

## P-CONTINUITY ON CLASSICAL BANACH SPACES

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ABSTRACT. Given a Banach space  $X$  and an integer  $n$ , the existence of an  $n$ -homogeneous polynomial which is not uniformly continuous with respect to the polynomial topology on  $B_X$  is investigated. We provide a complete characterization for some classical Banach spaces, while for others a surprising unresolved difficulty is encountered for a certain value of  $n$  (depending on  $X$ ).

The notion of P-continuity (definition below) was introduced and investigated in [ACL], [GGL] and [GL] where it was shown that every P-continuous linear operator is weakly compact. It was observed there that if  $X$  has a separating polynomial, i.e. a real valued polynomial which separates the origin from the unit sphere, or  $X$  has DPP and  $\ell_1 \not\hookrightarrow X$  (e.g.  $C(K)$  Asplund spaces), then every polynomial on  $B_X$  is P-continuous. Examples of 2-homogeneous polynomials on  $L_1[0, 1]$ , 3-homogeneous polynomials on  $C[0, 1]$  and  $L_\infty[0, 1]$  and  $n$ -homogeneous polynomials on  $\ell_1$ ,  $n \geq 2$ , which are not P-continuous were constructed.

In our present note we address the following general question: Given a (classical) Banach space  $X$ , for which  $n \in \mathbb{N}$  does there exist an  $n$ -homogeneous non-P-continuous polynomial on  $B_X$ ?

A simple method of construction of non-P-continuous polynomials gives us an almost complete answer (with the exception of at most one value of  $n$ ). The remaining case turns out to be surprisingly difficult, and we are able to settle the problem only for some classical Banach spaces, using various additional properties. The complete answer is described below for  $C(K)$ ,  $L_p[0, 1]$ ,  $p \geq 2$ , and  $\ell_p$ ,  $p \in I := \mathbb{N} \cup \{(\frac{2n}{2n-1})\}$ . In the remaining cases there is one value of  $n$  (indicated in the bracket) which remains unclear:  $L_p[0, 1]$ ,  $1 < p < 2$  ( $n = 2$ );  $\ell_p$ ,  $p \in [1, \infty) \setminus I$  ( $n = [p] + 1$ ).

In this note all polynomials and operators are assumed to be bounded, all spaces are real. Recall that, given Banach spaces  $X, Y$ , by  $(X \oplus Y)_\infty$  (resp.  $(X \oplus Y)_1$ ) we denote the direct sum together with the norm  $\|(x, y)\|_\infty = \max(\|x\|, \|y\|)$  (resp.  $\|(x, y)\|_1 = \|x\| + \|y\|$ ). By  $S_X$  we denote the unit sphere of  $X$ .

$\mathcal{P}^n(X)$  (resp.  $\mathcal{P}(X)$ ) denotes the space of all  $n$ -homogeneous (resp. all) polynomials on  $X$ .

Let us start with the definition of P-continuity.

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**Definition 1.** Let  $X, Y$  be Banach spaces, and let  $f : X \rightarrow Y$  be a mapping. We say that  $f$  is *P-continuous* if given  $\varepsilon > 0$  and a bounded set  $B \subset X$ , there exist scalar polynomials  $P_1, \dots, P_n$  on  $X$  and  $\delta > 0$  such that for all  $x, y \in B$ ,

$$|P_i(x - y)| < \delta, \quad i = 1, \dots, n, \quad \text{implies} \quad \|f(x) - f(y)\| < \varepsilon.$$

It is straightforward to observe that if  $f$  is a polynomial, we may restrict our attention only to  $B = B_X$  (the closed unit ball of  $X$ ) and instead of  $P_1, \dots, P_n$  use only one homogeneous polynomial.

Given  $r \in (0, 1]$  we put  $K_r = \{(x_i) \in B_{\ell_1}, x_i \geq 0, \sum x_k^r \leq 1\}$ . The following is a slight improvement of a well-known fact.

