT. Let $A, B \in B(H)$ be invertible bounded linear operators on a Hilbert space H satisfying $O \leq B \leq A$, and let p, r, s, t be real numbers satisfying $1 < s, 0 < t < 1, t \leq r, 1 \leq p$. Furuta showed that if $0 < \alpha \leq \frac{1-t+r}{(p-t)s+r}$, then $\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^\alpha \leq A^{\{(p-t)s+r\}\alpha}$. This inequality is called the grand Furuta inequality, which interpolates the Furuta inequality ($t = 0$) and the Ando-Hiai inequality ($t = 1, r = s$).

In this paper, we show the grand Furuta inequality is best possible in the following sense: that is, if $\frac{1-t+r}{(p-t)s+r} < \alpha$, then there exist invertible matrices A, B with $O \leq B \leq A$ which do not satisfy $\left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^\alpha \leq A^{\{(p-t)s+r\}\alpha}$.

1. INTRODUCTION

Let A, B be bounded linear operators on a Hilbert space H . The following operator inequality is well-known as the Löwner-Heinz inequality ([9],[10]).

Proposition 1 (Löwner-Heinz). *Let $A, B \in B(H)$ be bounded linear operators on a Hilbert space H satisfying $O \leq B \leq A$. If $0 \leq p \leq 1$, then $B^p \leq A^p$.*

Concerning the Löwner-Heinz inequality, Chan and Kwong ([1]) conjectured that $O \leq B \leq A$ will imply $B^2 \leq (BA^2B)^{\frac{1}{2}}$ and Furuta gave an affirmative answer, which is a very useful extension of the Löwner-Heinz inequality ([4]).

Proposition 2 (Furuta). *Let $A, B \in B(H)$ be bounded linear operators on a Hilbert space H satisfying $O \leq B \leq A$. If*

$$(1.1) \quad p + 2r \leq (1 + 2r)q \quad \text{and} \quad 1 \leq q,$$

then

$$(1.2) \quad (A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$$

and

$$(1.3) \quad B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}.$$

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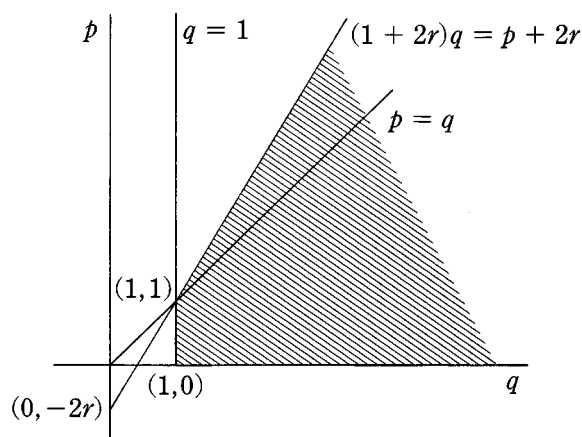


FIGURE 1

This inequality is called the Furuta inequality and many applications have been developed in the p -hyponormal operator theory ([2], [8]) and the relative entropy theory ([6], [7]). Recently, the Furuta inequality with negative powers has been studied by Yoshino ([13]), Fujii, Furuta, Kamei ([3]) and the author ([12]). Furthermore, Furuta ([7]) generalized the Furuta inequality.

Proposition 3 (Furuta). *Let $A, B \in B(H)$ be invertible bounded linear operators on a Hilbert space H satisfying $O \leq B \leq A$, and let p, r, s, t be real numbers satisfying $1 < s, 0 < t < 1, t \leq r, 1 \leq p$. If*

$$(1.4) \quad 0 < \alpha \leq \frac{1-t+r}{(p-t)s+r},$$

then

$$(1.5) \quad \left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^\alpha \leq A^{\{(p-t)s+r\}\alpha}.$$

This inequality is called the grand Furuta inequality, which interpolates the Furuta inequality ($t = 0$) and the Ando-Hiai inequality ($t = 1, r = s$). In [12], the author showed that the condition (1.1) is best possible in the following sense: that is, if $0 < q < 1$ or $(1+2r)q < p+2r$, then there exist matrices A, B with $O \leq B \leq A$ which do not satisfy the Furuta inequality (1.2). In this paper, by using the same technique as in [12], we show the condition (1.4) is best possible in the following sense: that is, if $\frac{1-t+r}{(p-t)s+r} < \alpha$, then there exist invertible matrices A, B with $O \leq B \leq A$ which do not satisfy the grand Furuta inequality (1.5).

