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A FATOU THEOREM FOR THE EQUATION $u_t = \Delta(u-1)_+$

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Dedicated to the memory of Eugene Fabes

ABSTRACT. In one space dimension and for a given function $u_I(x) \in C_0^\infty$ (say such that $u_I(x) > 1$ in some interval), the equation $u_t = \Delta(u-1)_+$ can be thought of as describing the energy per unit volume in a Stefan-type problem where the latent heat of the phase change is given by $1 - u_I(x)$. Given a solution $0 \le u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ to this equation, we prove that for a.e. $x_0 \in \mathbb{R}^n$, there exists $\lim_{(x,t)\in\Gamma^k_\beta(x_0),\,(x,t)\to x_0}(u(x,t)-1)_+ = (f(x_0)-1)_+$, where $f = \partial\mu/\partial|\;|$ is the Radon-Nikodym derivative of the initial trace μ with respect to Lebesgue measure and $\Gamma^k_\beta(x_0) = \{(x,t) : |x-x_0| < \beta\sqrt{t},\ 0 < t < k\}$ are the parabolic "non-tangential" approach regions. Since only $(u-1)_+$ is continuous, while u is usually not, $\lim_{(x,t)\in\Gamma^k_\beta(x_0),\,(x,t)\to x_0} u(x,t) = f(x_0)$ does not hold in general.

1. Introduction

In this contribution we want to address the existence of pointwise limits of non-negative solutions to the equation

$$u_t = \Delta(u-1)_+, \tag{1.1}$$

where $(x,t) \in \mathbb{R}^n \times (0,T)$ when (x,t) approaches the initial surface $\{t = 0\}$. In one space dimension and for a given function $u_I(x) \in C_0^\infty$ (say such that $u_I(x) > 1$ in some interval), equation (1.1) can be thought of as describing the energy per unit volume in a Stefan-type problem where the latent heat of the phase change is given by $1 - u_I(x)$. Note that discontinuous solutions should be expected for (1.1) (see [BKM]).

In previous papers ([AK], [K]) a-priori regularity of non-negative solutions in the sense of distributions $u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ was found, mainly u, $\nabla_x(u-1)_+$ and $\frac{\partial}{\partial t}(u-1)_+ \in L^2_{\text{loc}}(\mathbb{R}^n \times (0,T))$, and continuity of $(u-1)_+$ in $(\mathbb{R}^n \times (0,T))$. Also $(u-1)_+$ is a (weak) subsolution to the heat equation. From the weak Harnack inequality

$$\int_{\mathbb{R}^n} u(x,t)\exp(-c|x|^2)\,dx \le M(u,n,T), \tag{1.2}$$

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for some $c = c(T) \in \mathbb{R}_+$ and $0 < t < T/2$ (see [AK]), existence of a unique initial trace follows, which is a Radon measure satisfying the growth condition at infinity

$$\int_{\mathbb{R}^n} \exp(-c|x|^2)\, d\mu(x) < \infty, \tag{1.3}$$

and is taken in the appropriate sense. Each such measure gives rise to a solution to (1.1), which is unique (see [K]).

Fatou theorems for solutions to degenerate parabolic equations were proven in [DFK] (for the porous medium equation $u_t = \Delta u^m$, $m > 1$) and [H] (for generalized diffusions $u_t = \Delta\phi(u)$, $\phi'(u) > 0$ for $u > 0$ and growth conditions on ϕ near 0 and at infinity). In both situations the solutions u are continuous (see [DFK], [DB]) and the result reads as follows:

Let $\Gamma^k_\beta(x_0) = \{(x,t) : |x - x_0| < \beta\sqrt{t},\ 0 < t < k\}$ be the parabolic non-tangential approach "cones". For a.e. $x_0 \in \mathbb{R}^n$, there exists $\lim_{(x,t)\in\Gamma^k_\beta(x_0),\ (x,t)\to x_0} u(x,t) = f(x_0)$, where $f = \partial\mu/\partial|\ |$ is the Radon-Nikodym derivative of the initial trace with respect to Lebesgue measure.

An analogous result holds for non-negative solutions of the heat equation; in this case the result follows from estimates on the heat kernel.

