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AN OBSTRUCTION TO 3-DIMENSIONAL THICKENINGS

FRANCISCO F. LASHERAS

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Abstract. In this paper we give a characterization of those locally finite 2-dimensional simplicial complexes that have an orientable 3-manifold thickening. This leads to an obstruction for a fake surface X to admit such a thickening. The obstruction is defined in $H^1(\Gamma;\mathbf{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order three.

1. Introduction

A simplicial complex X "thickens" to a CW-complex Y if Y admits a CW-structure containing, as a subcomplex, a copy of a subdivision of X onto which Y collapses. For a *standard* 2-complex (a finite 2-complex with a single vertex) L. Neuwirth [6] exhibited an algorithm to decide whether the given 2-complex expands to an orientable 3-manifold. Later, H. Ikeda [4] introduced the concept of a *fake surface* (see §4). It is known that for this class of 2-complexes we always have a thickening to a *singular* 3-*manifold* [3, 7], i.e., a polyhedron in which the link of each point is a disk D^2 (boundary point), a sphere S^2 (inner point), or a projective plane P^2 (singular point). P. Wright [10] showed a sufficient condition for a (compact) fake surface X to embed into a 3-manifold; namely, if every simple closed curve C in the subgraph of all triple edges is *untwisted* (i.e., a regular neighborhood of C in X contains a T-bundle over C which embeds in $\mathbf{R}^3$), then X embeds into a 3-manifold.

In order to detect some kind of "twists" in a 2-dimensional locally finite simplicial complex X (with planar links), we work with a family of embeddings $\Phi = \{\phi_v : lk(v,X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ and we consider certain cyclic orderings around each vertex, determined by the family Φ. This way we associate to each family of embeddings Φ a cochain (cocycle) $\omega_\Phi \in C^1(\Gamma;\mathbf{Z}_2)$, where $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order ≥ 3. From the study of these cochains we obtain the main results of this paper. More precisely

Theorem 1.1. *Let X be a 2-dimensional connected locally finite simplicial complex. Then, X thickens to an orientable 3-manifold if and only if*

(i) $lk(v,X)$ is planar, for every vertex v of X, and

(ii) there exists a family of embeddings $\Phi = \{\phi_v : lk(v,X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ so that the associated cochain ω_Φ is trivial.

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A particular case of this result can be found in [1]; namely, for those finite 2-complexes in which the vertices of the subgraphs X^1 and $lk(v, X)$ all have valence at least 2, for every vertex $v \in X^0$.

If, in addition, X is a fake surface, then $\Gamma \subset X$ is the subgraph consisting of all the 1-simplexes of order 3, and Theorem 1.1 leads to the following theorem which gives us an obstruction to thickening.

Theorem 1.2. *Let X be a fake surface, and let $\Gamma \subset X$ be as above. There exists a well defined cohomology class $\xi_X \in H^1(\Gamma; \mathbf{Z}_2)$ with the property that $\xi_X = 0$ if and only if X thickens to an orientable 3-manifold.*

On the other hand, the cocycle $\omega_\Phi \in C^1(\Gamma; \mathbf{Z}_2)$ can be regarded as a cochain in the whole complex, i.e., $\omega_\Phi \in C^1(X; \mathbf{Z}_2)$. We will say that the family of embeddings Φ is *admissible* if the cochain ω_Φ can be completed to a cocycle in X, via the complementary graph Γ' of Γ in X^1 (see §4). We show that this property is intrinsic to X, i.e., if a family of embeddings is admissible, then so is any other family of embeddings for X. This gives rise to a sufficient condition for a fake surface X to thicken to a 3-manifold (not necessarily orientable). More explicitly

Theorem 1.3. *If a fake surface X has an admissible family of embeddings Φ, then X thickens to a 3-manifold M. If, in addition, we can choose $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\omega_\Phi + \eta \in C^1(X; \mathbf{Z}_2)$ is in fact a coboundary, then M is orientable.*

2. (Co)homology of infinite CW-complexes

Let R be a ring and let X be an oriented locally finite CW-complex. Let $R(e)$ be the free left R-module generated by the cell e in X, and let $C_n^\infty(X; R) = \prod_{dim(e)=n} R(e)$. Elements in $C_n^\infty(X; R)$ will be denoted by infinite sums, and will be called *infinite cellular n-chains with coefficients in R*. Note that the R-module of ordinary cellular n-chains in X, $C_n(X; R)$, is a submodule of $C_n^\infty(X; R)$. Since X is locally finite, the ordinary boundary homomorphism $\partial : C_n(X; R) \longrightarrow C_{n-1}(X; R)$ extends to a boundary homomorphism $\partial : C_n^\infty(X; R) \longrightarrow C_{n-1}^\infty(X; R)$. This way we have a chain complex $(C_*^\infty(X; R), \partial)$ whose homology modules $H_*^\infty(X; R)$ are called the *cellular homology modules of X based on infinite chains* [2].