2. RESULT

Theorem 4. *Let p, r, s, t be real numbers satisfying $1 < s, 0 < t < 1, t \leq r, 1 \leq p$. If*

$$(2.1) \quad \frac{1-t+r}{(p-t)s+r} < \alpha,$$

then there exist invertible matrices A, B with $O \leq B \leq A$ which do not satisfy

$$(2.2) \quad \left\{ A^{\frac{r}{2}} \left(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}} \right)^s A^{\frac{r}{2}} \right\}^\alpha \leq A^{\{(p-t)s+r\}\alpha}.$$

Proof. Let

$$A = \begin{pmatrix} a & \sqrt{\varepsilon(a-b-\delta)} \\ \sqrt{\varepsilon(a-b-\delta)} & b + \varepsilon + \delta \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

where

$$(2.3) \quad 0 < b < 1 < a, \quad 0 < \varepsilon, \quad 0 < \delta, \quad \varepsilon(1-b) \leq \delta(a-1+\varepsilon).$$

Then A and B are invertible and $O \leq B \leq A$. We shall show that if (2.2) is valid, then we have a contradiction by letting

$$\delta = \frac{1-b}{a-1}\varepsilon, \varepsilon \rightarrow +0, b \rightarrow +0.$$

Assume that (2.2) is valid. Let

$$\gamma = a - b + \varepsilon - \delta,$$

and

$$U = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} \sqrt{a-b-\delta} & \sqrt{\varepsilon} \\ \sqrt{\varepsilon} & -\sqrt{a-b-\delta} \end{pmatrix}.$$

Then U is unitary and

$$U^*AU = \begin{pmatrix} a + \varepsilon & 0 \\ 0 & b + \delta \end{pmatrix}.$$

Multiplying (2.2) by U, U^* , we have

$$U^*A^{\{(p-t)s+r\}\alpha}U \geq \left\{ U^*A^{\frac{r}{2}}U \left(U^*A^{-\frac{t}{2}}UU^*B^pUU^*A^{-\frac{t}{2}}U \right)^s U^*A^{\frac{r}{2}}U \right\}^\alpha$$

and

$$(2.4) \quad \begin{aligned} & \begin{pmatrix} (a + \varepsilon)^{\{(p-t)s+r\}\alpha} & 0 \\ 0 & (b + \delta)^{\{(p-t)s+r\}\alpha} \end{pmatrix} \\ & \geq \left[\begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} \left\{ \begin{pmatrix} (a + \varepsilon)^{-\frac{t}{2}} & 0 \\ 0 & (b + \delta)^{-\frac{t}{2}} \end{pmatrix} \right. \right. \\ & \quad \times \left. \left. U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} (a + \varepsilon)^{-\frac{t}{2}} & 0 \\ 0 & (b + \delta)^{-\frac{t}{2}} \end{pmatrix} \right\}^s \begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} \right]^\alpha. \end{aligned}$$

Write

$$\begin{aligned} & \begin{pmatrix} (a + \varepsilon)^{-\frac{t}{2}} & 0 \\ 0 & (b + \delta)^{-\frac{t}{2}} \end{pmatrix} U^* \begin{pmatrix} 1 & 0 \\ 0 & b^p \end{pmatrix} U \begin{pmatrix} (a + \varepsilon)^{-\frac{t}{2}} & 0 \\ 0 & (b + \delta)^{-\frac{t}{2}} \end{pmatrix} \\ & = \frac{1}{\gamma} D. \end{aligned}$$

