PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 713–715
S 0002-9939(99)05071-6
Article electronically published on March 3, 1999

QUASI-ISOMORPHISMS OF KOSZUL COMPLEXES

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(Communicated by Wolmer V. Vasconcelos)

ABSTRACT. Let $f : A \to B$ be a surjective homomorphism of noetherian local commutative rings that induces an isomorphism between the first Koszul homology modules and an epimorphism between the second Koszul homology modules. Then f induces isomorphisms between Koszul homology modules in all dimensions.

Let $(A, \mathfrak{m}, k)$ be a local noetherian (commutative with unit) ring, I an ideal of A, $B = A/I$, and $\mathfrak{n}$ the maximal ideal of B. Let $\{x_1, \dots, x_n\}$ be a minimal set of generators of the ideal $\mathfrak{m}$ of A, and $y_1, \dots, y_n$ the images of these elements in B. Let $\{v_1, \dots, v_r\}$ be a minimal set of generators of the ideal $\mathfrak{n}$ of B. Fix for each $1 \le j \le n$ elements $b_{j\alpha} \in B$ such that $y_j = \sum_{\alpha=1}^r b_{j\alpha} v_\alpha$. Let $E = A\langle X_1, \dots, X_n; dX_i = x_i\rangle$ be the Koszul complex associated to the elements $x_1, \dots, x_n$ of A, and $E' = B\langle V_1, \dots, V_r; dV_i = v_i\rangle$ be the Koszul complex associated to the elements $v_1, \dots, v_r$ of B. By a little abuse of language we denote $H_*(A) = H_*(E)$, $H_*(B) = H_*(E')$ (they do not depend, up to isomorphism, of the minimal set of generators of the maximal ideal). Let $f : E \to E'$ be the homomorphism of complexes extending the projection map $A \to B$ by sending X_i to $\sum_{\alpha=1}^r b_{j\alpha} V_\alpha$.

L. L. Avramov and E. S. Golod [4, Proposition 1] show that if the ideal I is generated by a regular sequence which is part of a minimal system of generators of the ideal $\mathfrak{m}$, then $H_*(f) : H_*(A) \to H_*(B)$ is an isomorphism. In [8] (see [9, (2.3.6)] S. S. Strogalov shows that the converse also holds. The following result shows that it is enough to consider the first two homology modules:

Proposition 1. *If $H_1(f)$ is an isomorphism and $H_2(f)$ is surjective, then the ideal I is generated by a regular sequence which is part of a minimal system of generators of the ideal $\mathfrak{m}$.*

Proof. Let $0 \to U \to F \xrightarrow{p} \mathfrak{m} \to 0$ be an exact sequence of A-modules with F free with basis $\{z_1, \dots, z_n\}$ and $p(z_i) = x_i$, $1 \le i \le n$, and let $0 \to U' \to F' \xrightarrow{p'} \mathfrak{n} \to 0$ be an exact sequence of B-modules with F' free with basis $\{z'_1, \dots, z'_r\}$ and $p'(z'_i) = v_i$, $1 \le i \le r$. Let $g : F \wedge F \to F$, $g(a \wedge b) = p(a)b - p(b)a$, and $V = \mathrm{Im}(g)$. Define similarly $g' : F' \wedge F' \to F'$ and $V' = \mathrm{Im}(g')$. We have isomorphisms $U/V = H_1(A)$, $U'/V' = H_1(B)$.

Received by the editors February 10, 1997 and, in revised form, May 5, 1998.
1991 *Mathematics Subject Classification.* Primary 13H10, 13D03.
Key words and phrases. Koszul homology, regular sequence, complete intersection.

Consider now the André-Quillen homology modules [1], [6]. We have a commutative diagram of k-vector spaces with exact rows [1, 15.12]

$$\begin{tikzcd}
0 \arrow[r] & H_2(A,k,k) \arrow[r, "\alpha"] \arrow[d, "\lambda"] & H_1(A) \arrow[r] \arrow[d, "H_1(f)"] & F/\mathfrak{m}F \arrow[r, "\beta"] \arrow[d, "\mu"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(B,k,k) \arrow[r, "\alpha'"] & H_1(B) \arrow[r] & F'/\mathfrak{n}F' \arrow[r, "\beta'"] & \mathfrak{n}/\mathfrak{n}^2 \arrow[r] & 0
\end{tikzcd}$$

where μ is the map such that $\mu(z_j + \mathfrak{m}F) = \sum_{\alpha=1}^{r} b_{j\alpha}(z'_\alpha + \mathfrak{n}F')$, $1 \le j \le n$, and λ is the canonical map.