We can also define a coboundary homomorphism $\delta : C_n^\infty(X; R) \longrightarrow C_{n+1}^\infty(X; R)$ as follows:

$$\delta\left(\sum_i \lambda_i e_i^n\right) = \sum_j \left(\sum_i \lambda_i [e_j^{n+1} : e_i^n]\right) e_j^{n+1},$$

where $[e_j^{n+1} : e_i^n]$ represents the incidence number corresponding to the (oriented) cells e_j^{n+1} and e_i^n. This gives us a cochain complex $(C_*^\infty(X; R), \delta)$ whose cohomology modules $H^*(X; R)$ are, indeed, the ordinary cellular cohomology modules of X with coefficients in R [2]. The cochain complex $(C_*^\infty(X; R), \delta)$ will also be denoted by $(C^*(X; R), \delta)$. Again, since X is locally finite, δ maps $C_n(X; R)$ into $C_{n+1}(X; R)$, giving us another cochain complex $(C_*(X; R), \delta)$, whose cohomology modules $H_f^*(X; R)$ are called the *cellular cohomology modules of X based on finite chains* or, as they are usually referred to, the cellular cohomology modules of X with compact support. The cochain complex $(C_*(X; R), \delta)$ will also be denoted by $(C_f^*(X; R), \delta)$.

For a subcomplex $A \subset X$ the corresponding relative (co)homology modules $H_*^\infty(X, A; R), H^*(X, A; R)$ and $H_f^*(X, A; R)$ of the pair (X, A) are defined in the usual way.

3. A thickening theorem

Our purpose in this section is to prove Theorem 1.1. For this, we need the following definition.

Definition 3.1. Let X be a 2-dimensional locally finite simplicial complex, and assume the link $lk(v, X)$ is planar for every vertex v of X. Let (v, w) be a 1-simplex of X. Given an embedding $\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2$, we denote by $\theta_{\phi_v}(w)$ the cyclic ordering determined by ϕ_v on $lk((v, w), X)$ as we go around $\phi_v(w)$ following the orientation in $\mathbf{R}^2$. Note that if the cardinality $|lk((v, w), X)|$ is ≤ 2, then there is only one cyclic ordering $\theta_{\phi_v}(w)$. Moreover, if $|lk((v, w), X)| = 3$, then there are only two possible orderings.

We will denote by $\Gamma \subset X$ the graph consisting of those 1-simplexes of X which are of order ≥ 3 (i.e., those which are the face of at least three 2-simplexes of X). Consider the cochain complex of Γ over $\mathbf{Z}_2$

$$0 \longrightarrow C^0(\Gamma; \mathbf{Z}_2) \xrightarrow{\delta} C^1(\Gamma; \mathbf{Z}_2) \longrightarrow 0.$$

Given a family $\Phi = \{\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ of embeddings, we can associate it to a cochain (cocycle)

$$\omega_\Phi = \sum_{\sigma \subset \Gamma} \omega_\Phi(\sigma) \cdot \sigma \in C^1(\Gamma; \mathbf{Z}_2),$$

where $\omega_\Phi(\sigma) = 0$ if $\theta_{\phi_{o(\sigma)}}(t(\sigma))$ and $\theta_{\phi_{t(\sigma)}}(o(\sigma))$ are opposite, and $\omega_\Phi(\sigma) = 1$ otherwise. Here $o(\sigma)$ and $t(\sigma)$ are the vertices of σ. By extension, we define $\omega_\Phi(\sigma) = 0$ for every 1-simplex σ of order ≤ 2.

Proof of Theorem 1.1. We will concentrate on the "sufficient" part, since the converse can be checked making use of some results of p.l. topology. More explicitly, if M collapses to a subdivision X' of X, we choose orientation-preserving homeomorphisms $\psi_v : lk(v, M) \longrightarrow S^2$ ($lk(v, M)$ inherits an orientation from M) whose restrictions $\phi_v : lk(v, X') \longrightarrow S^2$ induce a family Φ of embeddings for X satisfying $\omega_\Phi = 0$, since a regular neighborhood N in M of any 1-simplex (v, w) of X is a 3-ball and we can apply the Annulus Theorem (see [8]) to get a product $S^1 \times [0, 1] \subset \partial N$ with $S^1 \times \{0\} \subset lk(v, M)$ and $S^1 \times \{1\} \subset lk(w, M)$, showing opposite cyclic orderings from the two vertices of (v, w).