Then

$$D = \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix},$$

where

$$\begin{aligned} A_1 &= (a + \varepsilon)^{-t}(a - b - \delta + b^p\varepsilon), \\ A_2 &= (b + \delta)^{-t}(\varepsilon + b^p(a - b - \delta)), \\ A_3 &= (a + \varepsilon)^{-\frac{t}{2}}(b + \delta)^{-\frac{t}{2}}(1 - b^p)\sqrt{\varepsilon}\sqrt{a - b - \delta}. \end{aligned}$$

If $\varepsilon, \delta \rightarrow +0$, then

$$A_1 \rightarrow a^{-t}(a - b), \quad A_2 \rightarrow b^{p-t}(a - b).$$

Since we will consider the case $b \rightarrow +0, A_1 \rightarrow a^{1-t}$ and $A_2 \rightarrow 0$, we may assume

$$0 < A_2 < A_1.$$

Let

$$V = \frac{1}{\sqrt{A_1 - A_2 + 2\varepsilon_1}} \begin{pmatrix} \sqrt{A_1 - A_2 + \varepsilon_1} & \sqrt{\varepsilon_1} \\ \sqrt{\varepsilon_1} & -\sqrt{A_1 - A_2 + \varepsilon_1} \end{pmatrix},$$

where

$$2\varepsilon_1 = -A_1 + A_2 + \sqrt{(A_1 - A_2)^2 + 4A_3^2}.$$

Then V is unitary and

$$V^*DV = V^* \begin{pmatrix} A_1 & A_3 \\ A_3 & A_2 \end{pmatrix} V = \begin{pmatrix} A_1 + \varepsilon_1 & 0 \\ 0 & A_2 - \varepsilon_1 \end{pmatrix}.$$

Then (2.4) implies

$$\begin{aligned} & \begin{pmatrix} (a + \varepsilon)^{\{(p-t)s+r\}\alpha} & 0 \\ 0 & (b + \delta)^{\{(p-t)s+r\}\alpha} \end{pmatrix} \\ (2.5) \quad & \geq \left\{ \begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} VV^* \left(\frac{1}{\gamma} D \right)^s VV^* \begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} \right\}^\alpha \\ & = \left\{ \begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} V \frac{1}{\gamma^s} \begin{pmatrix} (A_1 + \varepsilon_1)^s & 0 \\ 0 & (A_2 - \varepsilon_1)^s \end{pmatrix} \right. \\ & \quad \left. \times V^* \begin{pmatrix} (a + \varepsilon)^{\frac{r}{2}} & 0 \\ 0 & (b + \delta)^{\frac{r}{2}} \end{pmatrix} \right\}^\alpha. \end{aligned}$$

Write the right-hand matrix as

$$\gamma^{-s\alpha}(A_1 - A_2 + 2\varepsilon_1)^{-\alpha} \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix}^\alpha;$$

then

$$\begin{aligned} B_1 &= (a + \varepsilon)^r \{ (A_1 + \varepsilon_1)^s (A_1 - A_2 + \varepsilon_1) + (A_2 - \varepsilon_1)^s \varepsilon_1 \}, \\ B_2 &= (b + \delta)^r \{ (A_1 + \varepsilon_1)^s \varepsilon_1 + (A_2 - \varepsilon_1)^s (A_1 - A_2 + \varepsilon_1) \}, \\ B_3 &= (a + \varepsilon)^{\frac{r}{2}} (b + \delta)^{\frac{r}{2}} \sqrt{\varepsilon_1} \sqrt{A_1 - A_2 + \varepsilon_1} \{ (A_1 + \varepsilon_1)^s - (A_2 - \varepsilon_1)^s \}. \end{aligned}$$

If $\varepsilon, \delta \rightarrow +0$, then

$$\varepsilon_1 \rightarrow 0, B_1 \rightarrow a^{r-st}(a - b)^{1+s}(a^{-t} - b^{p-t}), B_2 \rightarrow b^{r+(p-t)s}(a - b)^{1+s}(a^{-t} - b^{p-t}).$$

