

**A NOTE ON THE ANALYTIC FAMILIES  
OF COMPACT SUBMANIFOLDS  
OF COMPLEX MANIFOLDS**

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**ABSTRACT.** In this paper, we prove a result of the deformation of the complex structure of a submanifold. Our result is a modification of the result of Kodaira (Ann. Math 75(1), 146–162, 1962).

1. INTRODUCTIONS

It is a classical theorem of Kodaira [1] that if the first cohomological group of the normal bundle is zero, then the deformation of a compact complex submanifold within the ambient complex manifold is unobstructed. However, it is usually difficult to check if such a cohomological group is indeed zero. Furthermore, as showed in §2, in some cases, it will never be zero.

In this short note, we are going to prove: if the deformation of the complex structure of a compact complex manifold  $M$  is unobstructed in the sense of Kodaira and Spencer, and if  $M \rightarrow V$  is an embedding to the complex manifold  $V$  and  $H^1(M, T_V|_M) = 0$ , then any fiber in a neighborhood of the universal deformation space  $U$  at  $M$  can be embedded holomorphically to  $V$ .

Contrary to the case of normal bundle, it will be relatively easy to check the vanishing of the group  $H^1(M, T_V|_M)$ . For example, if the curvature of the manifold  $V$  has some kinds of positivity along  $M$ , then the group vanishes.

The typical examples of  $M$  are compact Calabi-Yau manifolds. Those manifolds admit Kähler metrics with zero Ricci curvature. In Tian [3], it is proved that the deformation of the complex structure of a Calabi-Yau manifold is unobstructed. The moduli space of a polarized Calabi-Yau manifold is then a complex orbifold.

We use the similar method as that of Kodaira [1]. That is, we construct a formal power series which gives the map we want. Then we prove the convergence of the power series. Fortunately, in our case, the original method in [1] of proving the convergence can be applied here.

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## 2. ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS

Suppose  $M$  is a compact complex manifold. We assume that the deformation of the complex structure of  $M$  is unobstructed. That is, the universal deformation space  $U$  of  $M$  is a complex manifold near  $M$  with complex dimension  $\dim H^1(M, T_M)$ , where  $T_M$  denotes the holomorphic tangent bundle of  $M$ .

If  $M$  is a Calabi-Yau manifold, then the deformation of the complex structure on  $M$  is unobstructed by the theorem of Tian [3].

We use the notations and definitions in [1].

**Definition 2.1.** Suppose  $N$  is a complex manifold of dimension  $r + n$ . By an analytic family of compact submanifolds of dimension  $n$  of  $N$  we shall mean a pair  $(\mathcal{M}, U)$  of a complex manifold  $U$  and a complex analytic submanifold  $\mathcal{M}$  of  $N \times U$  of co-dimension  $r$  which satisfies the following two conditions:

1). For Each Point  $t \in U$ , the intersection  $\mathcal{M} \cap N \times t$  is a connected, compact submanifold of  $N \times t$  of dimension  $n$ .

2). For each point  $p \in \mathcal{M}$ , there exist  $r$  holomorphic functions  $f_1 = f_1(w, t), \dots, f_r = f_r(w, t)$  defined on a neighborhood  $\mathcal{U}_p$  of  $p$  in  $N \times U$  such that

$$\text{rank} \frac{\partial(f_1, \dots, f_r)}{\partial(w^1, \dots, w^{r+d})} = r$$

and in  $\mathcal{U}_p$ , the submanifold  $\mathcal{M}$  is defined by the simultaneous equations

$$f_1(w, t) = f_2(w, t) = \dots = f_r(w, t) = 0.$$

We call  $U$  the parameter manifold or the base space of the family  $(N, U)$ . We denote the family  $(N, U)$  simply by  $N$  when we need not indicate the base space  $U$ . For each point  $t \in U$ , we set

$$M_t \times t = \mathcal{M} \cap N \times t;$$

the submanifold  $M_t$  of  $N$  thus defined will be called the fiber of  $\mathcal{M}$  over  $t$ . We may identify  $M_t \times t$  with  $M_t$  and consider  $M_t$  as a *family consisting of compact submanifolds*  $M_t, t \in U$ , of  $N$ .