**Lemma 2.** *Let  $r \in (0, 1]$ , and let  $X$  be a separable Banach space. Then there exists a linear operator  $T : \ell_1 \rightarrow X$ ,  $\|T\| \leq 1$ , such that  $\frac{1}{2}B_X \subseteq T(K_r)$ .*

*Proof.* Choose a dense sequence  $\{x_i\}$  in  $B_X$ . Put  $T(e_i) = x_i$ . Let  $x \in X$ ,  $\|x\| = 1$ . Given  $C > 0$ , find inductively a sequence  $i_n$  in  $\mathbb{N}$  such that

$$\begin{aligned} \|Te_{i_1} - x\| &< 2^{-\frac{C}{r}}, \\ \|2^{-\frac{C}{r}}Te_{i_2} + Te_{i_1} - x\| &< 2^{-2\frac{C}{r}}, \\ \left\| \sum_{n=1}^k 2^{-(n-1)\frac{C}{r}}Te_{i_n} - x \right\| &< 2^{-\frac{kC}{r}}. \end{aligned}$$

Put  $y = \sum_{n=1}^{\infty} 2^{-(n-1)\frac{C}{r}}e_{i_n}$ . Then  $Ty = x$ ,  $\|y\|_{\ell_1} \leq \sum_{i=1}^{\infty} |y_i|^r = \frac{1}{1-2^{-\frac{C}{r}}}$ . Thus, given any  $x' \in \frac{1}{2}B_X$ , we can find (for  $C$  large enough)  $y \in K_r$  such that  $Ty = x'$ .  $\square$

Given  $p \in \mathbb{R}$ , denote by  $\bar{p}$  the largest integer *strictly smaller* than  $p$ . For  $n \geq \bar{p} + 1$  we define  $I_{p,n} : \ell_p \rightarrow \ell_1$ ,  $I_{p,n}(\{x_i\}) = \{x_i^n\}$ . Clearly,  $I_{p,n}$  is an  $n$ -homogeneous polynomial,  $I_{p,n}(B_{\ell_p}) \supset K_{\frac{p}{n}}$ .

**Lemma 3.** *Let  $X$  be a separable Banach space. Suppose  $X \cong (X_1 \oplus X_2)_{\infty}$  where  $X_1 \cong \ell_p$ ,  $1 \leq p < \infty$ , and  $X_2$  does not admit a separating polynomial. Then for every  $n \geq \bar{p} + 2$  there exists a non-P-continuous  $n$ -homogeneous polynomial on  $X$ .*

*Proof.* Let  $n \geq \bar{p} + 2$ . Denote by  $i : \ell_p \rightarrow X_1$  an isomorphism,  $\|i\| \leq 1$ . Since  $X$  is separable, there exists a separable subspace  $Y$  of  $X^*$  which is 1-norming for  $X_2$ . Choose, using Lemma 2, an operator  $T : \ell_1 \rightarrow Y$  such that  $T(K_{\frac{p}{n-1}}) \supset 2B_Y$ . We claim that the  $n$ -homogeneous scalar polynomial  $P(x, y) : (X_1 \oplus X_2)_{\infty} \cong X \rightarrow \mathbb{R}$ ,

$$P(x, y) = \langle T \circ I_{p,n-1} \circ i^{-1}(x), y \rangle,$$

is not P-continuous (on  $B_{(X_1 \oplus X_2)_{\infty}}$ ).