Since we will consider the case $b \rightarrow +0, B_1 \rightarrow a^{1+r+s-t-st}$ and $B_2 \rightarrow 0$, we may assume

$$0 < B_2 < B_1.$$

Let

$$W = \frac{1}{\sqrt{B_1 - B_2 + 2\varepsilon_2}} \begin{pmatrix} \sqrt{B_1 - B_2 + \varepsilon_2} & \sqrt{\varepsilon_2} \\ \sqrt{\varepsilon_2} & -\sqrt{B_1 - B_2 + \varepsilon_2} \end{pmatrix},$$

where

$$2\varepsilon_2 = -B_1 + B_2 + \sqrt{(B_1 - B_2)^2 + 4B_3^2}.$$

Then W is unitary and

$$W^* \begin{pmatrix} B_1 & B_3 \\ B_3 & B_2 \end{pmatrix} W = \begin{pmatrix} B_1 + \varepsilon_2 & 0 \\ 0 & B_2 - \varepsilon_2 \end{pmatrix}.$$

Multiplying (2.5) by W^*, W , we have

$$(2.6) \quad \begin{aligned} & \gamma^{-s\alpha}(A_1 - A_2 + 2\varepsilon_1)^{-\alpha} \begin{pmatrix} (B_1 + \varepsilon_2)^\alpha & 0 \\ 0 & (B_2 - \varepsilon_2)^\alpha \end{pmatrix} \\ & \leq W^* \begin{pmatrix} (a + \varepsilon)^{\{(p-t)s+r\}\alpha} & 0 \\ 0 & (b + \delta)^{\{(p-t)s+r\}\alpha} \end{pmatrix} W. \end{aligned}$$

Write the right-hand matrix as

$$\frac{1}{B_1 - B_2 + 2\varepsilon_2} \begin{pmatrix} C_1 & C_3 \\ C_3 & C_2 \end{pmatrix};$$

then

$$\begin{aligned} C_1 &= (a + \varepsilon)^{\tilde{\alpha}}(B_1 - B_2 + \varepsilon_2) + (b + \delta)^{\tilde{\alpha}}\varepsilon_2, \\ C_2 &= (a + \varepsilon)^{\tilde{\alpha}}\varepsilon_2 + (b + \delta)^{\tilde{\alpha}}(B_1 - B_2 + \varepsilon_2), \\ C_3 &= \{(a + \varepsilon)^{\tilde{\alpha}} - (b + \delta)^{\tilde{\alpha}}\} \sqrt{\varepsilon_2} \sqrt{B_1 - B_2 + \varepsilon_2} \end{aligned}$$

where

$$\tilde{\alpha} = \alpha(r + sp - st).$$

Let

$$\tilde{\gamma} = \gamma^{s\alpha}(A_1 - A_2 + 2\varepsilon_1)^\alpha.$$

Then

$$O \leq \begin{pmatrix} \tilde{\gamma}C_1 - (B_1 - B_2 + 2\varepsilon_2)(B_1 + \varepsilon_2)^\alpha & \tilde{\gamma}C_3 \\ \tilde{\gamma}C_3 & \tilde{\gamma}C_2 - (B_1 - B_2 + 2\varepsilon_2)(B_2 - \varepsilon_2)^\alpha \end{pmatrix}.$$

Hence, by taking the determinant of the right-hand matrix, we have

$$\begin{aligned} 0 \leq & \tilde{\gamma}^2 C_1 C_2 - \tilde{\gamma} C_1 (B_1 - B_2 + 2\varepsilon_2) (B_2 - \varepsilon_2)^\alpha - \tilde{\gamma} C_2 (B_1 - B_2 + 2\varepsilon_2) (B_1 + \varepsilon_2)^\alpha \\ & + (B_1 - B_2 + 2\varepsilon_2)^2 (B_1 + \varepsilon_2)^\alpha (B_2 - \varepsilon_2)^\alpha - \tilde{\gamma}^2 C_3^2, \end{aligned}$$