In this paper we prove the following

1.1. Theorem. *Let $0 \le u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ be a solution in the sense of distributions of* (1.1). *Let μ be its initial trace and $f = d\mu/dx$ the Radon-Nikodym derivative of μ with respect to the Lebesgue measure. Then for almost every $x_0 \in \mathbb{R}^n$,*

$$\lim_{(x,t)\in\Gamma^k_\beta(x_0),\ (x,t)\to x_0} (u(x,t) - 1)_+ = (f(x_0) - 1)_+.$$

Theorem 1.1 is proven stepwise: In Section 2 we prove it for initial traces which are locally integrable functions satisfying (1.3). The proof uses a lemma which allows a reduction to the linear case due to Dahlberg, Fabes and Kenig (Lemma 2.2 from [DFK]). Then we follow an argument of [H], getting the necessary equicontinuity of a scaled family of comparison functions from the results of DiBenedetto ([DB]) which applies to (1.1) due to a priori integrabilities of u, $(u-1)_+$, $\frac{\partial(u-1)_+}{\partial t}$ and $\nabla(u-1)_+$ obtained in [K] and [AK]. In Section 3 we prove the result if the initial trace is a measure satisfying (1.3). We regularize the singular part of the initial measure and use the convergence of the solutions thus obtained. Here the uniform integrability of suitable heat functions plays a strong role. Then we use the uniqueness result of [K] to complete the proof. In Section 4 we exhibit an example which shows that

$$\lim_{(x,t)\in\Gamma^k_\beta(x_0),\ (x,t)\to x_0} u(x,t) = f(x_0)$$

does not hold in general for solutions to (1.1).

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2. Step 1: The initial trace μ is a locally integrable function

Let $0 \le \mu = u_I(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$, such that u_I satisfies (1.3) and let $u(x,t)$ be the (unique) solution to (1.1) with initial trace u_I; u is defined in $\mathbb{R}^n \times (0,T)$ where $T = T(u_I)$, and $\|u(x,t) - u_I(x)\|_{L^1(K)} \to 0$ for any compact $K \subset \mathbb{R}^n$ (see [AK]).

For $0 < \alpha, h$ let $\Gamma^h_\alpha(x_0) = \{(x,t) \in \mathbb{R}^n \times (0,\infty) : |x - x_0| \le \alpha\sqrt{t}, 0 < t < h\}$.

2.1. *Remark.* Let $\alpha,\ h > 0$. For a.e. $x_0 \in \mathbb{R}^n$,

$$0 \le \liminf_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} (u(x,t)-1)_+ \le \lim_{t\to 0}(u(x,t)-1)_+ = (u_I(x_0)-1)_+.$$

This is a consequence of the convergence of $u(.,t) \to u_I$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, the Lipchitz continuity of $g(u) := (u-1)_+$, and the continuity of $(u-1)_+$ in $\mathbb{R}^n \times (0,T)$. Therefore, for a.e. $x_0 \in E_0 := \{x_0 \in \mathbb{R}^n : u_I(x_0) \le 1\}$,

$$\lim_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} (u(x,t)-1)_+ = 0.$$

Here we write $u_I(x)$ for the precise representative of u_I; that is, we understand $u_I(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} u_I(y)dy$.

2.2. Lemma. *Let $E = \{x_0 \in \mathbb{R}^n : u_I(x_0) > 1\}$, and $0 < \alpha < T$. For a.e. $x_0 \in E$,*

$$\lim_{(x,t)\in\Gamma_1^\alpha(x_0),(x,t)\to x_0} u(x,t) = u_I(x_0).$$

Proof. Given $\epsilon > 0$, let $E_\epsilon = \{x_0 \in \mathbb{R}^n : (u_I(x_0)-1)_+ > \epsilon\}$. Then $E = \bigcup_{\epsilon>0} E_\epsilon$.

Let $E'_\epsilon = \{$density points of $E_\epsilon\}$. Let $x_0 \in E'_\epsilon$. Without loss of generality we may assume $x_0 = 0$ and $\alpha > 1$.