We now proceed to check that properties (i) and (ii) yield a thickening of X. Assume that X is oriented, with all vertices having orientation $+1$. We may regard the family Φ as a family of embeddings into the sphere S^2. We are going to build an orientable 3-manifold M, and a CW-structure on it, from X as follows. For every vertex $v \in X^0$, we consider a 3-ball $e_v^3 \subset \mathbf{R}^3$ with the usual orientation inherited from $\mathbf{R}^3$. Given $v \in X^0$, we construct a regular neighborhood N_v of $\phi_v(lk(v, X))$ in ∂e_v^3 (a 2-sphere) and give that neighborhood a CW-structure with a 2-cell ${e^2}_{\sigma,v}$ for each 1-simplex $\sigma \succ v$ in X, and a 1-cell ${e^1}_{\tau,v}$ for each 2-simplex $\tau \succ v$. For this, we thicken every vertex $w \in \phi_v(lk(v, X))$ to a disk $D_w \subset \partial e_v^3$ and complete the regular neighborhood by considering strips C_ν, one for each 1-cell ν in $\phi_v(lk(v, X))$, joining the corresponding disks D_w so that the core of C_ν is contained in ν. Thus,

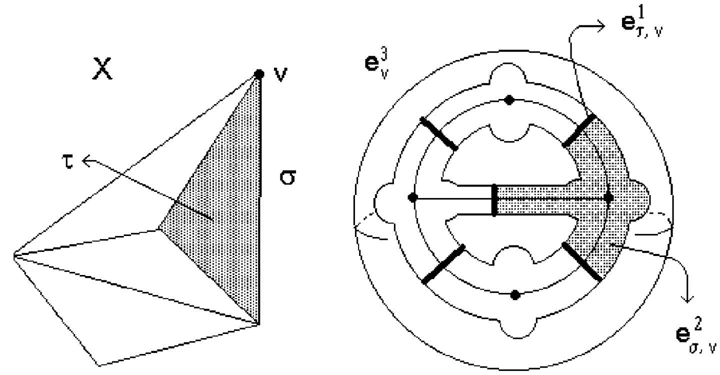


FIGURE 1

given a 1-cell ν in $\phi_v(lk(v,X))$, there is a 2-simplex τ in X containing v and the preimage of ν; and ${e^1}_{\tau,v}$ is going to be a 1-cell dividing C_ν in half, transverse to its core. Given a 1-simplex $\sigma = (v,w)$ in X, the 2-cell ${e^2}_{\sigma,v}$ is going to be the union of D_w together with those halves of the strips C_ν intersecting D_w. See Figure 1.

This way ∂e^3_v has been given a CW-structure (and so has been e^3_v) for every vertex $v \in X^0$, with special cells of the type ${e^2}_{\sigma,v}$ and ${e^1}_{\tau,v}$ as described above. Observe that if $lk(v,X)$ was not connected, we can still achieve such a CW-structure by introducing extra 1-cells in the closure of $\partial e^3_v - N_v$. Note that a cell ${e^1}_{\tau,v}$ is a face of ${e^2}_{\sigma,v}$ if and only if σ is a face of τ. Let's fix a vertex $v \in X^0$. Given a 1-simplex $\sigma \succ v$, we orient the 2-cell ${e^2}_{\sigma,v}$ in such a way that the incidence number $[e^3_v : {e^2}_{\sigma,v}]$ equals $[\sigma : v]$. Next, we orient the 1-cell ${e^1}_{\tau,v}$ so that $[{e^2}_{\sigma,v} : {e^1}_{\tau,v}]$ equals $[\tau : \sigma]$, where $\tau \succ \sigma \succ v$. We have to check that this orientation on ${e^1}_{\tau,v}$ does not depend on the choice of σ. For this, if σ' is the other face of τ containing v, then from the identity $\partial^2(\tau) = 0$, where ∂ stands for the boundary homomorphisms in $C_*(X;\mathbf{Z})$, it follows that

$$[\tau : \sigma]\ [\sigma : v] = -[\tau : \sigma']\ [\sigma' : v]$$

and hence

$$[\tau : \sigma]\ [e^3_v : {e^2}_{\sigma,v}] = -[\tau : \sigma']\ [e^3_v : {e^2}_{\sigma',v}]. \tag{3.1}$$

Also, $\partial^2(e^3_v) = 0$ (for the boundary in the chain complex of e^3_v) and similarly, assuming ${e^1}_{\tau,v}$ oriented as prescribed above with respect to $e^2_{\sigma,v}$, we get

$$[e^3_v : e^2_{\sigma,v}]\ [e^2_{\sigma,v} : e^1_{\tau,v}] = -[e^3_v : e^2_{\sigma',v}]\ [e^2_{\sigma',v} : e^1_{\tau,v}].$$

Thus, from (3.1) it readily follows that $[e^2_{\sigma',v} : e^1_{\tau,v}] = [\tau : \sigma']$.