and

$$\begin{aligned} 0 \leq & (B_1 - B_2 + 2\varepsilon_2) \{ \tilde{\gamma}^2 (\alpha + \varepsilon)^{\tilde{\alpha}} (b + \varepsilon)^{\tilde{\alpha}} (B_1 - B_2 + 2\varepsilon_2) \\ & - \tilde{\gamma} (\alpha + \varepsilon)^{\tilde{\alpha}} (B_1 - B_2 + 2\varepsilon_2) (B_2 - \varepsilon_2)^\alpha \\ & - \tilde{\gamma} (b + \varepsilon)^{\tilde{\alpha}} (B_2 - \varepsilon_2)^\alpha - \tilde{\gamma} (\alpha + \varepsilon)^{\tilde{\alpha}} \varepsilon_2 (B_1 + \varepsilon_2)^\alpha \\ & - \tilde{\gamma} (b + \varepsilon)^{\tilde{\alpha}} (B_1 - B_2 + \varepsilon_2) (B_1 + \varepsilon_2)^\alpha \\ & + (B_1 - B_2 + 2\varepsilon_2) (B_1 + \varepsilon_2)^\alpha (B_2 - \varepsilon_2)^\alpha \}. \end{aligned}$$

Since $0 < B_1 - B_2 + 2\varepsilon_2$, we have

$$(2.7) \quad \begin{aligned} & -\varepsilon_2 \left\{ \tilde{\gamma}(a + \varepsilon)^{\tilde{\alpha}} - (B_2 - \varepsilon_2)^\alpha \right\} \left\{ \tilde{\gamma}(b + \delta)^{\tilde{\alpha}} - (B_1 + \varepsilon_2)^\alpha \right\} \\ & \leq (B_1 - B_2 + \varepsilon_2) \left\{ \tilde{\gamma}(a + \varepsilon)^{\tilde{\alpha}} - (B_1 + \varepsilon_2)^\alpha \right\} \left\{ \tilde{\gamma}(b + \delta)^{\tilde{\alpha}} - (B_2 - \varepsilon_2)^\alpha \right\}. \end{aligned}$$

We estimate the first order of each terms in (2.7) with respect to ε, δ . In the following, o means $o(\varepsilon), o(\delta)$, that is

$$\frac{o}{\varepsilon}, \frac{o}{\delta} \rightarrow 0 \quad (\varepsilon, \delta \rightarrow +0).$$

Then

$$\begin{aligned} A_1 &= a^{-t} \left(1 - \frac{t}{a} \varepsilon + o \right) (a - b) \left(1 + \frac{b^p}{a - b} \varepsilon - \frac{1}{a - b} \delta + o \right) \\ &= a^{-t} (a - b) \left(1 - \frac{t}{a} \varepsilon + \frac{b^p}{a - b} \varepsilon - \frac{1}{a - b} \delta + o \right), \\ A_2 &= b^{-t} \left(1 - \frac{t}{b} \delta + o \right) b^p (a - b) \left(1 + \frac{1}{b^p (a - b)} \varepsilon - \frac{1}{a - b} \delta + o \right) \\ &= b^{p-t} (a - b) \left(1 + \frac{1}{b^p (a - b)} \varepsilon - \frac{t}{b} \delta - \frac{1}{a - b} \delta + o \right), \\ A_3^2 &= a^{-t} \left(1 - \frac{t}{a} \varepsilon + o \right) b^{-t} \left(1 - \frac{t}{b} \delta + o \right) (1 - b^p)^2 \varepsilon (a - b) \left(1 - \frac{1}{a - b} \delta \right) \\ &= a^{-t} b^{-t} (1 - b^p) (a - b) \varepsilon \left(1 - \frac{t}{a} \varepsilon - \frac{t}{b} \delta - \frac{1}{a - b} \delta + o \right), \\ \varepsilon_1 &= \frac{1}{2} (A_1 - A_2) \left\{ -1 + \sqrt{1 + \frac{4A_3^2}{(A_1 - A_2)^2}} \right\} \\ &= \frac{1}{2} (A_1 - A_2) \left\{ -1 + \left(1 + \frac{1}{2} \frac{4A_3^2}{(A_1 - A_2)^2} + o \right) \right\} \\ &= \frac{a^{-t} b^{-t} (1 - b^p)^2}{a^{-t} - b^{p-t}} \varepsilon \left(1 + \frac{o}{\varepsilon} \right). \end{aligned}$$

