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# TYPE $II_\infty$ FACTORS GENERATED BY PURELY INFINITE SIMPLE C*-ALGEBRAS ASSOCIATED WITH FREE GROUPS

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ABSTRACT. Let $\Gamma = G_1 * G_2 * ... * G_n * ...$ be a free product of at least two but at most countably many cyclic groups. With each such group $\Gamma$ we associate a family of C*-algebras, denoted $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ and generated by the reduced group C*-algebra $C_r^*\Gamma$ and a collection $\mathcal{P}_\Lambda$ of projections onto the $\ell^2$-spaces over certain subsets of $\Gamma$. We determine $W^*(\Gamma, \mathcal{P}_\Lambda)$, the weak closure of $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ in $\mathcal{L}(\ell^2(\Gamma))$, and use this result to show that many of the C*-algebras in question are non-nuclear.

## 0. INTRODUCTION

In recent years a great deal of progress has been made towards the classification of separable, simple C*-algebras. This program, initiated by George Elliott [6] and Mikael Rørdam [14], has been advanced significantly through combined efforts of a number of researchers. The progress has been particularly visible in the subcategory of unital, purely infinite and nuclear C*-algebras; e.g. see [7], [9], [13]. And yet a number of important related problems still remain open. One such outstanding problem is whether there exists a simple C*-algebra containing both finite and infinite projections. Partially motivated by the desire to find an example of such an algebra the second named author initiated investigations of a class of C*-algebras related to free-product groups [17], [18].

Let $\Gamma = G_1 * G_2 * ... * G_n * ...$ be a free product of finitely or countably many cyclic groups. For each $i$ we fix a generator $g_i$ of $G_i$. For any $h \in \Gamma \setminus \{e\}$ let $\Gamma(h)$ be the set of reduced words, in the fixed set $\{g_i\}$ of generators of $\Gamma$, whose initial segments coincide with $h$. Let $P_h$ denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma(h))$, and let $C_r^*(\Gamma, P_h)$ be the C*-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ generated by the reduced group C*-algebra $C_r^*\Gamma$ and the projection $P_h$ [17], [18]. This construction is somewhat similar, and actually related to those of [2], [12], [15].

Against the hope of finding a counterexample to the outstanding problem mentioned above, the second named author showed [17], [18] that all C*-algebras of the form $C_r^*(\Gamma, P_h)$ are either purely infinite (see [5] for the definition) and simple, or extensions of purely infinite, simple C*-algebras by the compacts. This result brought several natural questions about the finer structure of these algebras. The

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most important one is whether the algebras are nuclear, and thus fit into the classifiable category, or not. In this article we provide a partial answer to this question. Our investigations are carried out for a larger class of C*-algebras, containing those considered in [17], [18].

Let $\Lambda$ be a subset of $\Gamma$ and denote $\mathcal{P}_\Lambda = \{P_h : h \in \Lambda\}$. We define $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ as the C*-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ generated by $C_r^*\Gamma$ and $\mathcal{P}_\Lambda$. Let $\Gamma_\Lambda$ be the subgroup of $\Gamma$ generated by those of $\{g_i^{\pm 1}\}$ which do not occur as final letters of reduced forms of elements in $\Lambda$. We denote by $W^*(\Gamma, \mathcal{P}_\Lambda)$ the weak closure of $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ in $\mathcal{L}(\ell^2(\Gamma))$.

The main result of this article, Theorem I, states that there is an isomorphism

$$W^*(\Gamma, \mathcal{P}_\Lambda) \cong W^*(\Gamma_\Lambda) \otimes \mathcal{L}(\mathcal{H}),$$

where $\mathcal{H}$ is an infinite dimensional, separable Hilbert space, and $W^*(\Gamma_\Lambda)$ denotes the standard group von Neumann algebra. Since the subgroups $\Gamma_\Lambda$ themselves are isomorphic to free-products of cyclic groups, all of them, with the exceptions of $\Gamma_\Lambda \cong \mathbb{Z}, \mathbb{Z}_k, \mathbb{Z}_2 * \mathbb{Z}_2$, are non-amenable. We are then able to conclude that for non-amenable $\Gamma_\Lambda$ the von Neumann algebras $W^*(\Gamma, \mathcal{P}_\Lambda)$ are not injective and, consequently, the corresponding C*-algebras $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ are non-nuclear (Theorem II). Thus, our construction yields a large class of non-nuclear, purely infinite, simple C*-algebras, which are not classifiable by the existing $K$-theoretic invariants. It seems plausible that these algebras may become helpful examples when the classification program extends beyond the class of nuclear C*-algebras. Having this ultimate objective in mind we carry out a detailed investigation, in a separate paper [16], of the structure of the algebras $C_r^*(\Gamma, \mathcal{P}_\Lambda)$.

