PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 893–902
S 0002-9939(99)05023-6
Article electronically published on September 27, 1999

AN OBSTRUCTION TO 3-DIMENSIONAL THICKENINGS

FRANCISCO F. LASHERAS

(Communicated by Ronald A. Fintushel)

ABSTRACT. In this paper we give a characterization of those locally finite 2-dimensional simplicial complexes that have an orientable 3-manifold thickening. This leads to an obstruction for a fake surface X to admit such a thickening. The obstruction is defined in $H^1(\Gamma; \mathbf{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order three.

1. INTRODUCTION

A simplicial complex X "thickens" to a CW-complex Y if Y admits a CW-structure containing, as a subcomplex, a copy of a subdivision of X onto which Y collapses. For a *standard* 2-complex (a finite 2-complex with a single vertex) L. Neuwirth [6] exhibited an algorithm to decide whether the given 2-complex expands to an orientable 3-manifold. Later, H. Ikeda [4] introduced the concept of a *fake surface* (see §4). It is known that for this class of 2-complexes we always have a thickening to a *singular* 3-*manifold* [3, 7], i.e., a polyhedron in which the link of each point is a disk D^2 (boundary point), a sphere S^2 (inner point), or a projective plane P^2 (singular point). P. Wright [10] showed a sufficient condition for a (compact) fake surface X to embed into a 3-manifold; namely, if every simple closed curve C in the subgraph of all triple edges is *untwisted* (i.e., a regular neighborhood of C in X contains a T-bundle over C which embeds in $\mathbf{R}^3$), then X embeds into a 3-manifold.

In order to detect some kind of "twists" in a 2-dimensional locally finite simplicial complex X (with planar links), we work with a family of embeddings $\Phi = \{\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ and we consider certain cyclic orderings around each vertex, determined by the family Φ. This way we associate to each family of embeddings Φ a cochain (cocycle) $\omega_\Phi \in C^1(\Gamma; \mathbf{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order ≥ 3. From the study of these cochains we obtain the main results of this paper. More precisely

Theorem 1.1. *Let X be a 2-dimensional connected locally finite simplicial complex. Then, X thickens to an orientable 3-manifold if and only if*

(i) $lk(v, X)$ is planar, for every vertex v of X, and

(ii) there exists a family of embeddings $\Phi = \{\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ so that the associated cochain ω_Φ is trivial.

Received by the editors October 27, 1997 and, in revised form, April 17, 1998.
2000 *Mathematics Subject Classification.* Primary 57M20; Secondary 57Q35.

A particular case of this result can be found in [1]; namely, for those finite 2-complexes in which the vertices of the subgraphs X^1 and $lk(v, X)$ all have valence at least 2, for every vertex $v \in X^0$.

If, in addition, X is a fake surface, then $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order 3, and Theorem 1.1 leads to the following theorem which gives us an obstruction to thickening.

Theorem 1.2. *Let X be a fake surface, and let $\Gamma \subset X$ be as above. There exists a well defined cohomology class $\xi_X \in H^1(\Gamma; \mathbf{Z}_2)$ with the property that $\xi_X = 0$ if and only if X thickens to an orientable 3-manifold.*

On the other hand, the cocycle $\omega_\Phi \in C^1(\Gamma; \mathbf{Z}_2)$ can be regarded as a cochain in the whole complex, i.e., $\omega_\Phi \in C^1(X; \mathbf{Z}_2)$. We will say that the family of embeddings Φ is *admissible* if the cochain ω_Φ can be completed to a cocycle in X, via the complementary graph Γ' of Γ in X^1 (see §4). We show that this property is intrinsic to X, i.e., if a family of embeddings is admissible, then so is any other family of embeddings for X. This gives rise to a sufficient condition for a fake surface X to thicken to a 3-manifold (not necessarily orientable). More explicitly

Theorem 1.3. *If a fake surface X has an admissible family of embeddings Φ, then X thickens to a 3-manifold M. If, in addition, we can choose $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\omega_\Phi + \eta \in C^1(X; \mathbf{Z}_2)$ is in fact a coboundary, then M is orientable.*

2. (Co)homology of infinite CW-complexes

Let R be a ring and let X be an oriented locally finite CW-complex. Let $R(e)$ be the free left R-module generated by the cell e in X, and let $C_n^\infty(X; R) = \prod_{dim(e)=n} R(e)$. Elements in $C_n^\infty(X; R)$ will be denoted by infinite sums, and will be called *infinite cellular n-chains with coefficients in R*. Note that the R-module of ordinary cellular n-chains in X, $C_n(X; R)$, is a submodule of $C_n^\infty(X; R)$. Since X is locally finite, the ordinary boundary homomorphism $\partial : C_n(X; R) \longrightarrow C_{n-1}(X; R)$ extends to a boundary homomorphism $\partial : C_n^\infty(X; R) \longrightarrow C_{n-1}^\infty(X; R)$. This way we have a chain complex $(C_*^\infty(X; R), \partial)$ whose homology modules $H_*^\infty(X; R)$ are called the *cellular homology modules of X based on infinite chains* [2].