Hence

$$\begin{aligned} A_1 - A_2 + 2\varepsilon_1 &= (a - b)(a^{-t} - b^{p-t})(1 + c_1 \varepsilon + c_2 \delta + o), \\ A_1 - A_2 + \varepsilon_1 &= (a - b)(a^{-t} - b^{p-t})(1 + c_3 \varepsilon + c_4 \delta + o), \end{aligned}$$

and

$$\tilde{\gamma} = (a - b)^{\alpha + \alpha s} (a^{-t} - b^{p-t})^\alpha \left(1 + \frac{\alpha s}{a - b} \varepsilon + \alpha c_1 \varepsilon - \frac{\alpha s}{a - b} \delta + \alpha c_2 \delta + o \right),$$

where

$$\begin{aligned} c_1 &= \frac{1}{(a - b)(a^{-t} - b^{p-t})} \left(-t(a - b)a^{-t-1} + a^{-t}b^p - b^{-t} + \frac{2a^{-t}b^{-t}(1 - b^p)^2}{a^{-t} - b^{p-t}} \right), \\ c_2 &= \frac{1}{(a - b)(a^{-t} - b^{p-t})} \left(-a^{-t} + t(a - b)b^{p-t-1} + b^{p-t} \right), \\ c_3 &= c_1 - \frac{a^{-t}b^{-t}(1 - b^p)^2}{(a - b)(a^{-t} - b^{p-t})^2}, \\ c_4 &= c_2. \end{aligned}$$

Then

$$(A_1 + \varepsilon_1)^s = a^{-st}(a-b)^s \left(1 - \frac{st}{a}\varepsilon + \frac{sb^p}{a-b}\varepsilon + \frac{sb^{-t}(1-b^p)^2}{(a-b)(a^{-t}-b^{p-t})}\varepsilon - \frac{s}{a-b}\delta + o \right)$$

and

$$(A_2 - \varepsilon_1)^s = b^{sp-st}(a-b)^s \left(1 + \frac{s}{b^p(a-b)}\varepsilon - \frac{sa^{-t}(1-b^p)^2}{b^p(a-b)(a^{-t}-b^{p-t})}\varepsilon - \frac{st}{b}\delta - \frac{s}{a-b}\delta + o \right).$$

Hence

$$\begin{aligned} B_1 &= (a+\varepsilon)^r \{ (A_1 + \varepsilon_1)^s (A_1 - A_2 + \varepsilon_1) + (A_2 - \varepsilon_1)^s \varepsilon_1 \} \\ &= a^{r-st}(a-b)^{1+s}(a^{-t}-b^{p-t})(1+c_5\varepsilon+c_6\delta+o), \\ B_2 &= (b+\delta)^r \{ (A_1 + \varepsilon_1)^s \varepsilon_1 + (A_2 - \varepsilon_1)^s (A_1 - A_2 + \varepsilon_1) \} \\ &= b^{r+sp-st}(a-b)^{1+s}(a^{-t}-b^{p-t})(1+c_7\varepsilon+c_8\delta+o), \\ B_3^2 &= (a+\varepsilon)^r (b+\delta)^r \varepsilon_1 (A_1 - A_2 + \varepsilon_1) \{ (A_1 + \varepsilon_1)^s - (A_2 - \varepsilon_1)^s \}^2 \\ &= a^{r-t}b^{r-t}(a-b)^{1+2s}(a^{-st}-b^{sp-st})^2(1-b^p)^2\varepsilon \left(1 + \frac{o}{\varepsilon} \right), \\ \varepsilon_2 &= \frac{1}{2}(B_1 - B_2) \left\{ -1 + \sqrt{1 + \frac{4B_3^2}{(B_1 - B_2)^2}} \right\} \\ &= \frac{1}{2}(B_1 - B_2) \left\{ -1 + \left(1 + \frac{1}{2} \frac{4B_3^2}{(B_1 - B_2)^2} + o \right) \right\} \\ &= \frac{a^{r-t}b^{r-t}(a-b)^s(a^{-st}-b^{sp-st})^2(1-b^p)^2}{(a^{-t}-b^{p-t})(a^{r-st}-b^{r+sp-st})}\varepsilon \left(1 + \frac{o}{\varepsilon} \right), \end{aligned}$$

