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PRODUCT-TRACE-RINGS AND A QUESTION OF G. S. GARFINKEL

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Dedicated to H. Röhrl on the occasion of his 70th birthday

ABSTRACT. It is an open question as to whether every left coherent ring R satisfying the intersection property for finitely generated left ideals of R is a right-product-trace-ring or not. R is a right-product-trace-ring iff every product of trace-right-R-modules (= universally torsionless-right-R-modules) is a trace-right-R-module. This question is shown to have a negative answer. Furthermore, looking at all valuation domains, the complete product-trace-rings, the product-trace-rings and the product-content-rings are characterized.

INTRODUCTION

In this paper all rings have an identity and all modules are unitary. A right-R-module M over an arbitrary ring R is called universally torsionless (UTL) if the natural mapping $M \otimes A \to \mathrm{Hom}(M^*, A)$, where $M^* := \mathrm{Hom}(M, R)$, is monic for all left-$R$-modules A ([4], p. 119). Bass calls M torsionless if the above mapping is monic for $A := R$ and reflexive if the same mapping is an isomorphism. M is a UTL-right-R-module if and only if M is a trace-right-R-module, i.e. $x \in M \cdot M^*(x)$, where $M^*(x) := \{f(x) | f \in M^*\}$, for every $x \in M$ ([4], Th. 3.2). Various equivalent conditions for M to be a trace-right-R-module can be found in [4], Th. 3.2. Trace-modules over commutative rings were also investigated in [8] and [6]. In [6] a trace-module is called a flat strict Mittag-Leffler module.

A direct sum of right-modules is a trace-right-module, if and only if each summand is ([4], 5.1). The situation for products is much more complicated. R is a right-product-trace-ring (i.e. every product of trace-right-R-modules is a trace-right-R-module) if it is left Noetherian ([4], Cor., p. 133). If R is a right-product-trace-ring, then R is left coherent (i.e. each finitely generated (f.g.) left ideal of R is finitely presented) and satisfies the intersection property for f.g. left ideals (i.e. every intersection of f.g. left ideals is f.g.) ([4], Th. 5.13). G. S. Garfinkel asked in his paper ([4], p. 137) whether the converse of [4], Th. 5.13, is true or not. It is the aim of this paper to answer this question in the negative. Furthermore, looking

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at all valuation domains, the product-trace-rings, complete product-trace-rings and product-content-rings are characterized. □

$\mathbb{R}$ ($\mathbb{Z}$ resp.) is the set of real numbers (integers resp.) and we put $\mathbb{R}^+ := \{x \in \mathbb{R} | x \geq 0\}$ and $\mathbb{Z}^+ := \mathbb{Z} \cap \mathbb{R}^+$. A valuation ring R is a commutative ring with 1 whose ideals are totally ordered by inclusion. If, in addition, R is an integral domain, it is called a valuation domain. Every valuation domain R is the domain $R_v := \{x \in K | v(x) \geq 0\}$ of a surjective valuation $v : K \to \Gamma \cup \{\infty\}$ ([3], I, Prop. 3.1). Here, K is the quotient field of R, Γ is a totally ordered (additively written) abelian group, ∞ is a symbol regarded as larger than any element of Γ and v fulfills the well-known axioms. In the following R always denotes a valuation domain of a surjective valuation $v : K \to \Gamma \cup \{\infty\}$. R is called maximally complete if it does not admit any proper immediate extension ([3], I, p. 5). This is the case if and only if R is pure-injective (= algebraically compact) as an R-module, and this is equivalent to K being spherically complete (see [3] and [7]). For all standard definitions the reader is referred to [2], [3], [7] and [9].