We can also define a coboundary homomorphism $\delta : C_n^\infty(X; R) \longrightarrow C_{n+1}^\infty(X; R)$ as follows:

$$\delta\left(\sum_i \lambda_i e_i^n\right) = \sum_j \left(\sum_i \lambda_i [e_j^{n+1} : e_i^n]\right) e_j^{n+1},$$

where $[e_j^{n+1} : e_i^n]$ represents the incidence number corresponding to the (oriented) cells e_j^{n+1} and e_i^n. This gives us a cochain complex $(C_*^\infty(X; R), \delta)$ whose cohomology modules $H^*(X; R)$ are, indeed, the ordinary cellular cohomology modules of X with coefficients in R [2]. The cochain complex $(C_*^\infty(X; R), \delta)$ will also be denoted by $(C^*(X; R), \delta)$. Again, since X is locally finite, δ maps $C_n(X; R)$ into $C_{n+1}(X; R)$, giving us another cochain complex $(C_*(X; R), \delta)$, whose cohomology modules $H_f^*(X; R)$ are called the *cellular cohomology modules of X based on finite chains* or, as they are usually referred to, the cellular cohomology modules of X with compact support. The cochain complex $(C_*(X; R), \delta)$ will also be denoted by $(C_f^*(X; R), \delta)$.

For a subcomplex $A \subset X$ the corresponding relative (co)homology modules $H_*^\infty(X, A; R), H^*(X, A; R)$ and $H_f^*(X, A; R)$ of the pair (X, A) are defined in the usual way.

3. A thickening theorem

Our purpose in this section is to prove Theorem 1.1. For this, we need the following definition.

Definition 3.1. Let X be a 2-dimensional locally finite simplicial complex, and assume the link $lk(v, X)$ is planar for every vertex v of X. Let (v, w) be a 1-simplex of X. Given an embedding $\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2$, we denote by $\theta_{\phi_v}(w)$ the cyclic ordering determined by ϕ_v on $lk((v, w), X)$ as we go around $\phi_v(w)$ following the orientation in $\mathbf{R}^2$. Note that if the cardinality $|lk((v, w), X)|$ is ≤ 2, then there is only one cyclic ordering $\theta_{\phi_v}(w)$. Moreover, if $|lk((v, w), X)| = 3$, then there are only two possible orderings.

We will denote by $\Gamma \subset X$ the graph consisting of those 1-simplexes of X which are of order ≥ 3 (i.e., those which are the face of at least three 2-simplexes of X). Consider the cochain complex of Γ over $\mathbf{Z}_2$

$$0 \longrightarrow C^0(\Gamma; \mathbf{Z}_2) \stackrel{\delta}{\longrightarrow} C^1(\Gamma; \mathbf{Z}_2) \longrightarrow 0.$$

Given a family $\Phi = \{\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ of embeddings, we can associate it to a cochain (cocycle)

$$\omega_\Phi = \sum_{\sigma \subset \Gamma} \omega_\Phi(\sigma) \cdot \sigma \in C^1(\Gamma; \mathbf{Z}_2),$$

where $\omega_\Phi(\sigma) = 0$ if $\theta_{\phi_{o(\sigma)}}(t(\sigma))$ and $\theta_{\phi_{t(\sigma)}}(o(\sigma))$ are opposite, and $\omega_\Phi(\sigma) = 1$ otherwise. Here $o(\sigma)$ and $t(\sigma)$ are the vertices of σ. By extension, we define $\omega_\Phi(\sigma) = 0$ for every 1-simplex σ of order ≤ 2.

Proof of Theorem 1.1. We will concentrate on the "sufficient" part, since the converse can be checked making use of some results of p.l. topology. More explicitly, if M collapses to a subdivision X' of X, we choose orientation-preserving homeomorphisms $\psi_v : lk(v, M) \longrightarrow S^2$ ($lk(v, M)$ inherits an orientation from M) whose restrictions $\phi_v : lk(v, X') \longrightarrow S^2$ induce a family Φ of embeddings for X satisfying $\omega_\Phi = 0$, since a regular neighborhood N in M of any 1-simplex (v, w) of X is a 3-ball and we can apply the Annulus Theorem (see [8]) to get a product $S^1 \times [0, 1] \subset \partial N$ with $S^1 \times \{0\} \subset lk(v, M)$ and $S^1 \times \{1\} \subset lk(w, M)$, showing opposite cyclic orderings from the two vertices of (v, w).