**Definition 2.2.** We say  $(\mathfrak{X}, U)$  is the local total family of  $M$ , if  $U$  is a neighborhood of  $C^d$  with  $d = \dim H^1(M, T_M)$  where  $T_M$  is the holomorphic tangent bundle of  $M$ , and there exists a projection

$$\pi : \mathfrak{X} \rightarrow U$$

such that it is holomorphic, surjective, of rank  $d$  and such that for all  $t \in U$ ,  $\pi^{-1}(t)$  is a deformation of complex structure of the center fiber  $\pi^{-1}(0) = M$ .

We state the main result of this paper.

**Theorem 2.1.** *Suppose  $M$  is a compact complex manifold whose deformation of its complex structure is unobstructed. Let  $N$  be another complex manifold. Suppose that*

$$i : M \rightarrow N$$

*is a holomorphic embedding. If  $H^1(M, T_N|_M) = 0$  where  $T_N|_M$  is the restriction of the holomorphic tangent bundle of  $N$  to  $M$ , then there is a holomorphic map*

$$f : \mathfrak{X} \rightarrow N$$

*such that  $f|_M = i$ . Here  $(\mathfrak{X}, U)$  is the local total family of  $M$ . Furthermore,  $f|_{\pi^{-1}(t)}$  is an embedding of  $\pi^{-1}(t)$  to  $N$  for  $t \in U$ .*

*Proof.* Suppose  $\bigcup_{j \in I} U_j \supset \mathfrak{X}$  is an open covering. On each  $U_j$ ,  $j \in I$ , suppose  $(z_j^1, \dots, z_j^n, t)$  is a local coordinate such that

$$\pi(z_j^1, \dots, z_j^n, t) = t, \quad t \in U \subset C^d.$$

It is obvious that, for fixed  $t$ ,  $(z_j^1, \dots, z_j^n)$  will be local coordinate for  $\pi^{-1}(t) = M_t$ . We further assume that  $U_j$  is defined by

$$U_j = \{|z_j| = \text{Max}_\alpha |z_j^\alpha| < 1\}.$$

We have, however, holomorphic functions  $g_{jk}$  such that

$$z_j^\alpha = g_{jk}^\alpha(z_k, t), \quad j, k \in I,$$

for  $\alpha = 1, \dots, n$  and for  $U_j \cap U_k \neq \emptyset$ .

Now we suppose  $\bigcup V_A \supset N$  is an open covering of  $N$ . Suppose

$$i(U_j) \subset V_{A(j)}, \quad j \in I,$$

for some  $A(j)$  of  $j$ . Suppose  $(w_A^1, \dots, w_A^r)$  is the local holomorphic coordinate chart of  $N$  on  $V_A$ . And we have transition functions  $h_{AB}$  on  $V_A \cap V_B \neq \emptyset$ ,

$$w_A^s = h_{AB}^s(w_B)$$

for  $s = 1, \dots, r + n = \dim N$ . And again, we assume

$$V_j = \{|w_j| = \text{Max} |w_j^s| < 1\}.$$

For the sake of simplicity, we denote  $j$  for  $A(j)$ . In order to construct  $f$ , we need only have to construct holomorphic mappings  $(f_j)$  such that

$$f_j : U_j \rightarrow V_j, \quad j \in I,$$

satisfying

$$(2.1) \quad h_{jk}^s(f_k(z_k, t)) = f_j^s(g_{jk}(z_k, t), t) \quad \text{on } U_j \cap U_k \neq \emptyset, \quad j, k \in I,$$

for  $s = 1, \dots, r + n$ .

We set up some notations. Let

$$f_k(z, t) = f_{k|0} + f_{k|1} + \dots + f_{k|m} + \dots$$

be the decomposition of  $f_k(z_k, t)$  into homogeneous polynomials of  $t$  of degree  $m$ . Of course, each  $f_k$ 's and  $f_{k|m}$ 's are vector valued functions  $f_k = (f_k^s), f_{k|m} = (f_{k|m}^s), s = 1, \dots, r + n$ . Suppose

$$f_k^m = f_{k|0} + \dots + f_{k|m}, \quad k \in I,$$

and let  $a \equiv_m b$  mean  $a - b$  is a polynomial of  $t$  of degree bigger than or equal to  $m + 1$ .

