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SAMPLING SEQUENCES FOR HARDY SPACES OF THE BALL

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ABSTRACT. We show that a sequence $a := \{a_k\}_k$ in the unit ball of $\mathbb{C}^n$ is sampling for the Hardy spaces H^p, $0 < p < \infty$, if and only if the admissible accumulation set of a in the unit sphere has full measure. For $p = \infty$ the situation is quite different. While this condition is still sufficient, when $n > 1$ (in contrast to the one dimensional situation) there exist sampling sequences for H^∞ whose admissible accumulation set has measure 0. We also consider the sequence $a(\omega)$ obtained by applying to each a_k a random rotation, and give a necessary and sufficient condition on $\{|a_k|\}_k$ so that, with probability one, $a(\omega)$ is of sampling for H^p, $p < \infty$.

§1. Introduction

Let $\mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. Let S^n denote the unit sphere and $d\sigma$ its normalized Lebesgue measure. Recall that for any $0 < p \leq \infty$ the Hardy space $H^p(\mathbb{B}_n)$ is the set of functions f holomorphic in $\mathbb{B}_n$ such that

$$\|f\|_p := \left(\sup_{r<1} \int_{S^n} |f(r\zeta)|^p \, d\sigma(\zeta)\right)^{1/p} < \infty,$$

where the integral is replaced by a supremum in the case $p = \infty$.

Roughly speaking, we would like to say that a sequence $a := \{a_k\}_k$ of the unit ball is *sampling for the space* H^p when the values of any function $f \in H^p$, restricted to the sequence, determine the function uniquely, and moreover some inequalities between the H^p-norm and an appropriate norm on the space of functions on the sequence a hold. In [Th] it was shown that a natural notion of sampling for H^p is the following.

Given $\alpha > 1$, let

$$\Gamma_\alpha(\zeta) := \left\{z \in \mathbb{B}_n : |1 - \bar{\zeta} \cdot z| < \frac{\alpha}{2}(1 - |z|^2)\right\}$$

be the *admissible approach region* with vertex at $\zeta \in S^n$ and aperture α (see [Ru1, p. 72] for the properties of these regions).

The *admissible maximal function* on S^n is then defined, for every $\alpha > 1$, as

$$M^\alpha f(\zeta) := \sup_{z \in \Gamma_\alpha(\zeta)} |f(z)|.$$

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For any $0 < p \leq \infty$ and for any $\alpha > 1$ we have $\|M^\alpha f\|_p \leq C_{p,\alpha}\|f\|_p$, where, for functions defined on the unit sphere, $\|\cdot\|_p$ stands for the usual norm in the space $L^p(d\sigma)$ ([Ru1, 5.6.5]).

Following [Br-Ni-Oy] we also consider the corresponding maximal function associated to the sequence a:

$$M_a^\alpha(f)(\zeta) := \sup_{z\in a\cap\Gamma_\alpha(\zeta)} |f(z)| \ .$$

From the above it follows that $\|M_a^\alpha(f)\|_p \leq C_{p,\alpha}\|f\|_p$.

Definition. A sequence a is called a *sampling sequence for* H^p when there exists $\alpha > 1$ and a constant $C > 0$ such that $\|M_a^\alpha(f)\|_p \geq C\|f\|_p$ for every $f \in H^p$.

In the case where $p = \infty$ this simply says that $\sup_a |f| \geq C\|f\|_\infty$, and by taking powers of f we see that $\sup_a |f| = \|f\|_\infty$.

Given a sequence a let

$$AD_\alpha(a) = \{\zeta \in S^n : \zeta \in \overline{a \cap \Gamma_\alpha(\zeta)}\}$$

and define the *admissible accumulation set* as $AD(a) := \bigcup_{\alpha>1} AD_\alpha(a)$.

Brown, Shields and Zeller showed that the condition $\sigma(AD(a)) = 1$ characterizes the sampling sequences for H^∞ of the disk ([Br-Sh-Ze, Th. 3, (iii)-(iv)]). It will be important to keep in mind that $\sigma(AD(a)) = 1$ if and only if $\sigma_\alpha(AD(a)) = 1$ for some $\alpha > 1$ large enough (see [Th]). Recently the second author showed that the same condition is actually necessary and sufficient for a to be of sampling for any H^p of the disk, $p < \infty$ [Th, Theorem 1].

In this note we first prove that when $n > 1$, the condition $\sigma(AD(a)) = 1$ also characterizes the sampling sequences for H^p, $p < \infty$.