The maps β and β' are isomorphisms and so α and α' are also isomorphisms. Since $H_1(f)$ is an isomorphism by hypothesis, we see that λ is an isomorphism.

Consider the commutative diagram of exact rows [6, 10.4]:

$$\begin{tikzcd}
\bigwedge^2 H_1(A) \arrow[r] \arrow[d] & H_2(A) \arrow[r] \arrow[d, "H_2(f)"] & H_3(A,k,k) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
\bigwedge^2 H_1(B) \arrow[r] & H_2(B) \arrow[r] & H_3(B,k,k) \arrow[r] & 0
\end{tikzcd}$$

Since $H_2(f)$ is surjective we have that ε is surjective and so from the Jacobi-Zariski exact sequence [1, 5.1] associated to $A \to B \to k$

$$H_3(A,k,k) \xrightarrow{\varepsilon} H_3(B,k,k) \to H_2(A,B,k) \to H_2(A,k,k) \xrightarrow{\sim} H_2(B,k,k)$$

we have $H_2(A,B,k) = 0$, i.e., the ideal I is generated by a regular sequence [1, 6.25].

Finally, the same Jacobi-Zariski exact sequence

$$H_2(A,k,k) \xrightarrow{\sim} H_2(B,k,k) \to H_1(A,B,k) = I/\mathfrak{m}I \to H_1(A,k,k) = \mathfrak{m}/\mathfrak{m}^2$$

shows that $I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$ is injective.

Remarks. i) We can see in the proof that in order to assure that I is generated by a regular sequence, it is enough to assume that $H_1(f)$ is injective and $H_2(f)$ surjective.

ii) If the ideal I is of finite projective dimension and $H_2(f)$ is surjective, then I is generated by a regular sequence. The proof is the same, having in mind that by the assumption on the projective dimension of I, a result of L. L. Avramov [3] (see [7, Lemma 1]) gives us that $H_2(A,k,k) \to H_2(B,k,k)$ is injective.

Corollary 2. *If $H_1(f)$ is an isomorphism and B is a complete intersection ring, then A is complete intersection.*

Proof. It suffices to show that $H_2(f)$ is surjective. In the commutative diagram

$$\begin{tikzcd}
\bigwedge^2 H_1(A) \arrow[r, "\gamma_2"] \arrow[d, "\simeq"] & H_2(A) \arrow[d, "H_2(f)"] \\
\bigwedge^2 H_1(B) \arrow[r, "\gamma'_2"] & H_2(B)
\end{tikzcd}$$

we know that γ'_2 is an isomorphism [10, Theorem 6], and so $H_2(f)$ is surjective.

Remark. If A contains a field, B is complete intersection and $H_1(f)$ is injective, then A is complete intersection: from the diagram

$$\begin{array}{ccc} \bigwedge^* H_1(A) & \longrightarrow & H_*(A) \\ \Big\downarrow{\scriptstyle \bigwedge^* H_1(f)} & & \Big\downarrow \\ \bigwedge^* H_1(B) & \xrightarrow{\sim} & H_*(B) \end{array}$$

we deduce that $\bigwedge^* H_1(A) \to H_*(A)$ is injective. Then it follows from a result of W. Bruns [5], based on deep results on commutative algebra, that A is complete intersection.

We end this paper with a slight generalization of a result of E. F. Assmus. Let $(A, \mathfrak{m}, k)$ be a local noetherian ring. We know that A is complete intersection if and only if the canonical homomorphism $\bigwedge^* H_1(A) \to H_*(A)$ is an isomorphism, i.e., $H_*(A)$ is a free graded (anticommutative) k-algebra generated by the degree 1 elements. The "only if" part is [10, Theorem 6], and the "if" part is [2, Theorem 2.7] (see also [2] and [5] for related characterizations). This result by Assmus can be stated in a slightly stronger form:

Proposition 3. *If $H_*(A)$ is a free graded k-algebra, then A is complete intersection.*

Proof. Let $H_*(A) = \mathbf{S}^* M$, where M is a graded k-module and $\mathbf{S}^*$ denotes the graded symmetric k-algebra functor (i.e., symmetric k-algebra S on the even degree part, and exterior k-algebra $\wedge$ on the odd degree part). By [2, Theorem 2.7] it suffices to show that $M_2 = 0$. We have $S^n M_2 \subset H_{2n}(A)$ for all $n \geq 0$, and so if $M_2 \neq 0$ we would have $H_{2n}(A) \neq 0$ for all n.

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