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THE FIRST OCCURRENCE FOR THE IRREDUCIBLE MODULES OF GENERAL LINEAR GROUPS IN THE POLYNOMIAL ALGEBRA

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ABSTRACT. Let p be a prime number and let GL_n be the group of all invertible matrices over the prime field $\mathbb{F}_p$. It is known that every irreducible GL_n-module can occur as a submodule of P, the polynomial algebra with n variables over $\mathbb{F}_p$. Given an irreducible GL_n-module ρ, the purpose of this paper is to find out the first value of the degree d of which ρ occurs as a submodule of P_d, the subset of P consisting of homogeneous polynomials of degree d. This generalizes Schwartz-Tri's result to the case of any prime p.

1. Introduction

Let p be a prime number and let $GL_n = GL(n, \mathbb{F}_p)$ be the group of all invertible $(n \times n)$-matrices over the prime field $\mathbb{F}_p$. Denote by $P = \mathbb{F}_p[x_1, \ldots, x_n]$ the polynomial algebra with variables $x_1, \ldots, x_n$ over $\mathbb{F}_p$. There is then an action of GL_n on P given by

$$^\sigma x_i = \sum_{j=1}^{n} a_{ji} x_j, \quad 1 \le i \le n,$$
$$^\sigma f(x_1, \ldots, x_n) = f(^\sigma x_1, \ldots, ^\sigma x_n)$$

with $\sigma = (a_{ij})_{1 \le i,j \le n} \in GL_n, f(x_1, \ldots, x_n) \in P$.

For an irreducible GL_n-module ρ, it is known (see *e.g.* [1]) that ρ is isomorphic to a submodule of P. One may ask about the first value of the degree d for which ρ occurs as a submodule of P_d, the subset of P consisting of homogeneous polynomials of degree d. This first occurrence problem also arises from topology, as it is related to the description of the cohomology of certain spaces as modules over the Steenrod algebra (see *e.g.* [7]). For the prime 2, the problem has been solved independently by Schwartz [4] and Tri [6]. The purpose of this paper is to solve this problem by generalizing the argument used in [6] to the case of any prime p.

Let us recall (see *e.g.* [2, Chapter 8]) that the irreducible GL_n-modules are indexed, up to isomorphism, by column regular partitions, *i.e.*, by sequences of non-negative integers

$$\alpha_1, \ldots, \alpha_n$$

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satisfying $0 \leq \alpha_i - \alpha_{i+1} \leq p-1$ for $1 \leq i \leq n-1$ and $0 \leq \alpha_n \leq p-1$. Namely, the irreducible GL_n-modules are indexed by column regular partitions $\alpha_1, \dots, \alpha_n$ satisfying $\alpha_n \leq p-2$. Recently, in [5], they were also obtained by means of modular invariants, as follows. For $1 \leq i \leq n$, let

$$V_i = \prod_{\lambda_j \in \mathbb{F}_p} (\lambda_1 x_1 + \cdots + \lambda_{i-1} x_{i-1} + x_i)$$

be the Mui invariant and let

$$L_i = V_1 \cdot \cdots \cdot V_i$$

be the Dickson invariant. It is clear that

$$L_n = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ x_1^p & x_2^p & \dots & x_n^p \\ . & . & \dots & . \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \dots & x_n^{p^{n-1}} \end{vmatrix}$$

and ${}^{\sigma}L_n = \det \sigma \cdot L_n$ for $\sigma \in GL_n$. Let $\beta = (\beta_1, \dots, \beta_n)$ with $0 \leq \beta_i \leq p-1$ and set $L^\beta = \prod_{i=1}^n L_i^{\beta_i} \in P$. Denote by $H_\beta = H_\beta(GL_n)$ the GL_n-module generated by L^β (so H_β is nothing but the $\mathbb{F}_p$-vector space generated by the set $\{{}^{\sigma}L^\beta | \sigma \in GL_n\}$). We have

Theorem A ([5, Corollary 1.2])**.** *$\{H_\beta | \beta = (\beta_1, \dots, \beta_n), 0 \leq \beta_i \leq p-1, \beta_n \neq 0\}$ is a complete set of $(p-1)p^{n-1}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_n]$.*

By noting that $H_{(\beta_1, \dots, \beta_{n-1}, p-1)} \cong H_{(\beta_1, \dots, \beta_{n-1}, 0)}$, we can restate the theorem as follows.

