

## THE DYNKIN SYSTEM GENERATED BY BALLS IN $\mathbb{R}^d$ CONTAINS ALL BOREL SETS

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ABSTRACT. We show that for every  $d \in \mathbb{N}$  each Borel subset of the space  $\mathbb{R}^d$  with the Euclidean metric can be generated from closed balls by complements and countable disjoint unions.

Let  $X$  be a nonempty set and  $\mathcal{S} \subset 2^X$ . Following [B, p. 8] we say that  $\mathcal{S}$  is a *Dynkin system* if

- (D1)  $X \in \mathcal{S}$ ,
- (D2)  $A \in \mathcal{S} \Rightarrow X \setminus A \in \mathcal{S}$ ,
- (D3) if  $A_n \in \mathcal{S}$  are pairwise disjoint, then  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$ .

Some authors use the name  $\sigma$ -class instead of Dynkin system. The smallest Dynkin system containing a system  $\mathcal{T} \subset 2^X$  is denoted by  $\mathcal{D}(\mathcal{T})$ . Let  $P$  be a metric space. The system of all closed balls in  $P$  (of all Borel subsets of  $P$ , respectively) will be denoted by  $\text{Balls}(P)$  ( $\text{Borel}(P)$ , respectively).

We will deal with the problem of whether

$$(\star) \quad \mathcal{D}(\text{Balls}(P)) = \text{Borel}(P).$$

One motivation for such a problem comes from measure theory. Let  $\mu$  and  $\nu$  be finite Radon measures on a metric space  $P$  having the same values on each ball. Is it true that  $\mu = \nu$ ? If  $\mathcal{D}(\text{Balls}(P)) = \text{Borel}(P)$ , then obviously  $\mu = \nu$ . If  $P$  is a Banach space, then  $\mu = \nu$  again (Preiss, Tišer [PT]). But Preiss and Keleti ([PK]) showed recently that  $(\star)$  is false in infinite-dimensional Hilbert spaces.

We prove the following result.

**Theorem 1.** *Let  $d \in \mathbb{N}$ , and let  $\mathbb{R}^d$  be equipped with the Euclidean metric. Then  $\mathcal{D}(\text{Balls}(\mathbb{R}^d)) = \text{Borel}(\mathbb{R}^d)$ .*

This theorem was partially proved by Olejček ([O]), who proved it for  $d = 2, 3$  (the case  $d = 1$  is easy and well-known). Several of Olejček's ideas will also be used in our proof of the general statement. In fact, we prove a slightly more general result:

*Each Borel subset of the space  $\mathbb{R}^d$  (with the Euclidean metric) can be generated from closed balls by countable monotone unions, countable monotone intersections and countable disjoint unions.*

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Each open ball is a countable increasing union of closed balls; we could use the open balls as well.

The paper [PK] contains a remark on the result, proved independently by Jackson and Mauldin, that  $(\star)$  holds even for every Banach space of finite dimension.

We start with definitions of auxiliary notions. Let  $X$  be a nonempty set. We say that  $\mathcal{S} \subset 2^X$  is a *monotone system* if the following hold:

- (M1) if  $A_1 \subset A_2 \subset A_3 \subset \dots$ ,  $A_n \in \mathcal{S}$ , then  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$ ,  
 (M2) if  $A_1 \supset A_2 \supset A_3 \supset \dots$ ,  $A_n \in \mathcal{S}$ , then  $\bigcap_{n=1}^{\infty} A_n \in \mathcal{S}$ .