Theorem 1. *A sequence a is sampling for H^p, $p < \infty$, if and only if $\sigma(AD(a)) = 1$.*

In particular, sampling sequences for H^p are the same for all values $p < \infty$.

For $p = \infty$ the situation is more complicated.

On the one hand it is clear that if a is sampling for H^∞, then necessarily $\overline{a}\cap S^n = S^n$ (if a avoids an open set $\{\zeta \in S : |1 - \zeta \cdot \bar{\eta}| < \delta\}$, any peak function for η, for instance $f(z) = z \cdot \bar{\eta}$, violates the sampling condition). Although $\overline{a} \cap S^n = S^n$ is also sufficient for sampling in the ball algebra $A(\mathbb{B}_n)$, for general H^∞ functions this is far from being sufficient: there are sequences which are contained in an H^∞ zero set such that $\overline{a} \cap S^n = S^n$ (for example any sequence a with $\sum_k(1 - |a_k|) < \infty$ having S^n as cluster set).

The proof of Theorem 1 shows that $\sigma(AD(a)) = 1$ is as well sufficient for a to be sampling for H^∞. This condition is far from being necessary.

Definition. A set E in S^n is a *max-set* when $esssup_E|f| = \|f\|_\infty$ for all $f \in H^\infty$.

It is clear from the definition that if $AD(a)$ contains a max-set, then a must be sampling for H^∞. Since there exist max-sets of arbitrarily small measure [Ru2, 13.4] it is possible to construct, for every $\varepsilon > 0$, a sampling sequence a with $\sigma(AD(a)) < \varepsilon$. This can be pushed a little further:

Theorem 2. *If $n > 1$, there exist sampling sequences for H^∞ with $\sigma(AD(a)) = 0$.*

Several conjectures can be made regarding necessary or sufficient conditions for sampling in H^∞, although we have not been able to prove any of them. All the attempts to prove any of these conjectures have led us to the well-known Fatou problem on radial behaviour of holomorphic bounded functions in higher dimension (see [Ru1, Chapter 11]).

We also prove a probabilistic result on random sampling sequences for H^p with prescribed radii, along the lines of the results in [Bo], [Co] and [Ma].

Consider the probability space $\Omega = \prod_{k=1}^{\infty} \Omega_k$, where Ω_k is the unit sphere S^n for all k. $\mathcal{A}_k$ denotes the σ-algebra of Lebesgue measurable sets on S^n, and P_k denotes the normalized Lebesgue measure σ on the sphere. An element of Ω is denoted by $\omega = (\zeta_1, \zeta_2, ...)$, where $\zeta_k \in S^n$. Each $\zeta_k : \Omega \longrightarrow S^n$ can be viewed as a random variable defined on S^n, with values on S^n as well. To construct the space of probability, one can alternatively take $\Omega_k = O(2n)$, the group of rotations of $\mathbb{C}^n$, P_k the Haar measure on $O(2n)$ and $\mathcal{A}_k$ the σ-algebra of measurable sets with respect to the Haar measure in $O(2n)$. Then the elements of Ω are denoted by $\omega = (\mathcal{R}_1, \mathcal{R}_2, ...)$.

Given a sequence a we consider a sequence of independent and uniformly distributed random variables $\zeta_k(\omega)$ in S^n (resp. $\mathcal{R}_k^\omega$ in $O(2n)$) and define the associated *random sequence* as $a(\omega) := \{a_k(\omega)\}_k$, where $a_k(\omega) = |a_k|\zeta_k(\omega)$ (resp. $a_k(\omega) = \mathcal{R}_k^\omega(a_k)$). Notice that $|a_k| = |a_k(\omega)|$ for all ω and for all k.

Theorem 3. *Let a be a sequence in $\mathbb{B}_n$.*

(a) *If $\sum_{k=1}^{\infty}(1-|a_k|)^n = \infty$, then $P(\{\omega : \sigma(AD(a(\omega))) = 1\}) = 1$.*
(b) *If $\sum_{k=1}^{\infty}(1-|a_k|)^n < \infty$, then $\sigma(AD(a(\omega))) = 0$ for all ω.*

As a consequence of Theorem 1 we have the following:

Corollary. *Let a be a sequence in $\mathbb{B}_n$.*

(a) *If $\sum_{k=1}^{\infty}(1-|a_k|)^n = \infty$, then $P(\{\omega : a(\omega)$ is sampling for $H^p\}) = 1$ for any $p \leq \infty$.*
(b) *If $\sum_{k=1}^{\infty}(1-|a_k|)^n < \infty$, then $P(\{\omega : a(\omega)$ is sampling for $H^p\}) = 0$ for any $p < \infty$.*

Some remarks are in order.