## 1. Main results

**1.0.** Throughout this article $\Gamma$ denotes a free product of finitely or countably many cyclic groups $G_i$. Let $e$ denote the unit of $\Gamma$ and let

$$\mathcal{G} = \{g_1, g_2, ..., g_n, ...\}$$

be a fixed set of generators of $\Gamma$, where $g_i$ is a generator of $G_i$.

If $g_i$ has finite order $m$, then we make a convention that elements of $G_i$ be written as $e, g_i, g_i^2, ..., g_i^{m-1}$. Then each element of $\Gamma$ can be uniquely written as a reduced word

$$g_{i_1}^{\epsilon_1} g_{i_2}^{\epsilon_2} ... g_{i_k}^{\epsilon_k}, \quad \epsilon_j = +1 \text{ or } -1.$$

A word is called *reduced* if it does not contain a subword of the form $hh^{-1}$, $h \in \Gamma$.

Let $\{\xi_h : h \in \Gamma\}$ be the standard basis of the Hilbert space $\ell^2(\Gamma)$, where $\xi_h(s) = 1$ if $h = s$ and $\xi_h(s) = 0$ if $h \neq s$. Let $L : \Gamma \longrightarrow \mathcal{L}(\ell^2(\Gamma))$ be the left regular representation, i.e. $L_h(\xi_s) = \xi_{hs}$ for $h, s \in \Gamma$. The reduced group C*-algebra $C_r^*\Gamma$ is generated by $\{L_h : h \in \Gamma\}$, and the group von Neumann algebra $W^*(\Gamma)$ is the weak closure of $C_r^*\Gamma$ [8, Vol. II, 6.7].

For a reduced word $h \in \Gamma \setminus \{e\}$ let $\Gamma(h)$ be the set of all reduced words in $\Gamma$ whose initial segments coincide with $h$. Let $P_h$ be the orthogonal projection from $\ell^2(\Gamma)$ onto the subspace $\ell^2(\Gamma(h))$. For a subset $\Lambda$ of $\Gamma$ let $\mathcal{P}_\Lambda$ denote the set of projections $\{P_h : h \in \Lambda\}$. We define $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ as the C*-subalgebra of $\mathcal{L}(\ell^2(\Gamma))$ generated by $C_r^*\Gamma$ and $\mathcal{P}_\Lambda$. We denote by $W^*(\Gamma, \mathcal{P}_\Lambda)$ the weak closure of $C_r^*(\Gamma, \mathcal{P}_\Lambda)$.

It is shown in [17, 1.1] that if $h$ is a reduced word ending with $g_i^{\pm 1}$, then

$$C_r^*(\Gamma, P_h) = C_r^*(\Gamma, P_{g_i}) = C_r^*(\Gamma, P_{g_i^{-1}}).$$

Similarly, $P_h \in C_r^*(\Gamma, \mathcal{P}_\Lambda)$ if and only if $P_{g_i} \in C_r^*(\Gamma, \mathcal{P}_\Lambda)$. Thus, it suffices to consider the case when $\Lambda$ is a subset of $\mathcal{G}$, and from now on we assume that this is indeed the case.

Let $\Lambda \subseteq \mathcal{G}$. We define $\Gamma_\Lambda$ as the subgroup of $\Gamma$ generated by $\mathcal{G} \setminus \Lambda$. It turns out that the subgroup $\Gamma_\Lambda$ plays an important role in determining the structures of $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ and $W^*(\Gamma, \mathcal{P}_\Lambda)$. We denote by $\mathcal{H}$ an infinite dimensional, separable Hilbert space.

**1.1. Theorem I.** *There exists a $*$-isomorphism*

$$W^*(\Gamma, \mathcal{P}_\Lambda) \cong W^*(\Gamma_\Lambda) \otimes \mathcal{L}(\mathcal{H}).$$

*Furthermore, the commutant $W^*(\Gamma, \mathcal{P}_\Lambda)'$ is $*$-isomorphic to $W^*(\Gamma_\Lambda)$.*