Now let $Z = \bigsqcup_{v\in X^0} e^3_v$ (disjoint union). We form M, as a quotient of Z, as follows. For every 1-simplex $\sigma = (v,w)$ of X, we glue e^3_v and e^3_w along the 2-cells ${e^2}_{\sigma,v}$ and ${e^2}_{\sigma,w}$ via a (cellular) homeomorphism which is orientation-reversing with respect to the orientations on these 2-cells inherited from e^3_v and e^3_w , taking cells of the form ${e^1}_{\tau,v}$ to cells ${e^1}_{\tau,w}$, for $\tau \succ \sigma$, in such a way as to preserve the orientations defined above on these 1-cells. Observe that this homeomorphism is orientation-preserving with respect to the orientations defined on the 2-cells we glue along, since $[e^3_v : {e^2}_{\sigma,v}] = -[e^3_w : {e^2}_{\sigma,w}]$. Such a homeomorphism does exist since $\theta_{\phi_v}(w)$

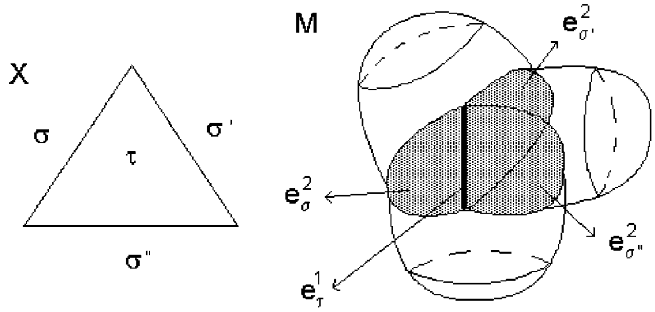


FIGURE 2

and $\theta_{\phi_w}(v)$ are opposite, by hypothesis, i.e., the cyclic orderings under which the cells ${e^1}_{\tau,v}$ appear on $\partial {e^2}_{\sigma,v}$, and the cells ${e^1}_{\tau,w}$ appear on $\partial {e^2}_{\sigma,w}$ are opposite. Let $q : Z \longrightarrow M$ be the quotient map. To simplify notation, we will write $e^3_v = q(e^3_v)$, $e^2_\sigma = q({e^2}_{\sigma,v})$ and $e^1_\tau = q({e^1}_{\tau,v})$. Notice that cells in M of the form e^3_v, e^2_σ and e^1_τ come with a well defined orientation from Z (via the quotient map). We give any other cell of M an arbitrary orientation. It can be checked that M is a 3-manifold with boundary. Moreover, the boundary of M, ∂M, consists of those points in the closure of a 2-cell of M which is not of the form e^2_σ, and the interior of M consists of those points in the interior of a cell of the form e^3_v, e^2_σ or e^1_τ.

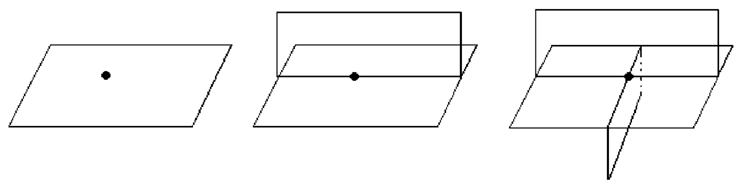
Next, we claim that M contains a copy of the first barycentric subdivision X' of X onto which M collapses. For this, we take points b_v, b_σ and b_τ in the interior of e^3_v, e^2_σ and e^1_τ respectively, for every triple $\tau \succ \sigma \succ v$ in X, and denote by $C(v,\sigma,\tau) \subset e^3_v$ the cone from b_v over an arc in e^2_σ from b_σ to b_τ. This cone can be identified with the 2-simplex $(b(v), b(\sigma), b(\tau))$ of X' (where $b(\rho)$ stands for the barycenter of ρ). Then, the 2-complex $\bigcup\limits_{\tau\succ\sigma\succ v} C(v,\sigma,\tau) \subset M$ is simplicially isomorphic to X', by construction of M. The 3-cell e^3_v, endowed with the cone structure $b_v \cdot \partial e^3_v$, collapses onto the cone $b_v \cdot N_v$, using as free faces those 2-cells in $\partial e^3_v - N_v$. Furthermore, since the regular neighborhood N_v collapses to $\phi_v(lk(v,X)) \cong \bigcup\limits_{\tau\succ\sigma} C(v,\sigma,\tau)$, for every $v \in X^0$, it follows that M collapses to $\bigcup\limits_{\tau\succ\sigma\succ v} C(v,\sigma,\tau) \subset M$. See Figure 2.