Let $v_I = (1+\epsilon)\ \chi_{E_\epsilon}$, whence $v_I \le u_I$ a.e. Let $v(x,t)$ be the solution of (1.1) evolving from the initial datum v_I. By the results in [B] and [AK] $v(x,t) \le u(x,t)$ for a.e. $(x,t) \in \mathbb{R}^n \times (0,T)$. For any $\lambda > 0$, let $v_\lambda(x,t) = v(\lambda x, \lambda^2 t)$, v_λ is a solution to (1.1). Let d_λ be its initial trace. Following [H], Lemma 1.6, we find $d_{\mu_\lambda} \to (\epsilon+1)$ weakly. Consider the family $(v_\lambda(x,t)-1)_+ = (v(\lambda x, \lambda^2 t)-1)_+$. Since $\{v_\lambda\}$ is locally bounded in $\mathbb{R}^n \times (0,T)$ and for any compact set $K \subset \mathbb{R}^n \times (0,T)$ the norms $\|(v_\lambda-1)_+\|_{L^2(K)}$, $\|\frac{\partial}{\partial t}(v_\lambda-1)_+\|_{L^2(K)}$, and $\|\frac{\partial}{\partial x_i}(v_\lambda-1)_+\|_{L^2(K)}$ can be estimated in terms of $\|(v_\lambda-1)_+\|_{L^2(K')}$, $K \subset K' \subset \mathbb{R}^n \times (0,T), K'$ compact (see [AK]), from the results in [DB] the equicontinuity of $(v_\lambda-1)_+$ on any compact subset of $\mathbb{R}^n \times (0,T)$ follows. Therefore there exists a subsequence (which we will still call $(v_\lambda-1)_+$) converging uniformly on compact subsets of $\mathbb{R}^n \times (0,T)$. Now take $K = \overline{B}_1(0) \times \{1\} = \{(x,t) \in \mathbb{R}^n \times (0,T) : |x| \le 1, t = 1\}$. We obtain

$$v_\lambda(x',1) > 1 + \frac{\epsilon}{2}, \qquad \text{for } |x'| < 1$$

if $\lambda < \lambda_0$, whence

$$u(x,t) \ge v(x,t) > 1 + \frac{\epsilon}{2}, \qquad |x|^2 < t,\ t < \lambda_0,$$

and

$$\liminf_{(x,t)\in\Gamma_1^T(x_0),\,(x,t)\to x_0} (u(x,t)-1)_+ > 0 \qquad \text{for a.e. } x_0 \in E.$$

In order to uniformize the height h of the "non-tangential" approach parabolas, let us split E_ϵ into a countable family of sets E_ϵ^k as follows:

Given $x_0 \in E_\epsilon$, there exists $h_{x_0} > 0$ such that $u(x,t) > 1 + \frac{\epsilon}{2}$ in $\Gamma_\alpha^{h_{x_0}}(x_0)$. Now let $k = \min\{k : h_{x_0} \ge 2^{-k}\}$.

Let us recall that by a familiar point of density argument (see [C]), given $\alpha, \beta > 0, h > 0, \epsilon > 0$ and a closed set F, there exists a closed set $F_0, |F \setminus F_0| < \epsilon$ and $k > 0$ such that $\bigcup_{x_0\in F_0} \Gamma_\beta^k(x_0) \subset \bigcup_{x_0\in F} \Gamma_\alpha^h(x_0)$.

Let $F \subset E$ be compact, and $R = \bigcup_{x_0 \in F} \Gamma_\alpha^h$. Applying Lemma 2.2 from [DFK] in R yields the existence and finiteness of the limit

$$\lim_{(x,t)\in\Gamma_\alpha^h(x_0),\ (x,t)\to x_0} u(x,t).$$

Then, the convergence $||u(.,t) - u_I||_{L^1} \to 0$ yields its equality to $u_I(x_0)$. (Alternatively, $u(x,t)$ solves the heat equation in R, whence its "non-tangential" limit equals $u_I(x_0)$, for a.e. point $x_0 \in F$.)

3. Step 2: The initial trace μ is a measure

3.1. Lemma. *Let $\mu = f + \mu_s$ be the initial trace of u, where f is absolutely continuous and μ_s is singular with respect to Lebesgue measure.*

For a.e. $x_0 \in \mathbb{R}^n$,

$$\lim_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} (u(x,t) - 1)_+ = (f(x_0) - 1)_+.$$

Proof. Take $\mu_s^\epsilon = \mu_s * \rho_\epsilon$, where ρ_ϵ are the usual compactly supported mollifiers. Writing u_g for the solution of (1.1) with initial datum g, by comparison (see [B], [AK]) we have

$$u_f \le u_{f+\mu^\epsilon} + \le w_\epsilon \quad \text{a.e. in } \mathbb{R}^n \times (0,T),$$

where w_ϵ is the solution of the heat equation with initial datum $f + \mu_s^\epsilon + 1$. From the results in Section 2 we have

$$\lim_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} (u_f(x,t) - 1)_+ = (f(x_0) - 1)_+, \quad \text{a.e. in } \mathbb{R}^n.$$