Theorem B (compare [1, Proposition 1.3])**.** *$\{H_\beta | \beta = (\beta_1, \dots, \beta_n), 0 \leq \beta_i \leq p-1, \beta_n \neq p-1\}$ is a complete set of $(p-1)p^{n-1}$ distinct irreducible modules for the algebra $\mathbb{F}_p[GL_n]$.*

We should note that, in terms of column regular partitions, for $0 \leq \beta_i \leq p-1, \beta_n \neq p-1$, H_β corresponds to the sequence $\lambda_1, \dots, \lambda_n$ with $\lambda_i = \beta_i + \cdots + \beta_n, 1 \leq i \leq n$.

Generalizing Schwartz-Tri's result ([4], [6]) to the case of any prime, we prove

Theorem C. *With β given in Theorem B, the first occurrence of H_β as a submodule of P_d is for $d = \deg L^\beta$.*

2. Preliminaries

Let T_n be the group consisting of upper triangular matrices with 1 on the diagonal. It follows from [3] that

$$P^{T_n} = \mathbb{F}_p[V_1, \dots, V_n].$$

We shall use the following notation. Given $g = g(x_1, \dots, x_n) \in P, \lambda \in \mathbb{F}_p$ and $1 \leq i, k \leq n$, then ${}^{\sigma_{i,k,\lambda}}g$ (resp. ${}^{\eta_{i,\lambda}}g$, ${}^{\tau_{i,k}}g$) denotes the polynomial obtained from g by replacing x_i by $x_i + \lambda x_k$ (resp. by replacing x_i by λx_i, by interchanging x_i and x_k). Consider a homogeneous element $f = f(x_1, \dots, x_n)$ of P^{T_n}. It is clear that V_i is a factor of f if and only if x_i is also. We have

Lemma 1. *If ${}^{\eta_{i,\lambda}}f = \lambda^\ell f$ with $1 \leq \ell < p-1$ and $\lambda \neq 0, 1$, then f contains x_i as a factor.*

Proof. Write $f = \sum_{r=0}^{m} x_i^r f_r(x_1, \ldots, x_n)$ with f_r free of x_i. It follows that $\sum_{r=0}^{m} x_i^r(\lambda^\ell f_r - \lambda^r f_r) = 0$. So $\lambda^\ell f_0 - f_0 = 0$, which implies that $f_0 = 0$. Therefore f contains x_i as a factor. The lemma follows. □

The case $p = 2$ in the following lemma was treated in [6]. However, our proof given here, for any prime p, is much simpler.

Lemma 2. *If $i < k$, then f contains x_i as a factor provided that one of the following conditions is satisfied:*

(i) $\sum_{r=0}^{p-1} {}^{\sigma_{i,k,r}}f = -\ {}^{\tau_{i,k}}f$ *and* ${}^{\eta_{i,\mu}}f = f$ *for every* $\mu \neq 0$*;*
(ii) f *contains* x_k *as a factor and* $\sum_{r=0}^{p-1} {}^{\sigma_{i,k,r}}f = \alpha \cdot {}^{\tau_{i,k}}f$ *with* $\alpha \in \{0, 1, -1\}$*.*

Proof. For $1 \leq r \leq p-1$, since f is T_n-invariant, we have

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, rx_k, x_{i+1}, \ldots, x_k, \ldots, x_n) \\ &\quad = f(x_1, \ldots, x_{i-1}, rx_k, x_{i+1}, \ldots, x_k - x_k, \ldots, x_n) \\ &\quad = f(x_1, \ldots, x_{i-1}, rx_k, x_{i+1}, \ldots, 0, \ldots, x_n) \\ &\quad = \begin{cases} f(x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, 0, \ldots, x_n), & \text{if } {}^{\eta_{i,r}}f = f\ , \\ 0, & \text{if } f \text{ contains } x_k \text{ as a factor.} \end{cases} \end{aligned}$$