Finally, consider the chain map $\psi_k : C^k(X;\mathbf{Z}) \longrightarrow C^\infty_{3-k}(M,\partial M;\mathbf{Z})$ determined by $\psi_0(v) = e^3_v$, $\psi_1(\sigma) = e^2_\sigma$ and $\psi_2(\tau) = e^1_\tau$. The map ψ_* is indeed a chain map, because of the choice of the orientations on the cells e^1_τ, e^2_σ and e^3_v, for every triple $\tau \succ \sigma \succ v$ in X. Moreover, ψ_* is a chain isomorphism, since the cells of the form e^3_v, e^2_σ and e^1_τ are the only ones which are not in ∂M and $C^\infty_0(M,\partial M;\mathbf{Z}) = 0$, by construction of M. Thus, $H^k(X;\mathbf{Z}) \cong H^\infty_{3-k}(M,\partial M;\mathbf{Z})$. In particular, $H^\infty_3(M,\partial M;\mathbf{Z}) \cong H^0(X;\mathbf{Z}) \cong \mathbf{Z}$, which proves that M is an orientable 3-manifold (see [2, 5]). □

4. Fake surfaces

The concept of a "fake surface" was introduced by H. Ikeda [4] in 1971. From now on, we will assume that X is a (closed) fake surface. Explicitly

Definition 4.1. A 2-dimensional locally finite simplicial complex X is called a *fake surface* if each vertex in X has a neighborhood of one of the following three types:



type I type II type III

In particular, $lk(v, X)$ is planar for every vertex v of X.

Remark 4.2. Although the combinatorial structure of a fake surface might appear as a quite simple structure, it is important to mention the fact that every compact 2-dimensional polyhedron has the (simple) homotopy type of a finite fake surface [9].

We will show that Theorem 1.1 together with the combinatorial structure of a fake surface (Lemma 4.3 and Proposition 4.4 below) lead to an obstruction to an orientable thickening (Theorem 1.2). For this, we recall that associated to a family of embeddings $\Phi = \{\phi_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$ we have a cocycle $\omega_\Phi \in C^1(\Gamma; \mathbf{Z}_2)$. Observe that if $\sigma = (v, w)$ and $\omega_\Phi(\sigma) \neq 0$, then $\theta_{\Phi_v}(w)$ and $\theta_{\Phi_w}(v)$ coincide, since such a simplex σ is always of order 3 in a fake surface. Given such a family Φ and a subset $S \subset X^0$, we can get another family of embeddings $\Phi^S = \{\phi'_v : lk(v, X) \longrightarrow \mathbf{R}^2, v \in X^0\}$, where $\phi'_v = \phi_v, v \in X^0 - S$, and $\phi'_v = h \circ \phi_v$, if $v \in S$, where h is a chosen orientation-reversing homeomorphism of $\mathbf{R}^2$. Note that if $v \in S$ and (v, w) is a 1-simplex of X, then $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. In this situation one can readily check the following lemma.

Lemma 4.3. $\omega_\Phi + \omega_{\Phi^S} = \delta\left(\sum_{v \in S \cap \Gamma} v\right) \in C^1(\Gamma; \mathbf{Z}_2)$.

Proposition 4.4. *Let* $\Phi = \{\phi_v, v \in X^0\}$ *and* $\Phi' = \{\phi'_v, v \in X^0\}$ *be two families of embeddings, and let* $S \subset X^0$ *be the subset*

$$S = \{v \in X^0 \mid \exists w \in lk(v, X) \cap \Gamma \text{ with } \theta_{\phi'_v}(w) \text{ opposite to } \theta_{\phi_v}(w)\}.$$

Then, $\omega_{\Phi'} = \omega_{\Phi^S}$.

