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UNIVERSAL $\mathbb{Z}$-LATTICES OF MINIMAL RANK

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Abstract. Let $U_{\mathbb{Z}}(n)$ be the minimal rank of n-universal $\mathbb{Z}$-lattices, by which we mean positive definite $\mathbb{Z}$-lattices which represent all positive $\mathbb{Z}$-lattices of rank n. It is a well known fact that $U_{\mathbb{Z}}(n) = n+3$ for $1 \leq n \leq 5$. In this paper, we determine $U_{\mathbb{Z}}(n)$ and find all n-universal lattices of rank $U_{\mathbb{Z}}(n)$ for $6 \leq n \leq 8$.

1. Introduction

A positive definite $\mathbb{Z}$-lattice (or simply a lattice) is said to be *n-universal* if it represents all positive definite $\mathbb{Z}$-lattices of rank n. It is well known that the ranks of n-universal lattices should be greater than or equal to $n+3$. In fact, for each n, $1 \leq n \leq 5$, the lattice I_{n+3} is n-universal because I_{n+3} has class number 1 and is universal over the p-adic integer ring $\mathbb{Z}_p$ for all p, where I_n is the lattice $\mathbb{Z}^n$ equipped with the standard inner product (see [10], [12] and [15]). For $n \geq 6$, however, no diagonal lattice can be n-universal. Moreover, there does not exist a lattice of rank $n+3$ which has class number 1 and represents all integral lattices of rank n over $\mathbb{Z}_p$ for all p (see [18], [20]). To be more precise, we define

$$U_{\mathbb{Z}}(n) = \min \{\text{rank } (L) \mid L \text{ is } n\text{-universal } \}.$$

Let $L_1, L_2, \ldots, L_k$ be all unimodular lattices of rank $n+3$ up to isometry. Then the lattice $L_1 \perp L_2 \perp \cdots \perp L_k$ is n-universal and therefore $U_{\mathbb{Z}}(n)$ exist for all n. As was mentioned above, $U_{\mathbb{Z}}(n) = n+3$ for $1 \leq n \leq 5$. In this paper, we investigate the minimal rank $U_{\mathbb{Z}}(n)$ of n-universal $\mathbb{Z}$-lattices for $6 \leq n \leq 10$. We prove that $U_{\mathbb{Z}}(n) = 13, 15, 16, 28, 30$ for $n = 6, 7, 8, 9, 10$ respectively, and find all $6, 7, 8$-universal $\mathbb{Z}$-lattices of rank $13, 15, 16$, respectively. For the complete list of $1, 2$-universal $\mathbb{Z}$-lattices of minimal rank, see [6], [7], [16] and [21].

In [1], Bannai proved that most unimodular lattices (even or odd) have trivial automorphism groups if the rank is sufficiently large, and that such lattices are indecomposable. If a lattice L is n-universal, then L must represent all indecomposable unimodular lattices of rank n as direct summands. So from this we may guess that $U_{\mathbb{Z}}(n)$ grows very quickly.

Remark. Note that if we define $U_{\mathbb{Q}}(n)$ to be the minimal rank of n-universal positive definite quadratic space over $\mathbb{Q}$, then $U_{\mathbb{Q}}(n) = n+3$ for all n.

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We adopt terminologies and notations from [2], [3] and [14]. By $l \to L$ we mean that the lattice L represents the lattice l. For a sublattice l of $L \perp M$ of the form $l = \mathbb{Z}(x_1 + y_1) + \mathbb{Z}(x_2 + y_2) + \cdots + \mathbb{Z}(x_n + y_n)$ for $x_i \in L$ and $y_i \in M$, we define sublattices $l(L) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$ and $l(M) = \mathbb{Z}y_1 + \mathbb{Z}y_2 + \cdots + \mathbb{Z}y_n$. A lattice l is said to be *additively indecomposable* if either $l(L) = 0$ or $l(M) = 0$ whenever $l \to L \perp M$.