We now proceed to check that properties (i) and (ii) yield a thickening of X. Assume that X is oriented, with all vertices having orientation $+1$. We may regard the family Φ as a family of embeddings into the sphere S^2. We are going to build an orientable 3-manifold M, and a CW-structure on it, from X as follows. For every vertex $v \in X^0$, we consider a 3-ball $e^3_v \subset \mathbf{R}^3$ with the usual orientation inherited from $\mathbf{R}^3$. Given $v \in X^0$, we construct a regular neighborhood N_v of $\phi_v(lk(v, X))$ in ∂e^3_v (a 2-sphere) and give that neighborhood a CW-structure with a 2-cell ${e^2}_{\sigma,v}$ for each 1-simplex $\sigma \succ v$ in X, and a 1-cell ${e^1}_{\tau,v}$ for each 2-simplex $\tau \succ v$. For this, we thicken every vertex $w \in \phi_v(lk(v, X))$ to a disk $D_w \subset \partial e^3_v$ and complete the regular neighborhood by considering strips C_ν, one for each 1-cell ν in $\phi_v(lk(v, X))$, joining the corresponding disks D_w so that the core of C_ν is contained in ν. Thus,

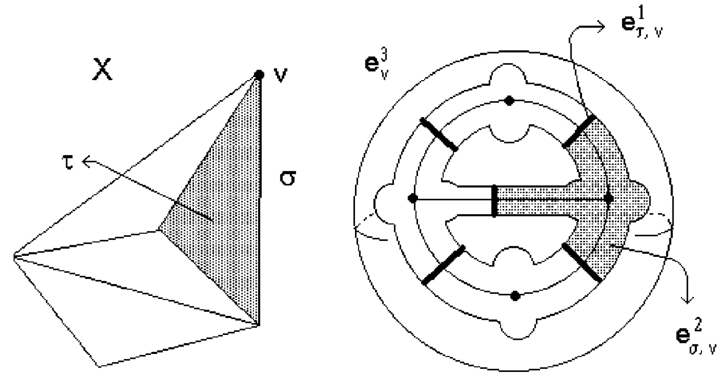


FIGURE 1

given a 1-cell ν in $\phi_v(lk(v, X))$, there is a 2-simplex τ in X containing v and the preimage of ν; and $e^1{}_{\tau,v}$ is going to be a 1-cell dividing C_ν in half, transverse to its core. Given a 1-simplex $\sigma = (v, w)$ in X, the 2-cell $e^2{}_{\sigma,v}$ is going to be the union of D_w together with those halves of the strips C_ν intersecting D_w. See Figure 1.

This way ∂e^3_v has been given a CW-structure (and so has been e^3_v) for every vertex $v \in X^0$, with special cells of the type $e^2{}_{\sigma,v}$ and $e^1{}_{\tau,v}$ as described above. Observe that if $lk(v, X)$ was not connected, we can still achieve such a CW-structure by introducing extra 1-cells in the closure of $\partial e^3_v - N_v$. Note that a cell $e^1{}_{\tau,v}$ is a face of $e^2{}_{\sigma,v}$ if and only if σ is a face of τ. Let's fix a vertex $v \in X^0$. Given a 1-simplex $\sigma \succ v$, we orient the 2-cell $e^2{}_{\sigma,v}$ in such a way that the incidence number $[e^3_v : e^2{}_{\sigma,v}]$ equals $[\sigma : v]$. Next, we orient the 1-cell $e^1{}_{\tau,v}$ so that $[e^2{}_{\sigma,v} : e^1{}_{\tau,v}]$ equals $[\tau : \sigma]$, where $\tau \succ \sigma \succ v$. We have to check that this orientation on $e^1{}_{\tau,v}$ does not depend on the choice of σ. For this, if σ' is the other face of τ containing v, then from the identity $\partial^2(\tau) = 0$, where ∂ stands for the boundary homomorphisms in $C_*(X; \mathbf{Z})$, it follows that

$$[\tau : \sigma]\ [\sigma : v] = -[\tau : \sigma']\ [\sigma' : v]$$

and hence

$$[\tau : \sigma]\ [e^3_v : e^2{}_{\sigma,v}] = -[\tau : \sigma']\ [e^3_v : e^2{}_{\sigma',v}]. \tag{3.1}$$

Also, $\partial^2(e^3_v) = 0$ (for the boundary in the chain complex of e^3_v) and similarly, assuming $e^1{}_{\tau,v}$ oriented as prescribed above with respect to $e^2_{\sigma,v}$, we get

$$[e^3_v : e^2_{\sigma,v}]\ [e^2_{\sigma,v} : e^1_{\tau,v}] = -[e^3_v : e^2_{\sigma',v}]\ [e^2_{\sigma',v} : e^1_{\tau,v}].$$

Thus, from (3.1) it readily follows that $[e^2_{\sigma',v} : e^1_{\tau,v}] = [\tau : \sigma']$.