Proof. We want to show that

$$S = \{v \in X^0 \mid \forall w \in lk(v, X) \cap \Gamma, \theta_{\phi'_v}(w) \text{ is opposite to } \theta_{\phi_v}(w)\}.$$

For this, let $v \in S$ and $w \in lk(v, X)$ such that $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Let $\mu : \phi_v(lk(v, X)) \longrightarrow \phi'_v(lk(v, X)) \subset \mathbf{R}^2$ be a (cellular) homeomorphism satisfying $\mu \circ \phi_v = \phi'_v$. The homeomorphism μ can be extended to a (cellular) homeomorphism $\hat{\mu} : S^2 \longrightarrow S^2$, regarding S^2 as the one-point compactification of $\mathbf{R}^2$ with a cell structure determined by the type of $lk(v, X)$, i.e., two 2-cells if type I, three 2-cells if type II, or four 2-cells if type III. Either $\hat{\mu}$ preserves the orientation of S^2 or it reverses it. The second is our case, since $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite. Thus, $\hat{\mu}$ is an orientation-reversing cellular homeomorphism of S^2, whence $\theta_{\phi'_v}(w)$ and $\theta_{\phi_v}(w)$ are opposite, for every vertex $w \in lk(v, X)$, as we wanted.

Let h be a (chosen) orientation-reversing homeomorphism of $\mathbf{R}^2$, and $v \in S$. Then, $\theta_{\phi_v}(w)$ is opposite to $\theta_{h\circ\phi_v}(w)$, which makes the latter coincide with $\theta_{\phi'_v}(w)$, for every $w \in lk(v,X)$. If the vertex v is not in S, then $\theta_{\phi'_v}(w)$ coincides with $\theta_{\phi_v}(w)$, for every $w \in lk(v,X)$. Therefore, $\omega_{\Phi^S} = \omega_{\Phi'}$. $\square$

Proof of Theorem 1.2. We define $\xi_X = [\omega_\Phi]$, where Φ is a family of embeddings as in Definition 3.1. We claim that $[\omega_\Phi] = [\omega_{\Phi'}] \in H^1(\Gamma;\mathbf{Z}_2)$, for any two such families of embeddings Φ, Φ'. Indeed, by Proposition 4.4, there is a subset $S \subset X^0$ such that $\omega_{\Phi'} = \omega_{\Phi^S}$. Therefore, by Lemma 4.3,

$$\omega_\Phi + \omega_{\Phi'} = \omega_\Phi + \omega_{\Phi^S} = \delta\left(\sum_{v\in S\cap\Gamma} v\right),$$

whence $[\omega_\Phi] = [\omega_{\Phi'}] = \xi_X$.

Now suppose that $\xi_X = 0$. We are to show that X thickens to an orientable 3-manifold. Since $\xi_X = 0$, given a family Φ there is a subset $S \subset X^0$ so that $\omega_\Phi = \delta\left(\sum_{v\in S} v\right)$. Now the family Φ^S satisfies $\omega_{\Phi^S} = 0$, since $\omega_\Phi + \omega_{\Phi^S} = \omega_\Phi$ by Lemma 4.3. The conclusion follows then from Theorem 1.1. The converse is clear, also from Theorem 1.1. $\square$

Next, we want to prove Theorem 1.3. For this we will denote by Γ' the complementary graph of Γ in X^1. Observe that the cochain ω_Φ can be regarded as a cochain in X, and $C^1(X;\mathbf{Z}_2) \cong C^1(\Gamma;\mathbf{Z}_2) \oplus C^1(\Gamma';\mathbf{Z}_2)$. We need the following definition.

Definition 4.5. We say that a family of embeddings Φ is *admissible* if ω_Φ can be completed, via Γ', to a cocycle in X; i.e., there is $\eta \in C^1(\Gamma';\mathbf{Z}_2)$ so that $\delta(\omega_\Phi+\eta) = 0$ in $C^2(X;\mathbf{Z}_2)$.

Lemma 4.6. *(a) For a fake surface X, either every family of embeddings is admissible or none is.*

(b) If X has an admissible family of embeddings, then so does any subdivision obtained by deriving X away from X^1.

Proof. (a) Let Φ, Φ' be two families of embeddings, and assume Φ is admissible; i.e., there is $\eta \in C^1(\Gamma';\mathbf{Z}_2)$ so that $\delta(\omega_\Phi+\eta) = 0$ in $C^2(X;\mathbf{Z}_2)$. There is $\eta' \in C^1(\Gamma';\mathbf{Z}_2)$ so that $\omega_\Phi + \omega_{\Phi'} + \eta'$ is a coboundary in X, since ω_Φ and $\omega_{\Phi'}$ are cohomologous in Γ, by Lemma 4.3 and Proposition 4.4. Then,

$$\delta(\omega_{\Phi'}+\eta+\eta') = \delta(\omega_{\Phi'}+\eta+\eta'+\omega_\Phi+\omega_\Phi) = \delta(\omega_\Phi+\omega_{\Phi'}+\eta') + \delta(\omega_\Phi+\eta) = 0.$$

Furthermore, if $\omega_\Phi + \eta$ is a coboundary, then so is $\omega_{\Phi'} + (\eta+\eta')$.