We construct  $f_{k|m}$  inductively. First, set

$$f_{j|0} = i_j(z_j), \quad j \in I.$$

It is easy to check that

$$h_{jk}(f_k^0(z_k, t)) \equiv_0 f_j^0(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset,$$

where  $h_{jk} = (h_{jk}^s)_{s=1, \dots, r+n}$ .

Now suppose for integer  $m$ ,  $f_k^m$  is constructed, and

$$(2.2) \quad h_{jk}(f_k^m(z_k, t)) \equiv_m f_j^m(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset.$$

Define

$$(2.3) \quad \Psi_{jk}(z_k, t) \equiv_{m+1} h_{jk}(f_k^m(z_k, t)) - f_j^m(g_{jk}(z_k, t), t), \quad U_j \cap U_k \neq \emptyset.$$

Then

*Claim.*

$$\Psi_{ik} = \Psi_{ij} + \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s \quad \text{on } U_i \cap U_j \cap U_k \neq \emptyset.$$

*Proof of the Claim.* By Equation (2.3), we have

$$f_i^m(g_{ik}(z_k, t), t) \equiv_{m+1} h_{ik}(f_k^m(z_k, t)) - \Psi_{ik}(z_k, t), \quad U_i \cap U_k \neq \emptyset.$$

Thus on  $U_i \cap U_j \cap U_k \neq \emptyset$ , we have

$$\begin{aligned} h_{ij}(f_j^m(g_{jk}(z_k, t), t)) &\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k, t)) - \Psi_{jk}(z_k, t)) \\ &\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k, t))) - \frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t) \\ &\equiv_{m+1} h_{ik}(f_k^m(z_k, t)) - \frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t). \end{aligned}$$

Note that  $\Psi_{jk}^s(z_k, t) \equiv_m 0$ . So

$$\frac{\partial w_i}{\partial w_j^s} \Psi_{jk}^s(z_k, t) \equiv_{m+1} \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t).$$

We have

$$\Psi_{ij}(g_{jk}(z_k, t), t) \equiv_{m+1} \Psi_{ik}(z_k, t) - \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t), \quad U_j \cap U_k \neq \emptyset.$$

On the other hand

$$g_{jk}(z_k, t) \equiv_0 z_j, \quad U_j \cap U_k \neq \emptyset.$$

So

$$\Psi_{ij}(z_j, t) \equiv_{m+1} \Psi_{ij}(g_{jk}(z_k, t), t).$$

Thus we have

$$(2.4) \quad \Psi_{ij}(z_j, t) = \Psi_{ik}(z_k, t) - \frac{\partial w_i}{\partial w_j^s} \Big|_{t=0} \Psi_{jk}^s(z_k, t), \quad U_j \cap U_k \neq \emptyset.$$

The claim is proved.  $\square$

Suppose  $\mathcal{U} = (U_j)$  is the covering of  $M$ ; we see from Equation (2.4)  $\{\Psi_{ij}\}_{i,j \in I}$  defined a cocycle of  $Z^1(\mathcal{U}, T_N|_M)$ . Thus  $\{\Psi_{ij}\}_{i,j \in I} \in H^1(\mathcal{U}, T_N|_M)$ . With a good covering  $\mathcal{U}$ , we have  $H^1(\mathcal{U}, T_N|_M) = H^1(M, T_N|_M)$  and by the assumption, the latter is zero. So we can find  $\{\Psi_j\}$  such that

$$\Psi_{jk} = \Psi_j - \frac{\partial w_j}{\partial w_k^s} \Big|_{t=0} \Psi_k^s, \quad U_j \cap U_k \neq \emptyset.$$

We then define  $f_k^{m+1}$  inductively as

$$(2.5) \quad f_k^{m+1}(z_k, t) = f_k^m(z_k, t) + \Psi_k(z_k, t), \quad k \in I.$$