We say that  $\mathcal{S} \subset 2^X$  is a  $\mathcal{D}^*$ -system if  $\mathcal{S}$  satisfies (M1), (M2) and (D3). The smallest  $\mathcal{D}^*$ -system containing  $\mathcal{T} \subset 2^X$  is denoted by  $\mathcal{D}^*(\mathcal{T})$ . We say that  $A \subset \mathbb{R}^d$  is a *sphere of dimension  $t$*  if there exists a closed ball  $B \subset \mathbb{R}^d$  with a center  $x \in \mathbb{R}^d$  and an affine subspace  $V \subset \mathbb{R}^d$  of dimension  $t+1$  containing  $x$  such that  $A = V \cap \partial B$ . We define

$$\mathcal{S}_t = \{A \subset \mathbb{R}^d; A \text{ is a Borel set, which can be covered by countably many spheres of dimension } t\}.$$

Let  $A, B \subset \mathbb{R}^d$ . We say that  $A$  is a subset of  $B$  modulo  $\mathcal{S}_t$  (the notation  $A \subset B \bmod \mathcal{S}_t$ ) if  $A \setminus B \in \mathcal{S}_t$ . We say that  $A = B \bmod \mathcal{S}_t$  if  $A \subset B \bmod \mathcal{S}_t$  and  $B \subset A \bmod \mathcal{S}_t$ . The Euclidean open ball with center  $x \in \mathbb{R}^d$  and radius  $r > 0$  is denoted by  $B(x, r)$ .

**Lemma 2.** *Let  $X$  be a nonempty set, and let  $\mathcal{V} \subset 2^X$  be a system closed with respect to finite unions and finite intersections. Then the smallest monotone system containing  $\mathcal{V}$  is closed with respect to countable unions and countable intersections.*

*Proof.* We define

$$\begin{aligned} \mathcal{M}(\mathcal{V}) &= \text{the smallest monotone system containing } \mathcal{V}, \\ \mathcal{M}_1 &= \{A \in \mathcal{M}(\mathcal{V}); \forall B \in \mathcal{V} : A \cup B \in \mathcal{M}(\mathcal{V}) \text{ and } A \cap B \in \mathcal{M}(\mathcal{V})\} \quad \text{and} \\ \mathcal{M}_2 &= \{A \in \mathcal{M}(\mathcal{V}); \forall B \in \mathcal{M}(\mathcal{V}) : A \cup B \in \mathcal{M}(\mathcal{V}) \text{ and } A \cap B \in \mathcal{M}(\mathcal{V})\}. \end{aligned}$$

It is easy to see that  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are monotone systems. We also have  $\mathcal{V} \subset \mathcal{M}_1 \subset \mathcal{M}(\mathcal{V})$ . Hence  $\mathcal{M}_1 = \mathcal{M}(\mathcal{V})$  and  $\mathcal{V} \subset \mathcal{M}_2 \subset \mathcal{M}(\mathcal{V})$ . This gives  $\mathcal{M}(\mathcal{V}) = \mathcal{M}_2$  and therefore  $\mathcal{M}(\mathcal{V})$  is closed with respect to finite unions and finite intersections. Thus  $\mathcal{M}(\mathcal{V})$  is closed with respect to countable unions and countable intersections.  $\square$

**Lemma 3.** *Each Dynkin system is a  $\mathcal{D}^*$ -system.*

**Lemma 4.** *Each Borel subset of a metric space  $P$  can be generated from open subsets of  $P$  by countable monotone unions and countable monotone intersections.*

The proof of Lemma 3 is obvious and will be omitted. Lemma 4 is well-known (see [K, p. 344] and Lemma 2).

**Lemma 5.** *Let  $m \in \mathbb{N}$ , let  $L \subset \mathbb{R}^d$  be a closed set, let  $t \in \mathbb{N}$ ,  $t \leq d$ , and let  $G$  be a subset of  $L$ . Suppose that there exists a sequence of systems of closed balls  $\{\mathcal{B}_n\}_{n=1}^{\infty}$  such that*

- (i)  $\mathcal{B}_n$  is a union of  $m$  systems such that each of them is disjoint,
- (ii) the center of each ball from  $\bigcup_{n=1}^{\infty} \mathcal{B}_n$  is in  $L$ ,
- (iii)  $L \cap \bigcup \mathcal{B}_n \subset G$  and  $G = L \cap \bigcup \mathcal{B}_n \bmod \mathcal{S}_{t-1}$  for every  $n \in \mathbb{N}$ ,