(b) Let Φ be an admissible family of embeddings for X. A derived subdivision Y of X away from X^1 does not have any new vertices of type II or III. Thus, the embeddings $\phi_v \in \Phi$ ($v \in \Gamma$) induce embeddings $\phi'_v : lk(v,Y) \longrightarrow \mathbf{R}^2$ having $\phi_v(lk(v,X))$ as the image set. For the new vertices (all of type I) we choose arbitrary embeddings. This gives us a family of embeddings Φ' for Y such that $\omega_\Phi = \omega_{\Phi'}$. Moreover, if there is $\eta \in C^1(\Gamma';\mathbf{Z}_2)$ so that $\omega_\Phi + \eta$ is a cocycle in X, we consider the complementary graph Γ'' of X^1 in Y^1 and we take $\eta' \in C^1(\Gamma'';\mathbf{Z}_2)$ as the sum of the new 1-simplexes σ'' of Y sharing a vertex with two 1-simplexes σ, σ' of X with coefficient 0 in $\omega_\Phi + \eta$. It is not hard to check that $\omega_\Phi + \eta + \eta'$ is a cocycle in Y. $\square$

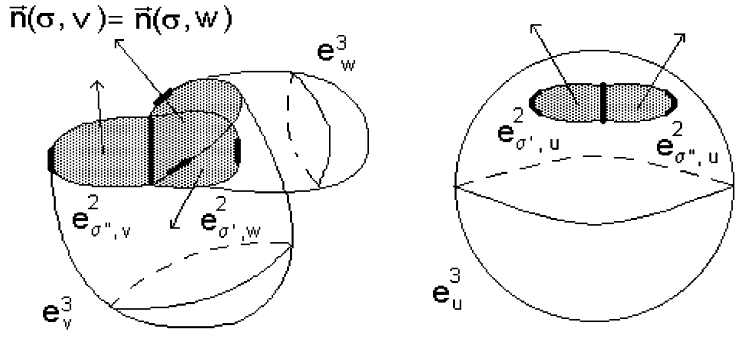


FIGURE 3

Proof of Theorem 1.3. Let Y be the first derived subdivision of X away from X^1, and let $\Gamma \subset Y$ be the corresponding subgraph of all 1-simplexes of order 3. By Lemma 4.6, if X has an admissible family of embeddings, then so does Y. Observe that any 2-simplex of Y contains at most one 1-simplex of order 3. This is why we subdivide.

Let $\Phi = \{\phi_v : lk(v, Y) \longrightarrow \mathbf{R}^2,\ v \in Y^0\}$ be an admissible family of embeddings. We consider the (oriented) CW-structure on $Z = \bigsqcup_{v \in Y^0} e^3_v$ given in the proof of Theorem 1.1. Recall that every 3-ball $e^3_v \subset \mathbf{R}^3$ had the inherited orientation from $\mathbf{R}^3$, and so we have a field of normal vectors defined on each sphere $\partial e^3_v, v \in Y^0$. In particular, each 2-cell $e^2_{\sigma,v},\ \sigma \succ v$, comes with a normal vector (independently of the orientation defined on $e^2_{\sigma,v}$). Denote this normal vector by $\vec{n}(\sigma, v)$.

We claim that we can glue these 3-cells e^3_v in such a way as to produce a 3-manifold M which collapses to a copy of the first barycentric subdivision Y' of Y. Let $\eta \in C^1(\Gamma'; \mathbf{Z}_2)$ so that $\delta(\omega_\Phi + \eta) = 0$ in Y, and let $\sigma = (v, w), \sigma' = (u, w)$ and $\sigma'' = (u, v)$ be the faces of the 2-simplex (u, v, w) of Y. We follow some rules for the gluing.

Rule (1). If $\omega_\Phi(\sigma) \neq 0$ (which implies that σ is of order 3), then we glue e^3_v and e^3_w along $e^2_{\sigma,v}$ and $e^2_{\sigma,w}$ via an orientation-preserving homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells $e^1_{\tau,v}$ to cells $e^1_{\tau,w},\ \tau \succ \sigma$. This way, we make the vectors $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ match. See Figure 3.

To complete the picture obtaining a 3-ball we could glue e^3_u (along $e^2_{\sigma',u}$ and $e^2_{\sigma'',u}$) in such a way that $\vec{n}(\sigma', u)$ matches $\vec{n}(\sigma', w)$, and $\vec{n}(\sigma'', u)$ does not match $\vec{n}(\sigma'', v)$, or vice versa. Rule (3) will give the criterion to follow, since the 1-simplexes σ' and σ'' are both of order 2.