Now let $Z = \bigsqcup_{v \in X^0} e^3_v$ (disjoint union). We form M, as a quotient of Z, as follows. For every 1-simplex $\sigma = (v, w)$ of X, we glue e^3_v and e^3_w along the 2-cells $e^2{}_{\sigma,v}$ and $e^2{}_{\sigma,w}$ via a (cellular) homeomorphism which is orientation-reversing with respect to the orientations on these 2-cells inherited from e^3_v and e^3_w , taking cells of the form $e^1{}_{\tau,v}$ to cells $e^1{}_{\tau,w}$, for $\tau \succ \sigma$, in such a way as to preserve the orientations defined above on these 1-cells. Observe that this homeomorphism is orientation-preserving with respect to the orientations defined on the 2-cells we glue along, since $[e^3_v : e^2{}_{\sigma,v}] = -[e^3_w : e^2{}_{\sigma,w}]$. Such a homeomorphism does exist since $\theta_{\phi_v}(w)$

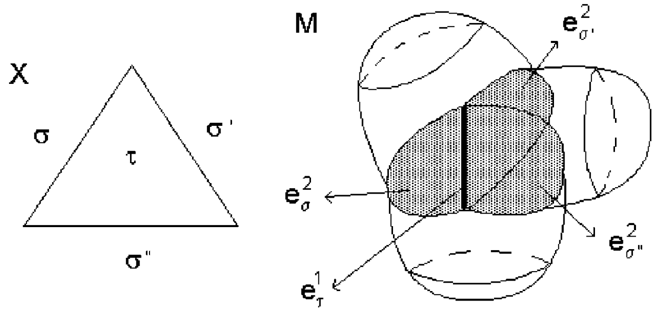


FIGURE 2

and $\theta_{\phi_w}(v)$ are opposite, by hypothesis, i.e., the cyclic orderings under which the cells $e^1{}_{\tau,v}$ appear on $\partial e^2{}_{\sigma,v}$, and the cells $e^1{}_{\tau,w}$ appear on $\partial e^2{}_{\sigma,w}$ are opposite. Let $q : Z \longrightarrow M$ be the quotient map. To simplify notation, we will write $e^3_v = q(e^3_v)$, $e^2_\sigma = q(e^2{}_{\sigma,v})$ and $e^1_\tau = q(e^1{}_{\tau,v})$. Notice that cells in M of the form e^3_v, e^2_σ and e^1_τ come with a well defined orientation from Z (via the quotient map). We give any other cell of M an arbitrary orientation. It can be checked that M is a 3-manifold with boundary. Moreover, the boundary of M, ∂M, consists of those points in the closure of a 2-cell of M which is not of the form e^2_σ, and the interior of M consists of those points in the interior of a cell of the form e^3_v, e^2_σ or e^1_τ.

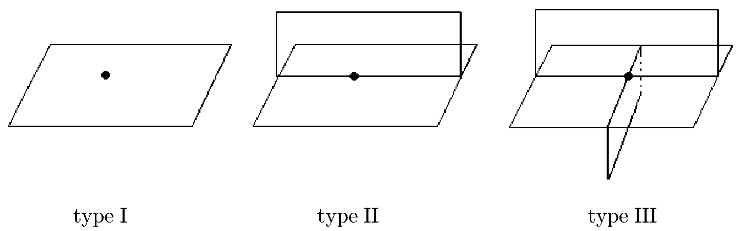
Next, we claim that M contains a copy of the first barycentric subdivision X' of X onto which M collapses. For this, we take points b_v, b_σ and b_τ in the interior of e^3_v, e^2_σ and e^1_τ respectively, for every triple $\tau \succ \sigma \succ v$ in X, and denote by $C(v,\sigma,\tau) \subset e^3_v$ the cone from b_v over an arc in e^2_σ from b_σ to b_τ. This cone can be identified with the 2-simplex $(b(v), b(\sigma), b(\tau))$ of X' (where $b(\rho)$ stands for the barycenter of ρ). Then, the 2-complex $\bigcup\limits_{\tau\succ\sigma\succ v} C(v,\sigma,\tau) \subset M$ is simplicially isomorphic to X', by construction of M. The 3-cell e^3_v, endowed with the cone structure $b_v \cdot \partial e^3_v$, collapses onto the cone $b_v \cdot N_v$, using as free faces those 2-cells in $\partial e^3_v - N_v$. Furthermore, since the regular neighborhood N_v collapses to $\phi_v(lk(v,X)) \cong \bigcup\limits_{\tau\succ\sigma} C(v,\sigma,\tau)$, for every $v \in X^0$, it follows that M collapses to $\bigcup\limits_{\tau\succ\sigma\succ v} C(v,\sigma,\tau) \subset M$. See Figure 2.