- (iv)  $\forall n, n' \in \mathbb{N}, n' > n \forall C \in \mathcal{B}_n \forall C' \in \mathcal{B}_{n'} : C' \subset C \text{ or } C' \cap C = \emptyset,$
- (v)  $\lim_{n \rightarrow +\infty} \sup\{\text{diam } C; C \in \mathcal{B}_n\} = 0.$

Then there exists  $\tilde{G} \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$  with  $\tilde{G} \subset G$  and  $\tilde{G} = G \text{ mod } \mathcal{S}_{t-1}.$

*Proof.* Let  $\mathcal{B}_n = \bigcup_{k=1}^m \mathcal{B}_n^k,$  where each  $\mathcal{B}_n^k$  is a disjoint system of closed balls. Put

$$D_{n,j}^k = \{C \in \mathcal{B}_n^k; C \not\subset \bigcup_{i=1}^{k-1} \bigcup_{s=j}^{\infty} \mathcal{B}_s^i\}, \quad D_{n,j}^k = \bigcup \mathcal{D}_{n,j}^k,$$

for every  $k \in \{1, \dots, m\}, j, n \in \mathbb{N},$  and

$$\tilde{G} = \bigcup_{k=1}^m \bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k, \quad Z = G \setminus \bigcap_{n=1}^{\infty} \bigcup \mathcal{B}_n.$$

Using (iii) we obtain  $Z \in \mathcal{S}_{t-1}.$  Let  $x \in G \setminus Z.$  Then there exist a sequence  $(C_n)_{n=1}^{\infty}$  of closed balls, an increasing sequence  $(p_n)_{n=1}^{\infty}$  of natural numbers,  $k \in \mathbb{N}$  and  $j \in \mathbb{N}$  such that

- $x \in C_n \in \mathcal{B}_{p_n}^k$  for every  $n \in \mathbb{N},$
- $x \notin \bigcup_{i=1}^{k-1} \bigcup_{s=j}^{\infty} \mathcal{B}_s^i.$

It is easy to verify that  $x \in \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k.$  Thus we have  $G \subset \tilde{G} \text{ mod } \mathcal{S}_{t-1}.$  On the other hand the conditions (ii) and (v) imply that  $\tilde{G} \subset L,$  since  $L$  is closed. Using this and the condition (iii) we obtain

$$\tilde{G} = L \cap \tilde{G} \subset L \cap \left( \bigcup_{n=1}^{\infty} \bigcup \mathcal{B}_n \right) = \bigcup_{n=1}^{\infty} \left( L \cap \bigcup \mathcal{B}_n \right) \subset G.$$

Now we want to prove that  $\tilde{G} \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d)).$  At first we prove that

$$\left( \bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^u \right) \cap \left( \bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^v \right) = \emptyset \text{ whenever } u \neq v.$$

We may assume that  $u < v.$  Suppose on the contrary that

$$x \in \left( \bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^u \right) \cap \left( \bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^v \right).$$

This means that there exist  $j \in \mathbb{N}$  and  $j' \in \mathbb{N}$  such that  $x \in \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^u$  and  $x \in \bigcap_{l'=1}^{\infty} \bigcup_{p'=l'}^{\infty} D_{p',j'}^v.$  There exist  $p > j'$  and a ball  $C \in \mathcal{D}_{p,j}^u$  with  $x \in C.$  There exist  $p' > p$  and a ball  $C' \in \mathcal{D}_{p',j'}^v$  with  $x \in C'.$  But this and (iv) imply that  $C' \subset C \subset \bigcup \mathcal{B}_p^u.$  This is a contradiction with  $C' \in \mathcal{D}_{p',j'}^v.$

Now it is sufficient to show that

$$\bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d)) \text{ for every } k = 1, \dots, m.$$