In case rank$(v) = 1$, $\prod_{i \in I} K := \{(z_i)_{i \in I} | z_i \in K \ (i \in I)$ and $\inf\{v(z_i) | i \in I\} > -\infty\}$ is the product in the category of K-Banach spaces for every set I; here the norm of $(z_i)_{i \in I} \in \prod_{i \in I} K$ is $\inf\{v(z_i) | i \in I\}$ ([9], p. 52). The existence of surjective valuations $v : K \to \mathbb{R} \cup \{\infty\}$ such that R is complete, but not maximally complete (or, equivalently, K is complete, but not spherically complete), is well-known ([9], Th. 1.3 and p. 11, Rem. (ii) and p. 25, Ex. 2. G). Hence the following proposition answers the question of G. S. Garfinkel in the negative.

Proposition 1. *Let $v : K \to \mathbb{R} \cup \{\infty\}$ be an arbitrary surjective valuation, such that R is complete, but not maximally complete. Then R is coherent and satisfies the intersection property for finitely generated ideals of R, but R is not a product-trace-ring.*

Proof. Since the value group $\mathbb{R}$ is complete as a lattice, according to [4], Ex. 5.3, we only have to show that R is not a product-trace-ring. Assume that $R^{\mathbb{N}}$ is a trace-R-module. Then, for some (every resp.) $\beta := (\beta_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ with $v(\beta_n) > v(\beta_{n+1})$ for every $n \in \mathbb{N}$, there exist $\gamma := (\gamma_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ and $f \in \text{Hom}(R^{\mathbb{N}}, R)$ with $\beta = f(\beta)\gamma$. (B, u_B) with $B := \prod_{n \in \mathbb{N}} K$ and $u_B((z_n)_{n \in \mathbb{N}}) := \inf\{v(z_n) | n \in \mathbb{N}\}$ is a K-Banach space. Define a mapping $g : B \to K$ as follows: For $x := (x_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ put $g(x) := f(x)$. For $x \in B \backslash R^{\mathbb{N}}$ there is a $\tau \in K \backslash \{0\}$ with $v(\tau) = \inf\{v(x_n) | n \in \mathbb{N}\} < 0$. Then one gets $\frac{1}{\tau} x = (\frac{1}{\tau} x_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$. In this case define $g(x) := \tau f(\frac{1}{\tau} x)$. For every $\sigma \in K \backslash \{0\}$ with $v(\sigma) \leq v(\tau)$ one has $\frac{1}{\sigma} x \in R^{\mathbb{N}}$, and from $\sigma f(\frac{1}{\sigma} x) = \tau f(\frac{1}{\tau} x)$ one easily sees that $g : B \to K$ is well-defined and K-linear. Take an arbitrary $x = (x_i)_{i \in \mathbb{N}} \in R^{\mathbb{N}}$ and assume the existence of $\lambda \in R$ with $v(f(x)) < v(\lambda) \leq inf\{v(x_i) | i \in \mathbb{N}\}$. This leads to the contradiction $v(f(x)) = v(\lambda f((\frac{x_i}{\lambda})_{i \in \mathbb{N}})) \geq v(\lambda)$. Thus $v(f(x)) \geq \inf\{v(x_i) | i \in \mathbb{N}\}$ follows for every $x = (x_i)_{i \in \mathbb{N}} \in R^{\mathbb{N}}$. Therefore, for each $x := (x_n)_{n \in \mathbb{N}} \in B \backslash R^{\mathbb{N}}$ with $u_B(x) = v(\tau)$

$$\begin{aligned} v(g(x)) = v(\tau f(\frac{1}{\tau} x)) = v(\tau) + v(f(\frac{1}{\tau} x)) &\geq v(\tau) + \inf\{v(\frac{1}{\tau} x_n) | n \in \mathbb{N}\} \\ &= \inf\{v(x_n) | n \in \mathbb{N}\} = u_B(x) \end{aligned}$$

holds. $v(g(x)) \geq u_B(x)$ is also valid for $x \in R^{\mathbb{N}}$. Hence $g : B \to K$ is a contraction. Since K is not spherically complete, there exist endomorphisms $g_n : K \to K$ of the K-Banach space K, $n \in \mathbb{N}$, with $(g_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} K'$, and for every $z := (z_n)_{n \in \mathbb{N}} \in B$, $g(z) = \sum_{j \in \mathbb{N}} g_j(z_j)$ holds because of [9], Th. 4.22(i). This implies $\beta = g(\beta)\gamma =$