Finally, consider the chain map $\psi_k : C^k(X;\mathbf{Z}) \longrightarrow C^\infty_{3-k}(M,\partial M;\mathbf{Z})$ determined by $\psi_0(v) = e^3_v$, $\psi_1(\sigma) = e^2_\sigma$ and $\psi_2(\tau) = e^1_\tau$. The map ψ_* is indeed a chain map, because of the choice of the orientations on the cells e^1_τ, e^2_σ and e^3_v, for every triple $\tau \succ \sigma \succ v$ in X. Moreover, ψ_* is a chain isomorphism, since the cells of the form e^3_v, e^2_σ and e^1_τ are the only ones which are not in ∂M and $C^\infty_0(M,\partial M;\mathbf{Z}) = 0$, by construction of M. Thus, $H^k(X;\mathbf{Z}) \cong H^\infty_{3-k}(M,\partial M;\mathbf{Z})$. In particular, $H^\infty_3(M,\partial M;\mathbf{Z}) \cong H^0(X;\mathbf{Z}) \cong \mathbf{Z}$, which proves that M is an orientable 3-manifold (see [2, 5]). □

4. Fake surfaces

The concept of a "fake surface" was introduced by H. Ikeda [4] in 1971. From now on, we will assume that X is a (closed) fake surface. Explicitly

Definition 4.1. A 2-dimensional locally finite simplicial complex X is called a *fake surface* if each vertex in X has a neighborhood of one of the following three types:



In particular, $lk(v, X)$ is planar for every vertex v of X.

Remark 4.2. Although the combinatorial structure of a fake surface might appear as a quite simple structure, it is important to mention the fact that every compact 2-dimensional polyhedron has the (simple) homotopy type of a finite fake surface [9].

We will show that Theorem 1.1 together with the combinatorial structure of a fake surface (Lemma 4.3 and Proposition 4.4 below) lead to an obstruction to an orientable thickening (Theorem 1.2). For this, we recall that associated to a family of embeddings $\Phi = \{\phi_v : lk(v,X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ we have a cocycle $\omega_\Phi \in C^1(\Gamma; \mathbf{Z}_2)$. Observe that if $\sigma = (v, w)$ and $\omega_\Phi(\sigma) \neq 0$, then $\theta_{\Phi_v}(w)$ and $\theta_{\Phi_w}(v)$ coincide, since such a simplex σ is always of order 3 in a fake surface. Given such a family Φ and a subset $S \subset X^0$, we can get another family of embeddings $\Phi^S = \{\phi'_v : lk(v,X) \longrightarrow \mathbf{R}^2, v \in X^0\}$, where $\phi'_v = \phi_v, v \in X^0 - S$, and $\phi'_v = h \circ \phi_v$, if $v \in S$, where h is a chosen orientation-reversing homeomorphism of $\mathbf{R}^2$. Note that if $v \in S$ and (v, w) is a 1-simplex of X, then $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. In this situation one can readily check the following lemma.

Lemma 4.3. $\omega_\Phi + \omega_{\Phi^S} = \delta\left(\sum_{v \in S\cap\Gamma} v\right) \in C^1(\Gamma; \mathbf{Z}_2)$.

Proposition 4.4. *Let $\Phi = \{\phi_v, v \in X^0\}$ and $\Phi' = \{\phi'_v, v \in X^0\}$ be two families of embeddings, and let $S \subset X^0$ be the subset*

$$S = \{v \in X^0 \mid \exists w \in lk(v,X) \cap \Gamma \textit{ with } \theta_{\phi'_v}(w) \textit{ opposite to } \theta_{\phi_v}(w)\}.$$

Then, $\omega_{\Phi'} = \omega_{\Phi^S}$.

Proof. We want to show that

$$S = \{v \in X^0 \mid \forall w \in lk(v,X) \cap \Gamma, \theta_{\phi'_v}(w) \text{ is opposite to } \theta_{\phi_v}(w)\}.$$

For this, let $v \in S$ and $w \in lk(v, X)$ such that $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Let $\mu : \phi_v(lk(v,X)) \longrightarrow \phi'_v(lk(v,X)) \subset \mathbf{R}^2$ be a (cellular) homeomorphism satisfying $\mu \circ \phi_v = \phi'_v$. The homeomorphism μ can be extended to a (cellular) homeomorphism $\hat{\mu} : S^2 \longrightarrow S^2$, regarding S^2 as the one-point compactification of $\mathbf{R}^2$ with a cell structure determined by the type of $lk(v, X)$, i.e., two 2-cells if type I, three 2-cells if type II, or four 2-cells if type III. Either $\hat{\mu}$ preserves the orientation of S^2 or it reverses it. The second is our case, since $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Thus, $\hat{\mu}$ is an orientation-reversing cellular homeomorphism of S^2, whence $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite, for every vertex $w \in lk(v, X)$, as we wanted.