When the generalized Blaschke condition $\sum_k (1-|a_k|)^n < \infty$ holds, the sequence is actually almost surely contained in an H^p zero set, for all $p < \infty$ [Ma, Theorem 1.2.]. Thus the generalized Blaschke condition distinguishes two sharply contrasting situations: either $a(\omega)$ is almost surely sampling for H^p or it is almost surely contained in an H^p zero set.

In the unit disk the Blaschke condition on a implies that every $a(\omega)$ is an H^p zero sequence, for all $p \leq \infty$. In particular, $a(\omega)$ is never a sequence of sampling for H^p, even for $p = \infty$. On the other hand, when $n > 1$ and $\sum_k (1-|a_k|)^n < \infty$ we have $\sigma(AD_\alpha(a(\omega))) = 0$ for all ω, but as seen in Theorem 2, this is not enough to deduce that $a(\omega)$ is not sampling for $H^\infty(\mathbb{B}_n)$.

As in [Co, Corollary 1] one can also randomize the moduli $|a_k|$ independently of $\{\zeta_k(\omega)\}_k$ and show that Theorem 3 also holds for $a_k(\omega) = r_k(\omega)\zeta_k(\omega)$, where the $\zeta_k(\omega)$ are as before and $\{r_k(\omega)\}_k$ satisfy:

(i) $r_k(\omega) \in (0,1)$ for all k and: in case (a) almost surely $\sum_k (1 - r_k(\omega))^n = \infty$; in case (b) almost surely $\sum_k (1 - r_k(\omega))^n < \infty$.
(ii) each $r_k(\omega)$ is independent of $\{\zeta_k(\omega)\}_k$.

In the following three sections we prove respectively Theorems 1, 2 and 3.

§2. Proof of Theorem 1

A function f defined on $\mathbb{B}_n$ is said to have *admissible limit* at $\zeta \in S^n$ when the limit $\lim_{\substack{z\to\zeta \\ z\in\Gamma_\alpha(\zeta)}} f(z)$ exists, is finite and is the same for all $\alpha > 1$. The limit is denoted by $f^*(\zeta)$.

The proof that $\sigma(AD(a)) = 1$ implies a sampling for H^p, $p \le \infty$, is essentially due to Brown, Shields and Zeller, and we include it for the sake of completeness.

For $\alpha > 1$, every $f \in H^p$ has admissible limit at almost every ζ and $\|f^*\|_p = \|f\|_p$ [Ru1, 5.6.8]. Thus for almost every $\zeta \in AD_\alpha(a)$,

$$M_a^\alpha(f) \ge \lim_{\substack{z\to\zeta \\ z\in a\cap\Gamma_\alpha(\zeta)}} |f(z)| = |f^*(\zeta)|.$$

Hence if $\sigma(AD_\alpha(a)) = 1$, then necessarily $\|M_a^\alpha(f)\|_p \ge \|f^*\|_p$.

Let us see now that $\sigma(AD(a)) = 1$ is also necessary, if $p < \infty$.

Assume $\sigma(AD_\alpha(a)) < 1$. By the same argument as in [Br-Sh-Ze], we may assume that there exist a compact $A \subset S^n$ and $N \in \mathbb{N}$ such that $\sigma(A) > 0$ and $\Gamma_\alpha(\zeta) \cap a \subset B(0, 1 - 1/N)$ for all $\zeta \in A$. We will use the following technical result.

Lemma. *For any $m \in \mathbb{N} \setminus \{0\}$ and $p > 0$, there exists a positive real function $\psi_m \in \mathcal{C}(\overline{\mathbb{B}_n})$ such that:*

(i) *$\psi_m(z) \le m$ for all $z \in \overline{\mathbb{B}_n}$, and $\psi_m(z) = m$ for $z \in A$;*
(ii) *$\psi_m(z) \le 1$ for $z \in B(0, 1 - 1/N)$;*
(iii) *$\sigma(\{\zeta \notin A : M^\alpha\psi_m(\zeta) \ge 1\}) \le m^{-p}$.*

Proof. Let $\varrho(\zeta, \eta) = |1 - \bar{\zeta} \cdot \eta|$ denote the non-isotropic pseudodistance on $\overline{\mathbb{B}_n}$ and let $\varrho(z, A) := \inf_{\zeta\in A} \varrho(z, \zeta)$. Define

$$\psi_m(z) := |z|^{\mu_m} \max\left(m(1 - \lambda_m \varrho(z, A))_+, 1/m\right),$$

where λ_m and μ_m are sequences of positive numbers increasing to infinity whose growth will be determined later on.