Indeed, let  $\delta > 0$  and  $Q$  be any homogeneous polynomial on  $X$ . Choose  $y \in S_{X_2}$  such that  $|Q(0, y)| < \delta$ . Since  $T \circ I_{p,n-1} \circ i^{-1}(B_{X_1}) \supset B_Y$ , there exists  $x \in B_{X_1}$  such that  $P(x, y) > \frac{1}{2}$  and  $P(x, 0) = 0$ . The proof is completed.  $\square$

Before passing to the next theorem, let us recall the well-known fact that every polynomial of degree less than  $p$  on  $\ell_p$  is weakly uniformly continuous when restricted to  $B_{\ell_p}$ . Thus the only possible values  $n$  for which there might exist a non-P-continuous homogeneous polynomial of degree  $n$  are  $n \geq \bar{p} + 1$ . Similarly, if  $p > 2$  every 2-homogeneous polynomial on  $X = L_p[0, 1]$  or  $X = C(K)$  is P-continuous. This follows from the fact [DJT] that every linear operator from  $X$  into  $X^*$  factors through a Hilbert space (and thus is P-continuous) combined with Proposition 1 from [GGL].

The case  $L_1[0, 1]$  will be treated later together with  $C(K)$ .

**Theorem 4.** *Let  $X = \ell_p$ ,  $p$ -noneven,  $p \geq 1$ ,  $n \geq \bar{p} + 2$ . Then there exists a non- $P$ -continuous  $n$ -homogeneous polynomial on  $X$ .*

*Let  $X = L_p[0, 1]$ ,  $p$ -noneven,  $p > 1$ ,  $n \geq 3$ . Then there exists a non- $P$ -continuous  $n$ -homogeneous polynomial on  $X$ .*

*Proof.* The  $\ell_p$  case follows immediately by Lemma 3. To resolve the  $L_p$  case, it is enough to recall that  $L_p$ ,  $p > 1$ , contains a complemented copy of  $\ell_2$  (see [D]).

We do not know the solution in case  $L_p[0, 1]$ ,  $1 < p < 2$ , and  $n = 2$ . Theorem 4 also leaves a gap for  $\ell_p$ ,  $p$ -noneven,  $n = \bar{p} + 1$ . We offer some partial solutions below, but the general case remains open and seems difficult to us. Before we proceed, let us recall the well-known fact that the space  $\ell_p$ ,  $p$ -even, does not admit a separating polynomial.  $\square$

**Proposition 5.** *Let  $1 < p$  be an odd integer. Then  $Q(x) = \sum_{i=1}^\infty x_i^p$  is non- $P$ -continuous on  $\ell_p$ .*

*Proof.* Given any  $\delta > 0$ ,  $P \in \mathcal{P}(\ell_p)$ , choose  $x \in \frac{1}{2}S_{\ell_p}$  such that  $|P(x)| < \delta$ . Write  $x = x^+ - x^-$ , where  $x_i^+ \geq 0$ ,  $x_i^- \geq 0$ ,  $\text{supp}(x^+) \cap \text{supp}(x^-) = \emptyset$ . Assume e.g.  $\|x^+\|_{\ell_p} \geq \|x^-\|_{\ell_p}$ , in particular  $\frac{1}{2^p} \geq \sum_{i=1}^\infty (x_i^+)^p \geq \frac{1}{2^{p+1}}$ . We have

$$|Q(x^+) - Q(x^+ + x)| \geq |Q(x^+ + x)| - |Q(x^+)| \geq \frac{1}{2} - \frac{1}{2^{p+1}} - \frac{1}{2^p} \geq \frac{1}{4}.$$

$\square$

**Proposition 6.** *Assume  $\frac{1}{p} + \frac{1}{2^m} = 1$  for  $m \in \mathbb{N}$ ,  $m > 1$ . Then every 2-homogeneous polynomial on  $\ell_p$  is  $P$ -continuous.*

*Proof.* Let  $Q \in \mathcal{P}(^2\ell_p)$  be given. Consider the associated operator  $T_Q : \ell_p \rightarrow \ell_p^* = \ell_{2^m}$  such that

$$Q(x) = \langle T_Q x, x \rangle.$$

By the result of [GGL], it is enough to show that  $T_Q$  is  $P$ -continuous. Consider  $R \in \mathcal{P}(^{2^m}\ell_p)$  defined as

$$R(x) = \sum_{i=1}^\infty (T_Q x)_i^{2^m}.$$

Whenever  $|R(x - y)| < \varepsilon$ , we have

$$\|T_Q(x) - T_Q(y)\|_{\ell_{2^m}} < \varepsilon^{\frac{1}{2^m}}.$$

$\square$

We now pass to a slight modification of the above method in case  $\ell_1$  imbeds into a Banach space  $X$ . As we will see, in this case the complementability assumption can be dropped.