Let h be a (chosen) orientation-reversing homeomorphism of $\mathbf{R}^2$, and $v \in S$. Then, $\theta_{\phi_v}(w)$ is opposite to $\theta_{h\circ\phi_v}(w)$, which makes the latter coincide with $\theta_{\phi'_v}(w)$, for every $w \in lk(v, X)$. If the vertex v is not in S, then $\theta_{\phi'_v}(w)$ coincides with $\theta_{\phi_v}(w)$, for every $w \in lk(v, X)$. Therefore, $\omega_{\Phi^S} = \omega_{\Phi'}$. □

Proof of Theorem 1.2. We define $\xi_X = [\omega_\Phi]$, where Φ is a family of embeddings as in Definition 3.1. We claim that $[\omega_\Phi] = [\omega_{\Phi'}] \in H^1(\Gamma; \mathbf{Z}_2)$, for any two such families of embeddings Φ, Φ'. Indeed, by Proposition 4.4, there is a subset $S \subset X^0$ such that $\omega_{\Phi'} = \omega_{\Phi^S}$. Therefore, by Lemma 4.3,

$$\omega_\Phi + \omega_{\Phi'} = \omega_\Phi + \omega_{\Phi^S} = \delta\left(\sum_{v\in S\cap\Gamma} v\right),$$

whence $[\omega_\Phi] = [\omega_{\Phi'}] = \xi_X$.

Now suppose that $\xi_X = 0$. We are to show that X thickens to an orientable 3-manifold. Since $\xi_X = 0$, given a family Φ there is a subset $S \subset X^0$ so that $\omega_\Phi = \delta\left(\sum_{v\in S} v\right)$. Now the family Φ^S satisfies $\omega_{\Phi^S} = 0$, since $\omega_\Phi + \omega_{\Phi^S} = \omega_\Phi$ by Lemma 4.3. The conclusion follows then from Theorem 1.1. The converse is clear, also from Theorem 1.1. □

Next, we want to prove Theorem 1.3. For this we will denote by Γ' the complementary graph of Γ in X^1. Observe that the cochain ω_Φ can be regarded as a cochain in X, and $C^1(X; \mathbf{Z}_2) \cong C^1(\Gamma; \mathbf{Z}_2) \oplus C^1(\Gamma'; \mathbf{Z}_2)$. We need the following definition.

Definition 4.5. We say that a family of embeddings Φ is *admissible* if ω_Φ can be completed, via Γ', to a cocycle in X; i.e., there is $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\delta(\omega_\Phi + \eta) = 0$ in $C^2(X; \mathbf{Z}_2)$.

Lemma 4.6. *(a) For a fake surface X, either every family of embeddings is admissible or none is.*

(b) If X has an admissible family of embeddings, then so does any subdivision obtained by deriving X away from X^1.

Proof. (a) Let Φ, Φ' be two families of embeddings, and assume Φ is admissible; i.e., there is $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\delta(\omega_\Phi + \eta) = 0$ in $C^2(X; \mathbf{Z}_2)$. There is $\eta' \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\omega_\Phi + \omega_{\Phi'} + \eta'$ is a coboundary in X, since ω_Φ and $\omega_{\Phi'}$ are cohomologous in Γ, by Lemma 4.3 and Proposition 4.4. Then,

$$\delta(\omega_{\Phi'} + \eta + \eta') = \delta(\omega_{\Phi'} + \eta + \eta' + \omega_\Phi + \omega_\Phi) = \delta(\omega_\Phi + \omega_{\Phi'} + \eta') + \delta(\omega_\Phi + \eta) = 0.$$

Furthermore, if $\omega_\Phi + \eta$ is a coboundary, then so is $\omega_{\Phi'} + (\eta + \eta')$.

(b) Let Φ be an admissible family of embeddings for X. A derived subdivision Y of X away from X^1 does not have any new vertices of type II or III. Thus, the embeddings $\phi_v \in \Phi$ ($v \in \Gamma$) induce embeddings $\phi'_v : lk(v, Y) \longrightarrow \mathbf{R}^2$ having $\phi_v(lk(v, X))$ as the image set. For the new vertices (all of type I) we choose arbitrary embeddings. This gives us a family of embeddings Φ' for Y such that $\omega_\Phi = \omega_{\Phi'}$. Moreover, if there is $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\omega_\Phi + \eta$ is a cocycle in X, we consider the complementary graph Γ'' of X^1 in Y^1 and we take $\eta' \in C^1(\Gamma''; \mathbf{Z}_2)$ as the sum of the new 1-simplexes σ'' of Y sharing a vertex with two 1-simplexes σ, σ' of X with coefficient 0 in $\omega_\Phi + \eta$. It is not hard to check that $\omega_\Phi + \eta + \eta'$ is a cocycle in Y. □