The property (i) is then clear, and we ensure (ii) by choosing μ_m large enough so that $m(1 - 1/N)^{\mu_m} \le 1$.

Let us now prove (iii). Take $\zeta \notin A$ and suppose that there exists $z \in \Gamma_\alpha(\zeta)$ such that $\psi_m \ge 1$. This implies that $m(1 - \lambda_m \varrho(z, A)) \ge 1$; thus $\varrho(z, A) \le 1/\lambda_m$.

Then, by the triangle inequality for $\varrho^{1/2}$,

$$\varrho(\zeta, A)^{1/2} \le \left(\frac{\alpha}{2}(1 - |z|^2)\right)^{1/2} + \varrho(z, A)^{1/2} \le (\sqrt{\alpha} + 1)\varrho(z, A)^{1/2} \le \frac{(\sqrt{\alpha} + 1)}{\sqrt{\lambda_m}}.$$

Since $S^n \setminus A$ is an open set of finite measure, we may end the proof by choosing λ_m large enough so that

$$\sigma\big(\{\zeta \notin A : \varrho(\zeta, A) \le (\sqrt{\alpha} + 1)^2/\lambda_m\}\big) \le m^{-p}.$$

□

This Lemma and [Ru2, Theorem 3.5] give us, for any $\varepsilon > 0$, a polynomial P_m such that $|P_m| \le \psi_m$ on the closed ball, and

$$\sigma(\{\zeta \in S^n : |P_m(\zeta)| < \psi_m(\zeta) - \varepsilon\}) < \varepsilon.$$

For a given $p > 0$, and taking ε small enough, we use Lemma (i) to obtain the following lower bound:

$$\int_{S^n} |P_m|^p d\sigma \geq \int_A |P_m|^p d\sigma \geq \frac{1}{2}\int_A \psi_m^p d\sigma = \frac{m^p}{2}\sigma(A).$$

On the other hand, $M_a^\alpha P_m(\zeta) \leq M^\alpha \psi_m(\zeta)$ for $\zeta \notin A$, and by Lemma (ii) also $M_a^\alpha P_m(\zeta) \leq \sup\limits_{z\in B(0,1-1/N)} \psi_m(z) \leq 1$ for $\zeta \in A$. This and Lemma (iii) yield:

$$\begin{aligned}\int_{S^n} (M_a^\alpha P_m)^p d\sigma &\leq \sigma(A) + \int_{S^n\setminus A} (M^\alpha \psi_m)^p d\sigma \\ &\leq \sigma(A) + \int_{\{\zeta\notin A: M^\alpha\psi_m > 1\}} (M^\alpha \psi_m)^p d\sigma + \int_{\{\zeta\notin A: M^\alpha\psi_m \leq 1\}} (M^\alpha \psi_m)^p d\sigma \\ &\leq \sigma(A) + m^{-p}(\sup_{\mathbb{B}_n} \psi_m)^p + \sigma(S^n \setminus A) \leq 2.\end{aligned}$$

Since this is bounded independently of m, a cannot be sampling for H^p. This finishes the proof of Theorem 1.

§3. Proof of Theorem 2

Let $\{\eta_k\}_k$ be a dense sequence on the sphere, and consider for each k the big circle $C_{\eta_k} = \{e^{i\theta}\eta_k : \theta \in [0, 2\pi)\}$. Denote $E = \bigcup_k C_{\eta_k}$.

Take next a dyadic decomposition of each big circle C_{η_k}: for any $m \in \mathbb{N}$ consider the intervals

$$I_{m,j}^{(k)} = \{e^{i\theta}\eta_k \in S^n : (j-1)2^{-m} \leq \frac{\theta}{2\pi} < j2^{-m}\}, \quad j = 1, \ldots, 2^m.$$

Let $\zeta_{m,j}^{(k)} = e^{i2\pi 2^{-m}(j-1/2)}\eta_k$ denote the center of the subinterval $I_{m,j}^{(k)}$.