With this definition, we have

$$\begin{aligned} h_{jk}(f_k^{m+1}(z_k, t)) &\equiv_{m+1} h_{jk}(f_k^m(z_k, t) + \Psi_k(z_k, t)) \\ &\equiv_{m+1} h_{jk}(f_k^m(z_k, t)) + \frac{\partial w_j}{\partial w_k^s} \Psi_k^s(z_k, t) \\ &\equiv_{m+1} \Psi_{jk}(z_k, t) + \frac{\partial w_j}{\partial w_k^s} \Psi_k^s(z_k, t) + f_j^m(g_{jk}(z_k, t), t) \\ &\equiv_{m+1} \Psi_j(z_j, t) + f_j^m(g_{jk}(z_k, t), t) \\ &\equiv_{m+1} f_j^{m+1}(g_{jk}(z_k, t), t). \end{aligned}$$

Now we have got a formal series

$$f_{k|0} + f_{k|1} + \dots + f_{k|m} + \dots, \quad k \in I,$$

which satisfies Equation (2.2) for any  $m$ . If it converges, then  $f_k$ 's and  $f$  will be holomorphic and  $f$  will satisfy Equation (2.1). We put off the proof of the convergence to the next section. At this moment, we assume the convergence is true.

By continuity, for fixed  $t$ ,  $t$  being sufficiently small,  $f$  will be an immersion on  $\pi^{-1}(t)$ . We claim that for sufficiently small  $t$ ,  $f$  is an embedding. Suppose not, then we can find  $t_n \rightarrow 0$  and  $x_n, y_n \in \pi^{-1}(t_n)$  such that  $x_n \neq y_n$  but  $f(x_n, t) = f(y_n, t)$ . Suppose  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . We see  $x = y$ ; otherwise it will contradict the fact that  $i$  is an embedding. But if  $x = y$ , it will contradict the fact that  $f$  is an immersion on each fiber.

Now we give an important example of manifolds such that  $H^1(M, T_N|_M) = 0$ .

**Proposition 2.1.** *Let  $M$  be a simply connected Calabi-Yau threefold.  $M \rightarrow CP^D$  is an embedding. Then we have*

$$H^1(M, T_{CP^D}|_M) = 0.$$

*Proof.* From the Euler exact sequence

$$(2.6) \quad 0 \rightarrow \mathcal{C} \rightarrow \oplus_{D+1} \mathcal{O}(1) \rightarrow T_{CP^D} \rightarrow 0.$$

We have the long exact sequence

$$\dots \rightarrow H^1(M, \oplus_{d+1} \mathcal{O}(1)) \rightarrow H^1(M, T_{CP^D}|_M) \rightarrow H^2(M, \mathcal{O}) \rightarrow \dots.$$

By the Kodaira Vanishing theorem

$$H^1(M, \oplus_{D+1} \mathcal{O}(1)) = 0.$$

By the Dolbeault theorem and Serre Duality  $H^2(M, \mathcal{O}) = 0$ , we have

$$H^1(M, T_{CP^D}|_M) = 0.$$

□

**Corollary 2.1.** *Suppose  $M$  is a simply connected Calabi-Yau threefold. If  $M$  is embedded to some  $CP^D$ , then  $M_t = \pi^{-1}(t)$  can also be embedded to the same  $CP^D$  for small  $t$ .*

The following example showed that, in general,  $H^1(M, T_{CP^D}|_M/T_M) \neq 0$ .

**Example.** Suppose  $\mathcal{N} = T_{CP^D}|_M/T_M$  is the normal bundle of  $M$  in  $CP^D$  in the previous proposition. Then, in general,  $H^1(M, \mathcal{N}) \neq 0$ .