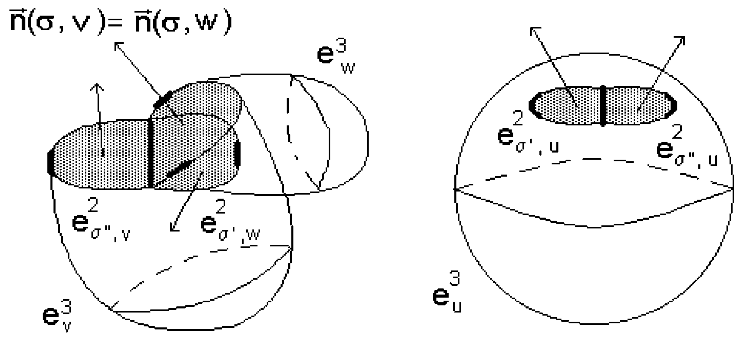


FIGURE 3

Proof of Theorem 1.3. Let Y be the first derived subdivision of X away from X^1, and let $\Gamma \subset Y$ be the corresponding subgraph of all 1-simplexes of order 3. By Lemma 4.6, if X has an admissible family of embeddings, then so does Y. Observe that any 2-simplex of Y contains at most one 1-simplex of order 3. This is why we subdivide.

Let $\Phi = \{\phi_v : lk(v, Y) \longrightarrow \mathbf{R}^2,\ v \in Y^0\}$ be an admissible family of embeddings. We consider the (oriented) CW-structure on $Z = \bigsqcup_{v \in Y^0} e_v^3$ given in the proof of Theorem 1.1. Recall that every 3-ball $e_v^3 \subset \mathbf{R}^3$ had the inherited orientation from $\mathbf{R}^3$, and so we have a field of normal vectors defined on each sphere $\partial e_v^3, v \in Y^0$. In particular, each 2-cell $e_{\sigma,v}^2,\ \sigma \succ v$, comes with a normal vector (independently of the orientation defined on $e_{\sigma,v}^2$). Denote this normal vector by $\vec{n}(\sigma, v)$.

We claim that we can glue these 3-cells e_v^3 in such a way as to produce a 3-manifold M which collapses to a copy of the first barycentric subdivision Y' of Y. Let $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\delta(\omega_\Phi + \eta) = 0$ in Y, and let $\sigma = (v, w), \sigma' = (u, w)$ and $\sigma'' = (u, v)$ be the faces of the 2-simplex (u, v, w) of Y. We follow some rules for the gluing.

Rule (1). If $\omega_\Phi(\sigma) \neq 0$ (which implies that σ is of order 3), then we glue e_v^3 and e_w^3 along $e_{\sigma,v}^2$ and $e_{\sigma,w}^2$ via an orientation-preserving homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells $e_{\tau,v}^1$ to cells $e_{\tau,w}^1,\ \tau \succ \sigma$. This way, we make the vectors $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ match. See Figure 3.

To complete the picture obtaining a 3-ball we could glue e_u^3 (along $e_{\sigma',u}^2$ and $e_{\sigma'',u}^2$) in such a way that $\vec{n}(\sigma', u)$ matches $\vec{n}(\sigma', w)$, and $\vec{n}(\sigma'', u)$ does not match $\vec{n}(\sigma'', v)$, or vice versa. Rule (3) will give the criterion to follow, since the 1-simplexes σ' and σ'' are both of order 2.

Rule (2). If $\omega_\Phi(\sigma) = 0$ and σ is of order 3, then we glue e_v^3 and e_w^3 along $e_{\sigma,v}^2$ and $e_{\sigma,w}^2$ via an orientation-reversing homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells of the form $e_{\tau,v}^1$ to cells $e_{\tau,w}^1,\ \tau \succ \sigma$. Observe that the gluing is done so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ do not match. See Figure 4.

Notice that the gluing of the 3-cell e_u^3 (along $e_{\sigma',u}^2$ and $e_{\sigma'',u}^2$) can be done in two different ways to obtain a 3-ball, depending on whether the pair $(\vec{n}(\sigma', w), \vec{n}(\sigma'', v))$ matches $(\vec{n}(\sigma', u), \vec{n}(\sigma'', u))$ or it does not, in which case neither $\vec{n}(\sigma', w)$ matches

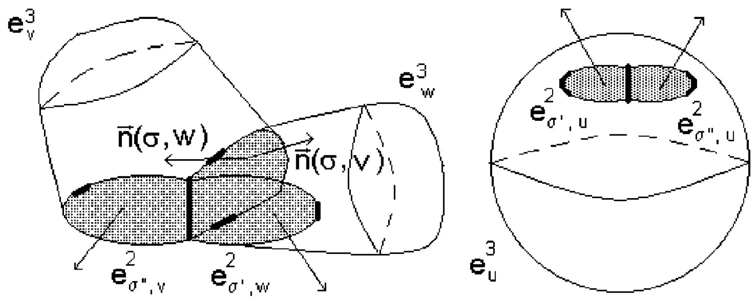


FIGURE 4

$\vec{n}(\sigma', u)$ nor $\vec{n}(\sigma'', v)$ matches $\vec{n}(\sigma'', u)$. Rule (3) will give the criterion to follow, since σ' and σ'' are again of order 2.