To show that $u_{f+\mu_s^\epsilon} \to u_{f+\mu}$ a.e. in $\mathbb{R}^n \times (0,T)$, let us recall that $\{u_\epsilon\}$ is bounded in $L^2(K)$, for any compact subset K of $\mathbb{R}^n \times (0,T)$), whence a subsequence of $\{u_\epsilon\}$, which we call again $\{u_\epsilon\}$, can be chosen such that $\lim_{\epsilon\to 0} u_\epsilon = u$, and $\lim_{\epsilon\to 0}(u_\epsilon - 1)_+ = h(x,t)$, a.e. in $\mathbb{R}^n \times (0,T)$. To show that $h(x,t) = (u-1)_+$, take arbitrary $0 < t_1 < t_2 < T$, $0 < R \in \mathbb{R}$, $B = B_R(0)$ and let $\epsilon \to 0$ in

$$\int_{t_1}^{t_2}\int_B u_\epsilon \chi_{\{u_\epsilon>1\}}\chi_{\{h>0\}} = \int_{t_1}^{t_2}\int_B (u_\epsilon - 1)\chi_{\{u_\epsilon>1\}}\chi_{\{h>0\}}$$

$$+ \int_{t_1}^{t_2}\int_B \chi_{\{u_\epsilon>1\}}\chi_{\{h>0\}},$$

to get

$$\int_{t_1}^{t_2}\int_B u \chi_{\{u_\epsilon>1\}}\chi_{\{h>0\}} = \int_{t_1}^{t_2}\int_B (u - 1)\chi_{\{u_\epsilon>1\}}\chi_{\{h>0\}} + \int_{t_1}^{t_2}\int_B \chi_{\{u>1\}}\chi_{\{h>0\}},$$

whence

$$\int_{t_1}^{t_2}\int_B (u-1)\chi_{\{h>0\}} = \int_{t_1}^{t_2}\int_B h\chi_{\{h>0\}},$$

and therefore $h = \lim(u_\epsilon - 1)_+ > 0$ implies $h = u - 1$.

To complete the proof use

$$\chi_{\{u_\epsilon\le 1\}} = 1 - \chi_{\{u_\epsilon>1\}} \to 1 - \chi_{\{h>0\}} = \chi_{\{h\le 1\}}, \quad \text{a.e. in } \mathbb{R}^n \times (0,T).$$

We need the following lemma, which follows from $u \in L^\infty((0,T-\delta) : L^1_{\text{loc}}(\mathbb{R}^n))$, $\forall\, \delta > 0$.

3.2. Lemma. *Let $0 \le u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$. Then u is a distributional solution to the Cauchy problem for* (1.1) *such that*

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^n} u(x,t)\ \phi(x)\ dx = \int_{\mathbb{R}^n} \phi(x)\ d\mu(x), \quad \forall \phi \in C_0^\infty(\mathbb{R}^n)$$

if and and only if u is a solution to (1.1) *in the sense of conservation laws, i.e.,*

$$\int_0^T \int_{\mathbb{R}^n} [u\phi_t + (u-1)_+ \Delta\phi]\ dx\ dt + \int_{\mathbb{R}^n} \phi(x,0)\ d\mu(x) = 0, \tag{3.1}$$

$\forall \phi \in C_0^\infty(\mathbb{R}^n \times (-\infty, T))$.

To see that $u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t)$ satisfies (3.1), take $\phi \in C_0^\infty(\mathbb{R}^n \times (0,T))$ and let $R > 0$ such that supp $\phi(.,t) \subset B_R$, $\forall t > 0$. From the uniform integrability in t of $\int_{B_r} w_\epsilon\ dx$ we get

$$\lim_{\delta\to 0} \int_0^\delta \int_{\mathbb{R}^n} u_\epsilon |\phi_t| dx\, dt = 0$$

and

$$\lim_{\delta\to 0} \int_0^\delta \int_{\mathbb{R}^n} (u_\epsilon - 1)_+ |\Delta\phi| dx\, dt = 0$$

uniformly in ϵ. From the discussion above also for any $\delta > 0$,

$$\int_\delta^T \int_{\mathbb{R}^n} u_\epsilon \phi_t dx\, dt \to \int_\delta^T u\phi_t dx\, dt,$$

and

$$\int_\delta^T \int_{\mathbb{R}^n} (u_\epsilon - 1)_+ \Delta\phi dx\, dt \to \int_\delta^T (u-1)_+ \Delta\phi dx\, dt, \qquad \epsilon \to 0.$$

This shows that u solves (3.1). Then by the uniqueness result in [K] this limit must be $u_{f+\mu}$. Now $(u_{f+\mu_\epsilon} - 1)_+ \le w_{(f+\mu_\epsilon - 1)_+} \le w_{(f-1)_+ + \mu_\epsilon}$, letting $\epsilon \to 0$, $(u_f - 1)_+ \le (u_{f+\mu} - 1)_+ \le w_{(f-1)_+ + \mu}$ a.e. in $\mathbb{R}^n \times (0,T)$, where w_g is the solution to the heat equation with initial datum g. But

$$\lim_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} w_{(f-1)_+ + \mu}(x,t) = (f(x_0) - 1)_+$$

for a.e. $x_0 \in \mathbb{R}^n$, whence the result follows.