Consider $\sum_{r=0}^{p-1} {}^{\sigma_{i,k,r}}f$ and set $x_i = 0$. (i) and the above equality imply that

$$\begin{aligned} \textstyle\sum_{r=0}^{p-1} {}^{\sigma_{i,k,r}}f &= f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k, \ldots, x_n) \\ &\qquad - f(x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, 0, \ldots, x_n) \\ &= -f(x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, 0, \ldots, x_n), \end{aligned}$$

so $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k, \ldots, x_n) = 0$; on the other hand, (ii) implies that

$$\begin{aligned} \textstyle\sum_{r=0}^{p-1} {}^{\sigma_{i,k,r}}f &= f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k, \ldots, x_n) \\ &= \alpha f(x_1, \ldots, x_{i-1}, x_k, x_{i+1}, \ldots, 0, \ldots, x_n) \\ &= 0, \qquad \text{since } f \text{ contains } x_k \text{ as a factor.} \end{aligned}$$

In any case, f contains x_i as a factor. The lemma follows. □

3. Proof of Theorem C

We proceed by induction on n. The proof is trivial for $n = 1$. Assume that the theorem holds for $n - 1$.

Let W be an irreducible GL_n-module of homogeneous polynomials. There exist then $\beta = (\beta_1, \ldots, \beta_n)$ with $0 \leq \beta_i \leq p-1, \beta_n \neq p-1$ and a GL_n-isomorphism $\zeta : W \to H_\beta(GL_n)$. Set $f = f(x_1, \ldots, x_n) = \zeta^{-1}(L^\beta)$. We need to prove that $\deg f \geq \deg L^\beta$. It is clear that f is invariant under the action of T_n and W is the GL_n-module generated by f. For convenience, write $W = W_f(GL_n)$.

Denote by $(\beta_{n_1}, \ldots, \beta_{n_k})$ the subsequence of β consisting of all non-zero elements, so $L^\beta = L_{n_1}^{\beta_1} \cdot \cdots \cdot L_{n_k}^{\beta_{n_k}}$. First, we prove that f contains L_{n_k} as a factor (note that $\beta_{n_k} < p-1$ if $n_k = n$). Let m be an integer satisfying $n_{k-1} < m \leq n_k$. For every $\mu \neq 0$, as ${}^{\eta_{m,\mu}}L^\beta = \mu^{\beta_{n_k}} L^\beta$, it follows that ${}^{\eta_{m,\mu}}f = \mu^{\beta_{n_k}} f$. If $\beta_{n_k} < p-1$, Lemma 1 implies that f contains x_m, therefore V_m, as a factor. If $\beta_{n_k} = p-1$ (which implies $n_k < n$), then ${}^{\eta_{m,\mu}}f = f$ and

$$\begin{aligned} {}^{\sigma_{m,n,r}}\ L_{n_k} &= L_{n_k}(x_1,\dots,x_{m-1},x_m + rx_n, x_{m+1},\dots,x_{n_k}) \\ &= L_{n_k}(x_1,\dots,x_{m-1},x_m,x_{m+1},\dots,x_{n_k}) \\ &\qquad + rL_{n_k}(x_1,\dots,x_{m-1},x_n,x_{m+1},\dots,x_{n_k}) \end{aligned}$$

for $r \in \mathbb{F}_p$. Since

$$\begin{aligned} &\sum_{r=0}^{p-1}(L_{n_k}(x_1,\dots,x_{m-1},x_m,x_{m+1},\dots,x_{n_k}) \\ &\qquad + rL_{n_k}(x_1,\dots,x_{m-1},x_n,x_{m+1},\dots,x_{n_k}))^{\lambda} \end{aligned}$$