Rule (3). If σ is a 1-simplex of order 2 (and hence $\omega_\Phi(\sigma) = 0$) and the coefficient of σ in $\omega_\Phi + \eta$ is 0, then we glue e^3_v and e^3_w so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ do not match. Otherwise, we do the gluing so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ match.

This way we make sure we are building up a 3-manifold each time we consider a 2-simplex of Y. Indeed, using the fact that $\omega_\Phi + \eta$ is a cocycle, i.e., the number of faces of a 2-simplex τ of Y which occur with non-zero coefficient in $\omega_\Phi + \eta$ is either 0 or 2, it can be checked that the points on the 1-cells of the form $e^1_{\tau,v}$ have euclidean neighborhoods in the space obtained after the gluing. Let M be the resulting 3-manifold built this way. The same argument used in the proof of Theorem 1.1 proves that M collapses to a copy of Y'.

Finally, if $[\omega_\Phi + \eta] = 0$ in $H^1(X; \mathbf{Z}_2)$, say $\omega_\Phi + \eta = \delta(z)$, $z \in C^0(X; \mathbf{Z}_2)$, then

$$\delta(i^*(z)) = i^*(\delta(z)) = i^*(\omega_\Phi) + i^*(\eta) = \omega_\Phi,$$

where i^* is induced by the inclusion $\Gamma \overset{i}{\hookrightarrow} X$. Therefore, the cohomology class $\xi_X = [\omega_\Phi] \in H^1(\Gamma; \mathbf{Z}_2)$ is trivial and the conclusion follows from Theorem 1.2. □

Remark 4.7. Notice that if $\xi_X = 0$, then any family of embeddings Φ for X is admissible. For if $\omega_\Phi = \delta(z)$ in Γ, then the coboundary of z in X, $\delta(z) = \omega_\Phi + \eta$ ($\eta \in C^1(\Gamma'; \mathbf{Z}_2)$), is in particular a cocycle in X. If $H^1(X; \mathbf{Z}_2) = 0$ (e.g., X simply connected), then we have the converse, since every cocycle is then a coboundary.

ACKNOWLEDGEMENTS

The author wishes to thank Professor R. Geoghegan for helping him throughout. He is also thankful to Professors R. Ayala and A. Quintero for their helpful comments. Most of this work is part of a thesis presented at the State University of New York at Binghamton, under the direction of Professor R. Geoghegan. This work was partially supported by the project DGICYT PB96-1374.

ADDED IN PROOF

Independently and essentially simultaneously these results were obtained by D. Repovš, N. B. Brodskij and A. B. Skopenkov; see *A classification of* 3*-thickenings of* 2*-polyhedra*, Topology and its Applications **94** (1999), 307-314.

References

[1] J.M. Corson, B. Trace, "Geometry and algebra of nonspherical 2-complexes". J. London Math. Soc. 54(1996), 180-198. MR **98b:**57007

[2] R. Geoghegan, "Topological methods in group theory". Book in preparation.

[3] C. Hog-Angeloni, W. Metzler, "Geometric aspects of two-dimensional complexes", pp. 1-50 of Two-dimensional Homotopy and Combinatorial Group Theory (C. Hog-Angeloni, W. Metzler, A.J. Sieradski ed.). Lecture Notes Series 197. London Math. Soc. Cambridge Univ. Press(1993). CMP 94:13

[4] H. Ikeda, "Acyclic fake surfaces". Topology 10(1971), 9-36. MR **48:**5078

[5] W.S. Massey, "Homology and Cohomology Theory". Marcel Dekker(1978). MR **58:**7594

[6] L. Neuwirth, "An algorithm for the construction of 3-manifolds from 2-complexes". Proc. Camb. Phil. Soc 64(1968), 603-613. MR **37:**2231

[7] F. Quinn, "Presentations and 2-complexes, fake surfaces and singular 3-manifolds". Virginia Polytechnic Institute. Blackburg Va.(1981). Preprint.

[8] C.P. Rourke, B.J. Sanderson, "Introduction to Piecewise-Linear Topology". Springer(1972). MR **50:**3236

[9] P. Wright, "Formal 3-deformations of 2-polyhedra". Proc. American Math. Soc. 37(1973), 305-308. MR **48:**9730

[10] P. Wright, "Covering 2-dimensional polyhedra by 3-manifold spines". Topology 16(1977), 435-439. MR **56:**16611

Departamento de Geometría y Topología, Universidad de Sevilla, Apdo 1160, 41080-Sevilla, Spain

E-mail address: `fjfer@cica.es`