$(\sum_{j\in\mathbb{N}} g_j(\beta_j)\gamma_n)_{n\in\mathbb{N}}$. Obviously, there exists a $j_0 \in \mathbb{N}$ with $v(\sum_{j\in\mathbb{N}} g_j(\beta_j)) \geq g_{j_0}(\beta_{j_0})$, and $g_{j_0} : K \to K$ is a contraction.

$$v(\beta_n) = v(\sum_{j\in\mathbb{N}} g_j(\beta_j)\gamma_n) \geq v(\sum_{j\in\mathbb{N}} g_j(\beta_j)) \geq v(g_{j_0}(\beta_{j_0})) \geq v(\beta_{j_0})$$

follows for every $n \in \mathbb{N}$. This contradicts the choice of $\beta \in R^{\mathbb{N}}$, i.e. R is not a product-trace-ring. □

Every field K is Noetherian, thus a product-trace-ring ([4], Cor., p. 133). Hence, in the following we may restrict our attention to valuation domains $R \neq K$, i.e. the valuation $v : K \to \Gamma \cup \{\infty\}$ is not trivial.

Theorem 2. *For every valuation domain $R \neq K$, the following are equivalent*:

(i) *R is a complete product-trace-ring.*

(ii) *R is coherent and satisfies the intersection property for finitely generated ideals of R, and the following condition a) or b) is fulfilled:*
   - *a) R is complete and v is discrete with rank$(v) = 1$.*
   - *b) $v(R\backslash\{0\}) = \mathbb{R}^+$ and R is maximally complete.*

(iii) *(R is complete and v is discrete with rank$(v) = 1$) or ($v(R\backslash\{0\}) = \mathbb{R}^+$ and R is maximally complete).*

(iv) *R is maximally complete and $v(R\backslash\{0\})$ is (isomorphic to) $\mathbb{Z}^+$ or $\mathbb{R}^+$.*

Proof. (i)⇒(ii): By [4], Th. 5.13, the product-trace-ring R is coherent and satisfies the intersection property for f.g. ideals of R. Assume rank$(v) > 1$. Then there exist $\alpha, \beta \in R\backslash\{0\}$ with $0 < v(\alpha^n) < v(\beta)$ for each $n \in \mathbb{N}$ ([5], Ex. 15, p. 203). Because of [4], Ex. 5.3, Γ is complete as a lattice. Thus $s := \sup\{v(\alpha^n)|n \in \mathbb{N}\}$ exists in Γ. From $s = \sup\{v(\alpha^{n+1})|n \in \mathbb{N}\} = s + v(\alpha)$ one gets the contradiction $v(\alpha) = 0$. Hence rank$(v) = 1$, and the remaining part of (ii) is a consequence of the above intersection property (resp. of the preceding proposition).

(ii)⇒(iii) is trivial.

(iii)⇒(i): If rank$(v) = 1$, R is complete and v is discrete, R is a complete product-trace-ring by [4], Cor., p. 133. But the following proof in the case $v(R\backslash\{0\}) = \mathbb{R}^+$ and R maximally complete is also valid in the first case. Let I be an arbitrary set and $0 \neq x := (x_i)_{i\in I} \in R^I$. There exists a $\gamma \in R\backslash\{0\}$ with $\inf\{v(x_i)|i \in I\} = v(\gamma)$. Since K is spherically complete, there is a contraction $g : \prod_{i\in I} K \to K$ with $g(x) = \gamma$ ([1], Cor. 2). The restriction $f : R^I \to R$ of g is a (well-defined) morphism of R-modules. From $(\frac{x_i}{\gamma})_{i\in I} \in R^I$ and $x = f(x)\ (\frac{x_i}{\gamma})_{i\in I}$ one concludes that R^I is a trace-R-module. Therefore, R is a product-trace-ring ([4], Th. 5.4) and R is obviously complete.