$$(*) \qquad = \begin{cases} 0, & \text{if } \lambda < p-1, \\ -L_{n_k}^{p-1}(x_1,\dots,x_{m-1},x_n,x_{m+1},\dots,x_{n_k}), & \text{if } \lambda = p-1, \end{cases}$$

we have $\sum_{r=0}^{p-1}\ {}^{\sigma_{m,n,r}}L_{n_k}^{p-1} = -L_{n_k}^{p-1}(x_1,\dots,x_{m-1},x_n,x_{m+1},\dots,x_{n_k})$ which implies that $\sum_{r=0}^{p-1}\ {}^{\sigma_{m,n,r}}L^{\beta} = -{}^{\tau_{m,n}}L^{\beta}$. Hence $\sum_{r=0}^{p-1}\ {}^{\sigma_{m,n,r}}f = -{}^{\tau_{m,n}}f$. It follows from Lemma 2 that f contains x_m, hence V_m, as a factor.

Set $n_0 = 0$. Suppose that f contains $x_{t+1},\dots,x_{n_k}$ as factors with $n_{\ell-1} < t \leq n_\ell < n_k$. It follows from $(*)$ that

$$\textstyle\sum_{r=0}^{p-1}\ {}^{\sigma_{t,n_\ell+1,r}}L_{n_\ell}^{\beta_{n_\ell}} = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ -{}^{\tau_{t,n_\ell+1}}L_{n_\ell}^{p-1}, & \text{if } \beta_{n_\ell} = p-1. \end{cases}$$

Therefore

$$\textstyle\sum_{r=0}^{p-1}\ {}^{\sigma_{t,n_\ell+1,r}}L^{\beta} = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ \alpha^{\tau_{t,n_\ell+1}}L^{\beta}, & \text{if } \beta_{n_\ell} = p-1, \end{cases}$$

with $\alpha = \pm 1$. So

$$\textstyle\sum_{r=0}^{p-1}\ {}^{\sigma_{t,n_\ell+1,r}}f = \begin{cases} 0, & \text{if } \beta_{n_\ell} < p-1, \\ \alpha^{\tau_{t,n_\ell+1}}f, & \text{if } \beta_{n_\ell} = p-1. \end{cases}$$

By Lemma 2, f contains x_t, hence V_t, as a factor. Therefore f contains L_{n_k} as a factor.

Write $f = L_{n_k}g$ with $g = g(x_1,\dots,x_n) \in P$. So $g \in P^{T_n}$. Consider GL_{n_k} as a subgroup of GL_n in the usual way. $H_\beta(GL_{n_k})$ and $W = W_f(GL_{n_k})$ are then GL_{n_k}-modules and $H_\beta(GL_{n_k}) \overset{\zeta}{\cong} W_f(GL_{n_k}) \cong \det \otimes W_g(GL_{n_k})$. Set $\beta' = (\beta_1,\dots,\beta_{n_k}-1)$ and $\beta'' = (\beta_1,\dots,\beta_{n_{k-1}})$. As GL_{n_k}-modules, $H_\beta(GL_{n_k}) \cong \det \otimes H_{\beta'}(GL_{n_k})$, so $W_g(GL_{n_k}) \cong H_{\beta'}(GL_{n_k})$. If $n_k < n$, it follows from the inductive hypothesis that $\deg g \geq \deg L^{\beta'}$, so $\deg f \geq \deg L^{\beta}$. Suppose that $n_k = n$. The argument used above shows that g contains L_n as a factor; hence f contains $L_n^{\beta_n}$ as a factor.

By writing $f = L_n^{\beta_n}h$, we have $W_h(GL_n) \cong H_{\beta''}(GL_n)$. By replacing f by h, arguing as in the case $n_k < n$ yields $\deg h \geq \deg L^{\beta''}$; hence $\deg f \geq \deg L^{\beta}$. The theorem is proved.

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