where

$$\begin{aligned} c_5 &= c_3 + \frac{r-st}{a} + \frac{sb^p}{a-b} + \frac{sb^{-t}(1-b^p)^2}{(a-b)(a^{-t}-b^{p-t})} + \frac{a^{-t+st}b^{-t+sp-st}(1-b^p)^2}{(a-b)(a^{-t}-b^{p-t})^2}, \\ c_6 &= c_4 - \frac{s}{a-b}, \\ c_7 &= c_3 + \frac{s}{b^p(a-b)} + \frac{a^{-t-st}b^{-t-sp+st}(1-b^p)^2}{(a-b)(a^{-t}-b^{p-t})^2} - \frac{sa^{-t}(1-b^p)^2}{b^p(a-b)(a^{-t}-b^{p-t})}, \\ c_8 &= c_4 + \frac{r-st}{b} - \frac{s}{a-b}. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\gamma}(b+\delta)^{\tilde{\alpha}} &= \gamma^{\alpha s}(A_1 - A_2 + 2\varepsilon_1)^\alpha (b+\delta)^{\alpha(r+sp-st)} \\ &= b^{\alpha(r+sp-st)}(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \\ &\quad \times \left(1 + \alpha c_1\varepsilon + \frac{\alpha s}{a-b}\varepsilon + \alpha c_2\delta - \frac{\alpha s}{a-b}\delta + \frac{\alpha(r+sp-st)}{b}\delta + o \right), \\ (B_1 + \varepsilon_2)^\alpha &= a^{\alpha(r-st)}(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha (1 + \alpha c_9\varepsilon + \alpha c_6\delta + o), \\ (B_2 - \varepsilon_2)^\alpha &= b^{\alpha(r+sp-st)}(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha (1 + \alpha c_{10}\varepsilon + \alpha c_8\delta + o), \end{aligned}$$

where

$$c_9 = c_5 + \frac{a^{-t+st}b^{r-t}(1-b^p)^2(a^{-st}-b^{sp-st})^2}{(a-b)(a^{-t}-b^{p-t})^2(a^{r-st}-b^{r+sp-st})},$$

$$c_{10} = c_7 - \frac{a^{r-t}b^{-t-sp+st}(1-b^p)^2(a^{-st}-b^{sp-st})^2}{(a-b)(a^{-t}-b^{p-t})^2(a^{r-st}-b^{r+sp-st})}.$$

Hence

$$\begin{aligned} & \tilde{\gamma}(b+\delta)^{\tilde{\alpha}} - (B_2 - \varepsilon_2)^\alpha \\ &= b^{\alpha(r+sp-st)}(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \\ & \quad \times \left(\alpha c_1 \varepsilon + \frac{\alpha s}{a-b} \varepsilon + \alpha c_2 \delta - \frac{\alpha s}{a-b} \delta + \frac{\alpha(r+sp-st)}{b} \delta \right. \\ & \quad \left. - \alpha c_{10} \varepsilon - \alpha \left(c_2 + \frac{r-st}{b} - \frac{s}{a-b} \right) \delta + o \right) \\ &= b^{\alpha(r+sp-st)}(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \varepsilon \left(c_{11} + \frac{\alpha sp}{b} \frac{\delta}{\varepsilon} + \frac{o}{\varepsilon} \right) \end{aligned}$$

where

$$c_{11} = -\frac{\alpha s(1-b^p)(a^{-t}-b^{-t})}{(a-b)(a^{-t}-b^{p-t})} - \frac{\alpha a^{-t}b^{-t}(1-b^p)^2(a^r-b^s)(a^{-st}-b^{sp-st})}{(a-b)(a^{-t}-b^{p-t})^2(a^{r-st}-b^{r+sp-st})}.$$