We fix  $k \in \{1, \dots, m\}$  and observe that the system  $\mathcal{V} = \{\bigcup \mathcal{X}; \mathcal{X} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n^k\}$  is contained in the system of all disjoint unions of closed balls in  $\mathbb{R}^d$  and is closed with respect to finite intersections and finite unions. Moreover,  $D_{p,j}^k \in \mathcal{V}$  for every

$p, j \in \mathbb{N}$ . Using Lemma 2 we obtain that the set  $\bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k$  is contained in the smallest monotone system containing  $\mathcal{V}$  and hence also

$$\bigcup_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcup_{p=l}^{\infty} D_{p,j}^k \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d)).$$

□

**Lemma 6.** *Let  $L \subset \mathbb{R}^d$  be an affine subspace of dimension  $t$  or a sphere of dimension  $t$ . Let  $G \subset L$  be relatively open in  $L$ . Then there exist  $m \in \mathbb{N}$  and a sequence of systems of closed balls  $\{\mathcal{B}_n\}_{n=1}^{\infty}$  satisfying the conditions (i) – (v) in Lemma 5.*

*Proof.* At first we prove the following Claim.

*Claim.* Let  $L \subset \mathbb{R}^d$  be as in Lemma 6,  $H \subset L$  be relatively open in  $L$  and  $\varepsilon > 0$ . Then there exist  $m \in \mathbb{N}$  (depending only on the dimension of  $\mathbb{R}^d$ ) and a system of closed balls  $\mathcal{A}$  such that

- (a)  $\mathcal{A}$  is a union of  $m$  systems such that each of them is disjoint,
- (b) each ball from  $\mathcal{A}$  has its center in  $L$ ,
- (c)  $L \cap \bigcup \mathcal{A} = H$ ,
- (d) each ball from  $\mathcal{A}$  has its diameter less than  $\varepsilon$ ,
- (e) for every  $x \in H$  there exists  $r > 0$  such that  $B(x, r)$  intersects only finitely many elements from  $\mathcal{A}$ .

*Proof of the Claim.* Let  $\mathcal{U}$  be a system of all closed balls  $B \subset \mathbb{R}^d$  such that the center of  $B$  is in  $H$ ,  $B \cap L \subset H$  and  $\min\{\frac{1}{2} \text{dist}(\text{center}(B), L \setminus H), \frac{1}{4}\varepsilon\} \leq \text{diam } B < \varepsilon$ . Since  $H$  is relatively open in  $L$ , we have  $L \cap \bigcup \mathcal{U} = H$ .

We use Besicovitch's theorem ([Z, p. 9]) to obtain  $m \in \mathbb{N}$  (depending only on the dimension of  $\mathbb{R}^d$ ) and disjoint systems  $\mathcal{A}_1, \dots, \mathcal{A}_m$  such that  $\bigcup_{k=1}^m \mathcal{A}_k \subset \mathcal{U}$  and  $L \cap (\bigcup_{k=1}^m \mathcal{A}_k) = H$ . It is easy to check that  $\mathcal{A} = \bigcup_{k=1}^m \mathcal{A}_k$  satisfies (a) – (e) and the proof of the Claim is over. □

We will construct the desired systems  $\mathcal{B}_n$ ,  $n \in \mathbb{N}$ . Put  $\varepsilon = 1$  and  $H = G$ . Using the Claim we obtain a system of closed balls  $\mathcal{B}_1$ , which satisfies (a) – (e) and hence also the conditions (i') – (vi') for  $l = 1$  below.