4. Counterexample

The following example shows that $\lim_{(x,t)\in\Gamma_\alpha^h(x_0),(x,t)\to x_0} u(x,t)$ need not exist:

4.1. Example. To construct a function $u_I : \mathbb{R} \to \{0,1\}$, supp $u_I \subset [0,1]$, such that u_I is discontinuous at every point of a set $D \subset [0,1]$, $|D| > 0$, with osc $u_I(x) = 1, \forall x \in D$, let $\mathcal{F} = \{I_\lambda, \lambda \in \mathbb{N}\}$ be the family of all subintervals of $[0,1]$ having rational endpoints. Let $0 < \epsilon < 1$. From each I_λ subtract a subinterval $\tilde{I}_\lambda$ with length $< \epsilon 2^{-\lambda}$. Let $E = \bigcup_\lambda \tilde{I}_\lambda$. If the $\tilde{I}_\lambda$ are open so is E, and $|E| \le \sum_{\lambda\in\mathbb{N}} \epsilon 2^{-\lambda} = \epsilon$, $|[0,1] \setminus E| \ge 1 - \epsilon$. Define the function $u_I(x) = \chi_{([0,1]\setminus E)}$, u_I is well defined, and osc $u_I = 1$. For $x_0 \in [0,1] \setminus E$, $u_I(x_0) = 1$, and since E is dense in $[0,1]$, there is a sequence $x_n \to x_0$ such that $u_I(x_n) = 0$. Of course, $u(x,t) \equiv u_I(x)$ is a distributional solution to (1.1). Then

$$\lim_{(x,t)\in\Gamma^h_\alpha(x_0),(x,t)\to x_0} u(x,t) = \lim_{x\to x_0} u_I(x_0),$$

which does not exist.

4.2. *Remark.* However it can be shown that if $\mu = f + \mu_s$ where $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous, then

$$\lim_{(x,t)\in\Gamma^k_\beta(x_0),\ (x,t)\to x_0} u(x,t) = f(x_0).$$

References

[AK] D. Andreucci, M.K. Korten, *Initial traces of solutions to a one-phase Stefan problem in an infinite strip*, Rev. Mat. Iberoamericana, Vol. 9, No. **2** (1993), 315-332. MR **94m:**35319

[B] J. E. Bouillet, *Signed solutions to diffusion-heat conduction equations*, Free Boundary Problems: Theory and Applications, Proc. Int. Colloq. Irsee/Ger. 1987, Vol. II, Pitman Res. Notes Math. Ser. 186 (1990), 480–485.

[BKM] J. E. Bouillet, M. K. Korten and V. Márquez, *Singular limits and the "Mesa" problem*, Rev. Union Mat. Argentina, Vol. 41 (1998), no. 1, 27–40.

[C] A. P. Calderón, *On the behaviour of harmonic functions at the boundary,* Trans. Amer. Math. Soc. 68 (1950), 47-54. MR **11:**357e

[DFK] B. E. J. Dahlberg, E. Fabes and C. E. Kenig, *A Fatou theorem for solutions of the porous medium equation*, Proc. Amer. Math. Soc. 91 (1984), 205-212. MR **85e:**35064

[DB] E. DiBenedetto, *Continuity of weak solutions to certain singular parabolic equations,* Ann. Mat. Pura Appl. (4), CXXX (1982), 131-176. MR **83k:**35045

[H] K. M. Hui, *Fatou theorem for the solutions of some nonlinear equations*, Jl. Math. Anal. Applic. 183 (1994), 37-52. MR **95c:**35125

[K] M. K. Korten, *Non-negative solutions of* $u_t = \Delta(u-1)_+$*: Regularity and uniqueness for the Cauchy problem*, Nonl. Anal., Th., Meth. and Appl, Vol. 27, No. 5 (1996), 589-603. MR **97h:**35089

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