*Proof.* We have the exact sequence:

$$0 \rightarrow T_M \rightarrow T_{CP^D}|_M \rightarrow \mathcal{N}|_M \rightarrow 0.$$

So the long exact sequence gives

$$\begin{aligned} \cdots \rightarrow H^1(M, T_{CP^D}|_M) &\rightarrow H^1(M, \mathcal{N}) \rightarrow H^2(M, T_M) \\ &\rightarrow H^2(M, T_{CP^D}|_M) \rightarrow \cdots \end{aligned}$$

However, by Serre Duality,  $H^2(M, T_M) = H^{1,1}(M)$ . And it is easy to see from the Euler Sequence (2.6) that  $\dim H^2(M, T_{CP^D}|_M) = 1$ . Thus in general  $\dim H^1(M, \mathcal{N}) \geq \dim H^{1,1}(M) - 1$  and is not zero.

### 3. THE CONVERGENCE

Now we shall show that the power series  $\sum f_k(z_k, t)$  in Equation (2.5) for all  $k \in I$ , converge for  $|t| < \varepsilon_0$ ,  $\varepsilon_0$  being a sufficiently small positive number, provided that we choose for each  $m$  the homogeneous polynomials  $f_{k|m+1}(z_k, t)$ ,  $k \in I$ , satisfying (2.3) in a proper manner.

For any vector  $\xi = (\xi^1, \xi^2, \dots, \xi^\lambda, \dots)$ , we define

$$|\xi| = \text{Max}_\lambda |\xi^\lambda|.$$

Consider a power series

$$\xi(z, u) = \sum \xi_{lm, \dots, n} u_1^l u_2^m \cdots u_q^n$$

in  $u_1, \dots, u_q$  whose coefficients  $\xi_{lm, \dots, n}$  are vector valued functions of  $z$  and a power series

$$a(u) = a_{lm, \dots, n} u_1^l u_2^m \cdots u_q^n, \quad a_{lm, \dots, n} \geq 0.$$

We indicate by writing  $\xi(z, u) \ll a(u)$  that

$$|\xi_{lm, \dots, n}(z)| \leq a_{lm, \dots, n}.$$

Let

$$A(t) = \frac{a}{16b} \sum_{n=1}^{\infty} \frac{1}{n^2} b^n (t_1 + \cdots + t_l)^n,$$

where  $a$  and  $b$  are positive constants. We have

$$A(t)^\gamma \ll \left(\frac{a}{b}\right)^{\gamma-1} A(t), \quad \gamma = 2, 3, \dots$$

For our purpose it suffices to prove the inequalities

$$f_k(z_k, t) \ll A(t), \quad i \in I.$$

In what follows we denote by  $c_0, c_1, c_2, \dots$  positive constants *which are greater than 1*. We may assume that

$$\left| \frac{\partial h_{ij}^\lambda}{\partial w_j^\mu} \right| < c_0, \quad c_0 > 1.$$

For the sake of simplicity, we denote  $f_k^m(z_k, t) - i(z_k)$  by  $f_k^m(z_k, t)$ . Then for sufficiently large  $a$ , we have

$$f_k^1(z_k, t) \ll \frac{a}{16} (t_1 + \cdots + t_d) \ll A(t), \quad k \in I.$$

Now assuming the inequalities

$$f_k^m(z_k, t) \ll A(t), \quad k \in I,$$

for an integer  $m \geq 1$ , we shall estimate the coefficients of the homogeneous polynomials  $\Psi_{ik}(z, t)$ . We expand  $h_{ik}$  and  $g_{ik}$  into power series, whose coefficients are vector valued holomorphic functions:

$$h_{ik}(w_k) \ll \sum_{\alpha=0}^{\infty} c_1^\alpha (w_k^1 + \dots + w_k^{r+n})^\alpha,$$

$$g_{ik}(z_k) \ll \sum_{\alpha=0}^{\infty} c_1^\alpha (z_k^1 + \dots + z_k^n)^\alpha$$

for  $i, k \in I$ .