In order to prove Theorem I we need two lemmas. Let $R : \Gamma \longrightarrow \mathcal{L}(\ell^2(\Gamma))$ be the right regular representation, i.e. $R_h(\xi_s) = \xi_{sh^{-1}}$ for $h, s \in \Gamma$. If $f \in \ell^2(\Gamma)$ and $R_f \in \mathcal{L}(\ell^2(\Gamma))$, then

$$R_f(\xi_s) = \sum_{h\in\Gamma} f(h)\xi_{sh^{-1}} \quad \forall s \in \Gamma.$$

**1.2. Lemma.** *Let $f \in \ell^2(\Gamma)$ and $R_f \in \mathcal{L}(\ell^2(\Gamma))$ be self-adjoint. Let $g \in \mathcal{G}$ (i.e. there exists $i$ such that $g = g_i$). Then $R_f P_g = P_g R_f$ if and only if $\{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma_{\{g\}}$.*

*Proof.* Clearly, $R_f = R_f^*$ if and only if $f(s) = \overline{f(s^{-1})}$ for all $s \in \Gamma$. Also, $R_f P_g = P_g R_f$ if and only if the subspace $\ell^2(\Gamma(g))$ is $R_f$-invariant (i.e. $R_f(\ell^2(\Gamma(g))) \subseteq \ell^2(\Gamma(g))$), i.e. $R_f(\xi_h) \in \ell^2(\Gamma(g))$ for all $h \in \Gamma(g)$. This is equivalent to

$$\{\Gamma(g)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g) \quad \text{whenever} \quad f(s) \neq 0.$$

Notice that $f(s) \neq 0$ if and only if $f(s^{-1}) \neq 0$. Thus, if $R_f P_g = P_g R_f$, then the function $f$ must satisfy

$$\{s \in \Gamma : f(s) \neq 0\} \subseteq \{s \in \Gamma : \{\Gamma(h)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g)\}.$$

We claim that $\{\Gamma(g)s^{-1}\} \cup \{\Gamma(g)s\} \subseteq \Gamma(g)$ if and only if neither $g$ nor $g^{-1}$ is contained as a factor in the reduced word $s$. In order to prove this claim we consider separately two cases:

(1) $g$ has infinite order,
(2) $g$ has finite order $m$.

*Case* (1). Suppose that $s$ contains a factor $g$ or $g^{-1}$, i.e. in reduced form

$$s = g_{i_1}^{\epsilon_1} ... g_{i_r}^{\epsilon_r} g^{\epsilon} g_{i_{r+1}}^{\epsilon_{r+1}} ... g_{i_k}^{\epsilon_k},$$

where $\epsilon, \epsilon_i = 1$ or $-1$. If $\epsilon = 1$, set $h = g g_{i_{r+1}}^{\epsilon_{r+1}} ... g_{i_k}^{\epsilon_k}$. Since $g_{i_{r+1}}^{\epsilon_{r+1}} \neq g^{-1}$, we have $h \in \Gamma(g)$. Then $hs^{-1} \notin \Gamma(g)$, because $g_{i_r}^{\epsilon_r} \neq g^{-1}$. If $\epsilon = -1$, set $h = g(g_{i_r}^{\epsilon_r})^{-1}(g_{i_{r-1}}^{\epsilon_{r-1}})^{-1}...(g_{i_1}^{\epsilon_1})^{-1}$, again an element of $\Gamma(g)$. Also $hs \notin \Gamma(g)$, due to reasoning as above. Thus, $\{\Gamma(g)s\} \cup \{\Gamma(g)s^{-1}\} \not\subseteq \Gamma(g)$.

Conversely, suppose that the reduced form of $s$ contains neither $g$ nor $g^{-1}$ as a factor, and let $h \in \Gamma(g)$. Then the initial letter $g$ of $h$ survives all cancellations in $hs$ (or $hs^{-1}$) and, thus, $\{\Gamma(g)s\} \cup \{\Gamma(g)s^{-1}\} \subseteq \Gamma(g)$.

*Case* (2). By our convention, all elements of the subgroup $G_i$ $(g = g_i)$ are written as $e, g, g^2, ..., g^{m-1}$. Suppose first that the reduced form of $s$ contains $g$ as a factor, i.e.

$$s = g_{i_1}^{\epsilon_1} ... g_{i_r}^{\epsilon_r} g^n g_{i_{r+1}}^{\epsilon_{r+1}} ... g_{i_k}^{\epsilon_k},$$

where $1 \le n \le m-1$, and $g_{i_r}$ and $g_{i_{r+1}}$ are different from $g$. Set

$$h = g^{m-n}(g_{i_r}^{\epsilon_r})^{-1}(g_{i_{r-1}}^{\epsilon_{r-1}})^{-1} ... (g_{i_1}^{\epsilon_1})^{-1}.$$

Then $hs \in \Gamma(g_{i_{r+1}}^{\epsilon_{r+1}})$ and, hence, $hs \notin \Gamma(g)$.