**Proposition 7.** *Let  $X$  be a Banach space,  $\ell_1 \hookrightarrow X$ ,  $n \geq 3$ . Then there exists  $P \in \mathcal{P}(^n X)$  which is not  $P$ -continuous.*

*Proof.* Let  $Y \hookrightarrow X$ ,  $Y \cong \ell_1$ . It is convenient to split  $Y$  into  $Y \cong (Y_1 \oplus Y_2)_1$ ,  $Y_1, Y_2 \cong \ell_1$ . There exists a linear operator  $L : Y \rightarrow \ell_2$  such that  $L(B_{Y_2}) \supseteq B_{\ell_2}$  and  $L(B_{Y_1}) = 0$ . As  $L$  is 2-summing, it can be extended to  $\tilde{L} : X \rightarrow \ell_2$  ([DJT]). Then  $I_{2,n-1} \circ \tilde{L}(B_{Y_2}) \supseteq K_{\frac{2}{n-1}}$ . There exists a separable subspace  $Z$  of  $X^*$ , which

is 1-norming for  $Y_1$ . By Lemma 2 there exists a linear operator  $S : \ell_1 \rightarrow Z$ , such that  $S(K_{\frac{2}{n-1}}) \supseteq 2B_Z$ .

Finally, consider the  $(n-1)$ -homogeneous polynomial  $T = S \circ I_{2,n-1} \circ \tilde{L} : X \rightarrow Z$ . By construction,  $T(B_{Y_2}) \supseteq 2B_Z$ . We claim that the  $n$ -homogeneous scalar polynomial

$$P(x) := \langle T(x), x \rangle$$

is non-P-continuous.

Indeed, given any scalar homogeneous polynomial  $R$  on  $X$ ,  $\varepsilon > 0$ , there exists  $x_0 \in B_{Y_1}$  such that  $|R(x_0)| < \varepsilon$ . However, for some  $y_0 \in B_{Y_2}$ ,  $\langle T(y_0), x_0 \rangle \geq 1$ . Thus

$$|P(x_0 + y_0) - P(y_0)| = |\langle T(y_0), x_0 + y_0 \rangle - \langle T(y_0), y_0 \rangle| \geq 1.$$

The proof is finished.  $\square$

Proposition 7 together with the remarks before Theorem 4 and in the introduction implies the following.

**Corollary 8.** *Let  $X = C(K)$ ,  $\ell_1 \hookrightarrow X$ . Then there exists a non-P-continuous  $n$ -homogeneous polynomial of degree  $n$  if and only if  $n \geq 3$ .*

*Let  $X = L_1[0, 1]$ . Then there exists a non-P-continuous  $n$ -homogeneous polynomial if and only if  $n \geq 2$ .*

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#### REFERENCES

- [ACL] R. M. Aron, Y. S. Choi and J. G. Llavona, *Estimates by Polynomials*, Bull. Austral. Math. Soc. **52** (1995), 475–486. MR **96k**:46073
- [D] J. Diestel, *Sequences and series in Banach spaces*, Grad. Texts in Math. 92, Springer, Berlin, 1984. MR **85i**:46020
- [DJT] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, 1996. MR **96i**:46001
- [GGL] M. González, J. M. Gutiérrez and J. G. Llavona, *Polynomial continuity on  $\ell_1$* , Proc. Amer. Math. Soc. **125**, no. 5 (1997), 1349–1353. MR **97g**:46024
- [GL] J. M. Gutiérrez and J. G. Llavona, *Polynomially continuous operators*, Israel J. Math. **102** (1997), 179–183. MR **98k**:46073

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