Our sequence is defined as $a = \{a_{m,j}^{(k)}\}_{k,m,j}$, where $a_{m,j}^{(k)} = (1 - \frac{\alpha_k}{2^m})\zeta_{m,j}^{(k)}$ and $\{\alpha_k\}_k$ is such that $\sum_k (k\alpha_k)^{n-1} < \infty$.

Let us see first that a is sampling for H^∞. By construction, and according to the theorem of Brown, Shields and Zeller, on each slice $D_{\eta_k} =: \eta_k\mathbb{D}$ the sequence $\{a_{m,j}^{(k)}\}_{m,j} \subset D_{\eta_k}$ is sampling for $H^\infty(D_{\eta_k})$, since the non-tangential accumulation set is all C_{η_k}. Thus, given $f \in H^\infty$, every slice function $f_{\eta_k}(\lambda) = f(\lambda\eta_k)$, $\lambda \in \mathbb{D}$, has radial limits $f^*_{\eta_k} \in L^\infty(\mathbb{T})$ satisfying $\|f^*_{\eta_k}\|_{L^\infty(\mathbb{T})} \leq S_f$, where $S_f = \sup_a |f|$. The maximum principle then yields $|f| \leq S_f$ in $\bigcup_k D_{\eta_k}$, which by the density of $\{\eta_k\}_k$ in S^n already implies $\|f\|_\infty \leq S_f$.

It remains to prove that $\sigma(AD(a)) = 0$. Define

$$F = \{\zeta \in S^n : \varrho(\zeta, C_{\eta_k}) < k\alpha_k \text{ for infinitely many } k\},$$

where ϱ is the non-isotropic pseudodistance defined at the beginning of the proof of the Lemma. Since $\sigma(F) \leq \sum_{k\geq p}(k\alpha_k)^{n-1}$ for all $p \in \mathbb{N}$, we deduce that $\sigma(F) = 0$.

On the other hand, for every $\zeta \notin E \cup F$, the quotient $\alpha_k/\varrho(\zeta, C_{\eta_k})$ (which is bounded by $1/k$ for k big enough) tends to 0, so ζ is not approachable within an admissible region by points of a. Hence $AD(a)$ is contained in the zero measure set $E \cup F$.

§4. Proof of Theorem 3

Proof of (a). It will be enough to show that for some $\alpha > 1$

$$\int_\Omega \sigma\big(AD_\alpha(a(\omega))\big)\, dP(\omega) = 1.$$

Notice that

$$(1) \qquad AD_\alpha(a(\omega)) = \bigcap_{p\in\mathbb{N}} \bigcup_{k\geq p} I_\alpha(a_k(\omega)) = \{\zeta \in S : \zeta \in I_\alpha(a_k(\omega)) \text{ for infinitely many } k\},$$

where $I_\alpha(a_k(\omega)) = \{\zeta \in S^n : a_k(\omega) \in \Gamma_\alpha(\zeta)\}$.

Since the random variables $\zeta_k(\omega)$ are uniformly distributed one has:

$$P(\{\omega : \zeta \in I_\alpha(a_k(\omega))\}) = \sigma(I_\alpha(a_k)) = C(1-|a_k|^2)^n$$

for some constant $C > 0$ depending only on α and the dimension.

Now $\sum_k P(\{\omega : \zeta \in I_\alpha(a_k(\omega))\}) = \infty$, so the Borel-Cantelli lemma yields:

$$P(\{\omega : \zeta \in I_\alpha(a_k(\omega)) \text{ for infinitely many } k\}) = 1.$$

In particular

$$P\big(\{\omega : \zeta \in \bigcup_{k\geq p} I_\alpha(a_k(\omega))\}\big) = 1$$

for all $\zeta \in S^n$ and all $p \in \mathbb{N}$. Thus

$$\int_\Omega \sigma\big(\bigcup_{k\geq p} I_\alpha(a_k(\omega))\big)\, dP(\omega) = \int_{S^n}\int_\Omega \mathbb{1}_{\bigcup_{k\geq p} I_\alpha(a_k(\omega))}(\zeta)\, dP(\omega) d\sigma(\zeta) = 1$$

for all $p \in \mathbb{N}$. This together with (1) shows that the required equality holds. □

Proof of (b). This is immediate from (1) and the fact that $\sigma(I_\alpha(a_k(\omega))) = C(1-|a_k|^2)^n$ for all ω. □

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