Conversely, if $s$ does not contain a factor $g$, then $hs, hs^{-1}$ are in $\Gamma(g)$ for any $h \in \Gamma(g)$, i.e. $\{\Gamma(g)s\} \cup \{\Gamma(g)s^{-1}\} \subseteq \Gamma(g)$.

Consequently, $R_f P_g = P_g R_f$ if and only if $\{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma_{\{g\}}$, as desired.

□

**1.3. Lemma.** $W^*(\Gamma, \mathcal{P}_\Lambda)' \cong W^*(\Gamma_\Lambda)$.

*Proof.* It is obvious that $W^*(\Gamma, \mathcal{P}_\Lambda)' = W^*(\Gamma)' \cap (\bigcap_{h \in \Lambda} \{P_h\}')$. By the classical theory [8, Vol. II, 6.7.2], the commutant $W^*(\Gamma)'$ in $\mathcal{L}(\ell^2(\Gamma))$ coincides with the von Neumann algebra generated by $\{R_s : s \in \Gamma\}$. Also, each element of $W^*(\Gamma)'$ can be written in the form $R_f$, for some $f \in \ell^2(\Gamma)$. It is immediate that

$$W^*(\Gamma, \mathcal{P}_\Lambda)' = \{R_f \in \mathcal{L}(\ell^2(\Gamma)) : \ R_f P_h = P_h R_f \ \ \forall h \in \Lambda\}.$$

Let us consider the self-adjoint part of $W^*(\Gamma, \mathcal{P}_\Lambda)'$, which of course generates the whole $W^*(\Gamma, \mathcal{P}_\Lambda)'$. If $R_f = R_f^*$ and $R_f \in W^*(\Gamma, \mathcal{P}_\Lambda)'$, then by Lemma 1.2

$$\{s \in \Gamma : f(s) \neq 0\} \subseteq \Gamma_\Lambda.$$

It follows that $W^*(\Gamma, \mathcal{P}_\Lambda)'$ equals the von Neumann algebra generated by the unitaries $\{R_s : s \in \Gamma_\Lambda\}$. Therefore, $W^*(\Gamma, \mathcal{P}_\Lambda)'$ is $*$-isomorphic to $W^*(\Gamma_\Lambda)$ [8, Vol. II, 6.7]. □

**1.4.** *Proof of Theorem I.* We write $\tilde{\Gamma}$ for a fixed set of representatives of cosets $\Gamma/\Gamma_\Lambda$. Then the disjoint union $\Gamma = \bigcup_{h \in \tilde{\Gamma}} h\Gamma_\Lambda$ induces a decomposition

$$\ell^2(\Gamma) = \bigoplus_{h \in \tilde{\Gamma}} \ell^2(h\Gamma_\Lambda).$$

For any pair $h_1, h_2 \in \tilde{\Gamma}$ we define a partial isometry $V_{h_1 h_2}$ from $\ell^2(h_1\Gamma_\Lambda)$ onto $\ell^2(h_2\Gamma_\Lambda)$ by

$$V_{h_1 h_2}(\xi_{h_1 s}) = \xi_{h_2 s}, \quad \forall s \in \Gamma_\Lambda.$$

It is clear that $\{V_{h_1, h_2} : h_1, h_2 \in \Gamma\}$ form a system of matrix units which generates a type $I$ factor, which is $*$-isomorphic to $\mathcal{L}(\mathcal{H})$. Furthermore, all $V_{h_1 h_2}$ are in the double commutant $W^*(\Gamma, \mathcal{P}_\Lambda)''$. Let $P_0$ be the projection onto $\ell^2(\Gamma_\Lambda)$. Then it is clear from [8, Vol. II, 6.7] that $P_0 W^*(\Gamma, \mathcal{P}_\Lambda)'' P_0$ is the von Neumann algebra generated by the unitaries $\{L_h : h \in \Gamma_\Lambda\}$. Thus, $P_0 W^*(\Gamma, \mathcal{P}_\Lambda)'' P_0 \cong W^*(\Gamma_\Lambda)$. Consequently, the Double Commutant Theorem [8, Vol. I, 5.3.1] implies

$$W^*(\Gamma, \mathcal{P}_\Lambda) \cong W^*(\Gamma, \mathcal{P}_\Lambda)'' \cong W^*(\Gamma_\Lambda) \otimes \mathcal{L}(\mathcal{H}).$$