- (i') For every  $n = 1, \dots, l$ ,  $\mathcal{B}_n$  is a union of  $m$  systems such that each of them is disjoint,
- (ii') the center of each ball from  $\bigcup_{n=1}^l \mathcal{B}_n$  is in  $L$ ,
- (iii')  $L \cap \bigcup \mathcal{B}_n \subset G$  and  $G = L \cap \bigcup \mathcal{B}_n \text{ mod } \mathcal{S}_{t-1}$  for every  $n = 1, \dots, l$ ,
- (iv')  $\forall n, n' \in \{1, \dots, l\}$ ,  $n' > n \forall C \in \mathcal{B}_n \forall C' \in \mathcal{B}_{n'} : C' \subset C$  or  $C' \cap C = \emptyset$ ,
- (v')  $\sup\{\text{diam } C; C \in \mathcal{B}_n\} \leq 1/n$  for every  $n = 1, \dots, l$ ,
- (vi') for every  $n = 1, \dots, l$  and for every  $x \in \bigcup \mathcal{B}_n$  there exists  $r > 0$  such that  $B(x, r)$  intersects only finitely many elements from  $\mathcal{B}_n$ .

Suppose that we have defined systems  $\mathcal{B}_n$ ,  $n = 1, \dots, l$ , satisfying (i') – (vi'). Put

$$H = \left( \bigcup \mathcal{B}_l \setminus \bigcup \{\partial B; B \in \mathcal{B}_l\} \right) \cap L \quad \text{and} \quad \varepsilon = \frac{1}{l+1}.$$

The condition (vi') gives that  $H$  is a relatively open subset of  $L$ . Applying the Claim to  $H$  and  $\varepsilon$  we obtain a system of closed balls  $\mathcal{B}_{l+1}$ .

Let  $B$  be a closed ball with the center in  $L$ . Then  $\partial B \cap L$  is a sphere of dimension  $t - 1$  or a singleton or an empty set since we use Euclidean balls. This implies that  $\bigcup\{\partial B; B \in \mathcal{B}_l\} \cap L \in \mathcal{S}_{t-1}$ . Thus we have

$$L \cap \bigcup \mathcal{B}_l = L \cap \bigcup \mathcal{B}_{l+1} \text{ mod } \mathcal{S}_{t-1}.$$

Since  $G = L \cap \bigcup \mathcal{B}_l \text{ mod } \mathcal{S}_{t-1}$  we conclude  $G = L \cap \bigcup \mathcal{B}_{l+1} \text{ mod } \mathcal{S}_{t-1}$ . The remaining part of condition (iii'\_{l+1}) and conditions (i'\_{l+1}) - (ii'\_{l+1}), (iv'\_{l+1}) - (vi'\_{l+1}) are easy to check.  $\square$

**Proposition 7.** *Let  $d \in \mathbb{N}$ , and let  $\mathbb{R}^d$  be equipped with the Euclidean metric. Then  $\mathcal{D}^*(\text{Balls}(\mathbb{R}^d)) = \text{Borel}(\mathbb{R}^d)$ .*

*Proof.* We will prove by induction over  $t$  the following Claim, which implies Proposition 7.

*Claim.* Let  $L \subset \mathbb{R}^d$  be an affine subspace of  $\mathbb{R}^d$  of dimension  $t$  or a sphere of dimension  $t$  and let  $B \subset L$  be a Borel set. Then  $B \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$ .

Let  $t = 0$ . Then the Claim obviously holds since  $B$  is empty or a singleton or a two point set. Suppose that we have proved the statement for  $t = p - 1$ . We will deal with the case  $t = p$ . If  $G \subset L$  is a relatively open set in  $L$ , then Lemmas 5 and 6 show that  $G = G_1 \cup G_2$ , where  $G_1 \cap G_2 = \emptyset$ ,  $G_1 \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$  and  $G_2 \in \mathcal{S}_{p-1}$ . The induction hypothesis easily implies that  $G_2 \in \mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$ . Thus each relatively open subset of  $L$  is contained in  $\mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$ . This fact and Lemma 4 show that each Borel subset of  $L$  is contained in  $\mathcal{D}^*(\text{Balls}(\mathbb{R}^d))$ . Thus the Claim is proved.  $\square$

*Proof of Theorem 1.* Theorem 1 follows immediately from Proposition 7 and Lemma 3.  $\square$

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