Also, we have

$$\begin{aligned} & \tilde{\gamma}(a+\varepsilon)^{\tilde{\alpha}} - (B_2 - \varepsilon_2)^\alpha \\ &= (a^{\alpha(r+sp-st)} - b^{\alpha(r+sp-st)})(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \left(1 + \frac{o}{\varepsilon} \right), \\ & \tilde{\gamma}(b+\delta)^{\tilde{\alpha}} - (B_1 + \varepsilon_2)^\alpha \\ &= -(a^{\alpha(r-st)} - b^{\alpha(r+sp-st)})(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \left(1 + \frac{o}{\varepsilon} \right), \\ & B_1 - B_2 + \varepsilon_2 \\ &= (a-b)^{1+s}(a^{-t}-b^{p-t})(a^{r-st}-b^{r+sp-st}) \left(1 + \frac{o}{\varepsilon} \right), \\ & \tilde{\gamma}(a+\varepsilon)^{\tilde{\alpha}} - (B_1 + \varepsilon_2)^\alpha \\ &= a^{\alpha(r-st)}(a^{\alpha sp} - 1)(a-b)^{\alpha(1+s)}(a^{-t}-b^{p-t})^\alpha \left(1 + \frac{o}{\varepsilon} \right). \end{aligned}$$

Hence, by (2.4),

$$\begin{aligned} & (a^{-st}-b^{sp-st})^2(a^{\alpha(r+sp-st)} - b^{\alpha(r+sp-st)})(a^{\alpha(r-st)} - b^{\alpha(r+sp-st)})(1-b^p)^2 \\ & \leq a^{\alpha(r-st)+t-r}b^{\alpha(r+sp-st)+t-r}(a-b)(a^{-t}-b^{p-t})^2(a^{r-st}-b^{r+sp-st})^2(a^{\alpha sp} - 1) \\ & \quad \times \left(c_{11} + \frac{\alpha sp}{b} \frac{\delta}{\varepsilon} + \frac{o}{\varepsilon} \right). \end{aligned}$$

We remark (2.3) implies

$$\frac{1-b}{a-1+\varepsilon} \leq \frac{\delta}{\varepsilon}.$$

Hence, to get a contradiction, it seems reasonable to assume

$$\delta = \frac{1-b}{a-1} \varepsilon.$$

Then, by letting $\varepsilon \rightarrow +0$, we have

$$\begin{aligned} & (a-1)(a^{-st} - b^{sp-st})^2(1-b^p)^2(a^{\alpha(r+sp-st)} - b^{\alpha(r+sp-st)})(a^{\alpha(r-st)} - b^{\alpha(r+sp-st)}) \\ & \leq a^{\alpha(r-st)+t-r}b^{\alpha(r+sp-st)-1+t-r}(a^{r-st} - b^{r+sp-st})(a^{\alpha sp} - 1) \\ & \times \left\{ -\alpha sb(1-b^p)(a^{-t} - b^{-t})(a^{-t} - b^{p-t})(a^{r-st} - b^{r+sp-st})(a-1) \right. \\ & \quad - \alpha a^{-t}b^{1-t}(1-b^p)^2(a^r - b^s)(a^{-st} - b^{sp-st})(a-1) \\ & \quad \left. + \alpha sp(1-b)(a-b)(a^{-t} - b^{p-t})^2(a^{r-st} - b^{r+sp-st}) \right\}. \end{aligned}$$

Since (2.1) implies

$$0 < \alpha(r+sp-st) - 1 + t - r,$$

we have, by letting $b \rightarrow +0$,

$$0 < (a-1)a^{-2st+\alpha(2r+sp-2st)} \leq 0.$$

This is a contradiction. \square

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