□

**1.5. Corollary.** (i) *If $\Gamma_\Lambda$ contains at least two generating cyclic groups and $\Gamma_\Lambda \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then $W^*(\Gamma_\Lambda)$ is a factor of type $II_1$ and, hence, $W^*(\Gamma, \mathcal{P}_\Lambda)$ is a factor of type $II_\infty$. In particular, if $\Gamma$ is a free group $\mathcal{F}_n$ on $n$ generators and $\Lambda = \{g_{n-m}, g_{n-m+1}, ..., g_n\}$, where $3 \leq n \leq +\infty$ and $n - m \geq 1$, then*

$$W^*(\Gamma, \mathcal{P}_\Lambda) \cong \begin{cases} \mathcal{L}^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}) & \text{if } n-m=1, \\ W^*(\mathcal{F}_{n-m}) \otimes \mathcal{L}(\mathcal{H}) & \text{if } n-m>1, \end{cases}$$

*where $+\infty - 1$ is understood as $+\infty$.*

(ii) *If $\Gamma = \mathbb{Z} * \mathbb{Z}_m$, where $2 \leq m < +\infty$, and $\Lambda = \{g_i\}$, then*

$$W^*(\Gamma, P_\Lambda) \cong \begin{cases} \mathcal{L}^\infty(S^1) \otimes \mathcal{L}(\mathcal{H}) & \text{if } G_i \cong \mathbb{Z}_m, \\ \underbrace{\mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \oplus ... \oplus \mathcal{L}(\mathcal{H})}_{m \text{ times}} & \text{if } G_i \cong \mathbb{Z}. \end{cases}$$

(iii) *If $\Gamma = \mathbb{Z}_n * \mathbb{Z}_m$, where $2 \leq m, n < +\infty$, and $\Lambda = \{g_i\}$, then*

$$W^*(\Gamma, P_\Lambda) \cong \underbrace{\mathcal{L}(\mathcal{H}) \oplus \mathcal{L}(\mathcal{H}) \oplus ... \oplus \mathcal{L}(\mathcal{H})}_{m \text{ times}}, \text{ if } G_i \cong \mathbb{Z}_n.$$

As the most important application of Theorem I we determine non-nuclearity of certain C*-algebras of the form $C_r^*(\Gamma, \mathcal{P}_\Lambda)$. At first we recall some relevant known results.

**1.6.** (a) *Assume that $G$ is a discrete group. Then $C_r^*G$ is amenable (nuclear) if and only if the group $G$ is amenable* ([1], [11]).

(b) *Assume that $G$ is a countable discrete i.c.c. group. Then $W^*(G)$ is $*$-isomorphic to the unique hyperfinite type $II_1$ factor $\mathcal{R}$ if and only if $G$ is amenable* [4]. *Also, $W^*(G) \otimes \mathcal{L}(\mathcal{H})$ is $*$-isomorphic to the unique injective type $II_\infty$ factor $\mathcal{R} \otimes \mathcal{L}(\mathcal{H})$ if and oly if $G$ is amenable* [4].

(c) *A C*-algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is nuclear, and again if and only if the enveloping von Neumann algebra $\mathcal{A}^{**}$ is injective* [3], [4], [10].

**1.7. Theorem II.** *If $\Gamma_\Lambda$ contains at least two different generators of $\Gamma$ but $\Gamma_\Lambda \neq \mathbb{Z}_2 * \mathbb{Z}_2$, then the C*-algebra $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ is non-nuclear.*

*Proof.* It is well known that our assumptions imply that the group $\Gamma_\Lambda$ is non-amenable (e.g. cf. [12, Theorem 1.1]). Consequently, Theorem I and (b) above imply that $W^*(\Gamma, \mathcal{P}_\Lambda)$, the weak closure of $C_r^*(\Gamma, \mathcal{P}_\Lambda)$, is not injective. Thus, the enveloping von Neumann algebra $C_r^*(\Gamma, \mathcal{P}_\Lambda)^{**}$ is not injective either, and (c) above implies that the C*-algebra $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ is non-nuclear, as desired. □

***1.8.*** *Remark.* The problem of nuclearity of those algebras $C_r^*(\Gamma, \mathcal{P}_\Lambda)$ for which the group $\Gamma_\Lambda$ is amenable (i.e. $\Gamma_\Lambda$ is either cyclic or $\mathbb{Z}_2 * \mathbb{Z}_2$) has already been answered in the affirmative in [16].

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