Recall that in Equation (2.3)

$$\Psi_{jk} = [h_{jk}(f_k^m(z_k, t))]_{m+1} - [f_j^m(g_{jk}(z_k, t), t)]_{m+1},$$

where  $[a]_{m+1}$  is a polynomial of  $t$  of degree bigger than  $m$ . First we estimate  $[h_{jk}(f_k^m(z_k, t))]_{m+1}$ . The terms which are linear in  $h_{jk}$  contribute nothing to  $[h_{jk}(f_k^m(z_k, t))]_{m+1}$ . So we have

$$[h_{jk}(f_k^m(z_k, t))]_{m+1} \ll \sum_{\alpha=2}^{\infty} c_1^\alpha (r+n)^\alpha A(t)^\alpha$$

$$\ll c_1(r+n)A(t) \sum_{\alpha=1}^{\infty} \left(\frac{c_1(r+n)a}{b}\right)^\alpha.$$

Assuming that

$$b > 2c_1(r+n)a,$$

we obtain therefore

$$(3.1) \quad [h_{jk}(f_k^m(z_k, t))]_{m+1} \ll 2c_1^2 r^2 a b^{-1} A(t).$$

On the other hand

$$[f_j^m(g_{jk}(z_k, t), t)]_{m+1} = [f_j^m(g_{jk}(z_k, t), t) - f_j^m(z_j, t)]_{m+1}.$$

Denote by  $U_i^\delta$  the subdomain of  $U_i$  consisting of all points  $z_j = (z_j, 0)$ ,  $|z_j| < 1 - \delta$ . We fix a positive number  $\delta$  such that  $\{U_i^\delta | i \in I\}$  forms a covering of  $M$ . Take a point  $z \in U_k \cap U_i^\delta$  and let  $z_k$  and  $z_j$  be the local coordinates of  $z$  on  $U_k$  and  $U_j$  respectively. Obviously, we have

$$z_j = g_{jk}(z_k, 0), \quad |z_k| < 1, |z_j| < 1 - \delta.$$

Letting  $y = (y_1, \dots, y_n)$ , we expand the coefficients of polynomial  $f_i^m(z_j + y, t)$  into power series. Suppose  $|y| < \delta$ ; we have

$$[f_j^m(z_j + y, t) - f_j^m(z_j, t)]_{m+1} \ll A(t) \left( \prod_{\alpha=1}^n \left(1 - \frac{|y_\alpha|}{\delta}\right)^{-1} - 1 \right).$$

Now if  $t$  is small, and  $\mu = \text{Max}_{j,k} |g_{jk}(z_k, t) - z_j| < \delta$ , then

$$(3.2) \quad [f_j^m(g_{jk}(z_k, t), t)]_{m+1} \ll \left( \left(1 - \frac{\mu}{\delta}\right)^{-n} - 1 \right) A(t).$$

So from Equation (3.1) and (3.2), we have

$$\Psi_{jk} \ll (2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))A(t).$$

We take an arbitrary point  $z \in U_k \cap U_i$  and choose a domain  $U_j^\delta$ , which contains  $z$ ; then

$$\Psi_{ik} = \frac{\partial w_i}{\partial w_j} \Psi_{jk} - \frac{\partial w_i}{\partial w_j} \Psi_{ji}.$$

Thus

$$\Psi_{ik} \ll 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))A(t).$$

Let  $c_3 = 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1 - \frac{\mu}{\delta})^{-n} - 1))$ . Then

$$(3.3) \quad \Psi_{jk} \ll c_3A(t), \quad z \in U_k \cup U_i.$$

The following lemma can be proved by an elementary consideration [2].

**Lemma 3.1.** *We can choose the homogeneous polynomials  $\Psi_i, i \in I$ , satisfying*

$$\Psi_{ik} = \Psi_i - \frac{\partial w_i}{\partial w_k^s} \Psi_k^s$$

*in such a way that*

$$\Psi_i \ll c_3c_4A(t)$$

*where  $c_4 > 1$  is a constant independent of  $m$ .* □

Since  $\mu$  is independent of  $m$ , and  $\mu \rightarrow 0$  as  $t \rightarrow 0$ , we can choose  $t$  small enough such that  $2c_0((1 - \mu/\delta)^{-n} - 1) < \frac{1}{2}$ . Choosing  $b$  large enough so that  $4c_0c_1^2(r+n)^2ab^{-1} < \frac{1}{2}$ , we have

$$f_j^{m+1} \ll A(t), \quad j \in I.$$

The convergence of the power series is proved.

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