(iii)⇔(iv) is a consequence of [9], 2.4. □

The conditions in (ii) ((iii), (iv) resp.) of the theorem above do not characterize all valuation domains which are product-trace-rings. Indeed, every non-complete valuation domain R with $v(R\backslash\{0\}) = \mathbb{Z}^+$ is Noetherian, and therefore, by [4], Cor., p. 133, a product-trace-ring. But we are able to prove the following

Theorem 3. *For every valuation domain $R \neq K$ the following are equivalent:*

(i) *R is a product-trace ring.*

(ii) *$v(R\backslash\{0\}) = \mathbb{Z}^+$ or ($v(R\backslash\{0\}) = \mathbb{R}^+$ and R is maximally complete).*

Proof. (ii)⇒(i): If $v(R\backslash\{0\}) = \mathbb{Z}^+$, R is Noetherian, hence a product-trace-ring ([4], Cor., p. 133); otherwise, because of Theorem 2, R is a product-trace-ring.

(i)⇒(ii): The proof of Theorem 2, (i)⇒(ii), shows that $v(R\backslash\{0\}) = \mathbb{Z}^+$ or $v(R\backslash\{0\}) = \mathbb{R}^+$ holds. We may assume $v(R\backslash\{0\}) = \mathbb{R}^+$. If R is complete, R is maximally complete by Proposition 1. Assume that R fails to be complete, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in R, such that $(v(x_n))_{n\in\mathbb{N}}$ is strictly decreasing. Because of (i), there are a $\mathbf{Mod}_R$-morphism $g : R^{\mathbb{N}} \to R$ and $y = (y_n)_{n\in\mathbb{N}} \in R^{\mathbb{N}}$ with $x = g(x)y$. Since K is obviously countably generated and R is not complete, the R-module R is slender ([3], XIV, Cor. 7.9), i.e., there is an $m \in \mathbb{N}$ with $g(e_n) = 0$ for $n > m$. Put $z := x - \sum_{i=1}^m x_i e_i$ and let $h : \prod_{i=m+1}^\infty R \to R$ be the restriction of $g : R^{\mathbb{N}} \to R$. Since R is slender and because $h(Re_i) = Rg(e_i) = 0$ for $i > m$, one obtains $h = 0$ ([3], XIV, Th. 7.4(ii)). Hence $x = g(x)y = g(z + \sum_{i=1}^m x_i e_i)y = (h(z) + \sum_{i=1}^m x_i g(e_i))y = \sum_{i=1}^m x_i g(e_i)y$ follows, which implies the contradiction $v(x_n) = v(\sum_{i=1}^m x_i g(e_i)y_n) \geq v(x_{i_0} g(e_{i_0})y_n) \geq v(x_{i_0})$ for some $i_0 \in \mathbb{N}_m$ and every $n \in \mathbb{N}$. This finishes the proof. □

For each right-R-module M $c(x) := \bigcap\{I | I$ is a left ideal of R with $x \in MI\}$ is the content-ideal of $x \in M$ ([4], p. 137). M is called a content-right-R-module if $x \in Mc(x)$ for every $x \in M$. We call R a right-product-content-ring if each product of content-right-R-modules is a content-right-R-module. A right-R-module M is a trace-right-R-module if and only if M is a content-right-R-module with $c(x) = \{f(x) | f \in M^*\}$ for every $x \in M$ ([4], 5.14 (4); [8], Rem. p. 66). For example, in case rank$(v) = 1$ and v is discrete, an R-module M is a content-R-module if and only if M is division free (i.e. the only divisible element of M is 0) ([8], Prop. 2.1). The following theorem is a simple consequence of [4], Ex. 5.3, Th. 5.15, and of the first part of the proof of "(i)⇒(ii)" in Theorem 2.

Theorem 4. *For any valuation domain $R \neq K$ the following are equivalent:*

(i) *R is a product-content-ring.*

(ii) *rank$(v) = 1$ and (v is discrete or $v(R\backslash\{0\}) = \mathbb{R}^+$).* □

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