Rule (2). If $\omega_\Phi(\sigma) = 0$ and σ is of order 3, then we glue e^3_v and e^3_w along $e^2_{\sigma,v}$ and $e^2_{\sigma,w}$ via an orientation-reversing homeomorphism (with respect to the inherited orientations from the 3-balls), taking cells of the form $e^1_{\tau,v}$ to cells $e^1_{\tau,w},\ \tau \succ \sigma$. Observe that the gluing is done so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ do not match. See Figure 4.

Notice that the gluing of the 3-cell e^3_u (along $e^2_{\sigma',u}$ and $e^2_{\sigma'',u}$) can be done in two different ways to obtain a 3-ball, depending on whether the pair $(\vec{n}(\sigma', w), \vec{n}(\sigma'', v))$ matches $(\vec{n}(\sigma', u), \vec{n}(\sigma'', u))$ or it does not, in which case neither $\vec{n}(\sigma', w)$ matches

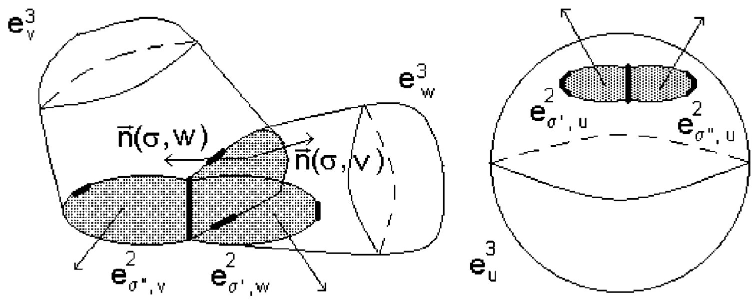


FIGURE 4

$\vec{n}(\sigma', u)$ nor $\vec{n}(\sigma'', v)$ matches $\vec{n}(\sigma'', u)$. Rule (3) will give the criterion to follow, since σ' and σ'' are again of order 2.

Rule (3). If σ is a 1-simplex of order 2 (and hence $\omega_\Phi(\sigma) = 0$) and the coefficient of σ in $\omega_\Phi + \eta$ is 0, then we glue e^3_v and e^3_w so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ do not match. Otherwise, we do the gluing so that $\vec{n}(\sigma, v)$ and $\vec{n}(\sigma, w)$ match.

This way we make sure we are building up a 3-manifold each time we consider a 2-simplex of Y. Indeed, using the fact that $\omega_\Phi + \eta$ is a cocycle, i.e., the number of faces of a 2-simplex τ of Y which occur with non-zero coefficient in $\omega_\Phi + \eta$ is either 0 or 2, it can be checked that the points on the 1-cells of the form $e^1_{\tau,v}$ have euclidean neighborhoods in the space obtained after the gluing. Let M be the resulting 3-manifold built this way. The same argument used in the proof of Theorem 1.1 proves that M collapses to a copy of Y'.

Finally, if $[\omega_\Phi + \eta] = 0$ in $H^1(X; \mathbf{Z}_2)$, say $\omega_\Phi + \eta = \delta(z)$, $z \in C^0(X; \mathbf{Z}_2)$, then

$$\delta(i^*(z)) = i^*(\delta(z)) = i^*(\omega_\Phi) + i^*(\eta) = \omega_\Phi,$$

where i^* is induced by the inclusion $\Gamma \overset{i}{\hookrightarrow} X$. Therefore, the cohomology class $\xi_X = [\omega_\Phi] \in H^1(\Gamma; \mathbf{Z}_2)$ is trivial and the conclusion follows from Theorem 1.2. $\square$

Remark 4.7. Notice that if $\xi_X = 0$, then any family of embeddings Φ for X is admissible. For if $\omega_\Phi = \delta(z)$ in Γ, then the coboundary of z in X, $\delta(z) = \omega_\Phi + \eta$ ($\eta \in C^1(\Gamma'; \mathbf{Z}_2)$), is in particular a cocycle in X. If $H^1(X; \mathbf{Z}_2) = 0$ (e.g., X simply connected), then we have the converse, since every cocycle is then a coboundary.

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Added in proof

Independently and essentially simultaneously these results were obtained by D. Repovš, N. B. Brodskij and A. B. Skopenkov; see *A classification of* 3*-thickenings of* 2*-polyhedra*, Topology and its Applications **94** (1999), 307-314.

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Departamento de Geometría y Topología, Universidad de Sevilla, Apdo 1160, 41080-Sevilla, Spain

E-mail address: fjfer@cica.es