2. Determination of $U_{\mathbb{Z}}(6)$

We assume that L is a 6-universal $\mathbb{Z}$-lattice. Since L must represent the lattice I_6, it decomposes into $I_6 \perp L'$. Furthermore, since the root lattice E_6 is additively indecomposable, it should be represented by L'. Therefore $U_{\mathbb{Z}}(n) \geq 12$. Suppose rank $L = 12$; then $L = I_6 \perp E_6$. But this cannot represent the root lattice A_6. On the other hand, the lattice $E_6 \perp I_{10}$ is 6-universal because E_6 is the unique additively indecomposable lattice of rank 6 and $E_6^{(2)}$ is represented by I_8, where $E_6^{(2)}$ is the lattice obtained from scaling E_6 by 2 (see [8] and [11]). Therefore $13 \leq U_{\mathbb{Z}}(n) \leq 16$. If L is 6-universal and rank $L = 13$, then L must be equal to $E_6 \perp I_7$ or $E_7 \perp I_6$ because the only lattice of rank 7 which represents both A_6 and E_6 is E_7. In this section, we prove that $E_6 \perp I_7$ and $E_7 \perp I_6$ are indeed 6-universal lattices of rank 13.

Lemma 2.1. *If a lattice L of rank $n+3$ has a square free determinant and its quadratic norm $Q(L)$ is not contained in $2\mathbb{Z}$, then every lattice l of rank n is represented by a lattice in the genus of L.*

Proof. The local lattice L_p is n-universal over $\mathbb{Z}_p$ by [13]. So the lemma follows directly from [14, 102:5]. (See also [4].) $\square$

Now, we prove the following technical lemma, which is useful in the sequel.

Lemma 2.2. *Let l be a lattice of rank n which is represented by I_m, $m \geq 7$.*

(1) If $5 \leq n \leq m-2$, then l is represented by $D_{n-i} \perp I_{m-n+i}$ for some $i = 1, 2, \ldots, n-1$, where D_k is the root lattice of type D for $k \geq 4$, $D_3 = A_3$, $D_2 = A_1 \perp A_1$, and $D_1 = \langle 4 \rangle$.

(2) If $n = m-1$, then l is represented by $D_{n+1-i} \perp I_i$ for some $i = 0, 1, 2, \ldots, n$.

Furthermore, if l is represented by $D_{n+1-i} \perp I_i$ only for $i = 0$ or 1 and $n \equiv i \pmod 2$, then $dl \equiv n - i + 1 \pmod 4$.

Proof. We only prove (1). The proof of (2) is quite similar to that of (1). It suffices to show this when m is equal to $n+2$. We may assume that $l = \bigoplus_{i=1}^{n} \mathbb{Z}(\sum_{k=1}^{m} a_{ik} e_k)$ is a sublattice of I_m, where the e_i's are the standard orthonormal basis of I_m. By suitable base change, we may also assume that $a_{ij} = 0$ for all i, j satisfying $i \geq 2$ and $j \geq m + 2 - i$, and that the $a_{k(m+1-i)}$'s are even for $1 \leq k \leq i-1$ if $a_{i(m+1-i)}$ is odd for some $i \geq 2$. For a subset $J = \{j_1, j_2, \ldots, j_r\} \subset \{1, 2, \ldots, m\}$, we define the lattice $M_J = \bigoplus_{i=1}^{n} \mathbb{Z}(\sum_{t=1}^{r} a_{ij_t} e_{j_t})$. We let $J = \{m\}$ if a_{1m} is even. Then $M_J = \mathbb{Z}(a_{1m} e_m) \to D_1$. Assume that $a_{i(m+1-i)}$ is even for some i, $2 \leq i \leq n-1$. Let i be the smallest such. Let J be the set containing $(m+1-i)$ and all $(m-k+1)$'s, $1 \leq k \leq i-1$, for which the $a_{k(m+1-i)}$'s are odd. Then $M_J \to D_{|J|}$ and hence $l \to D_{|J|} \perp I_{m-|J|}$. Therefore we may assume that the $a_{i(m+1-i)}$'s are odd for all i, $1 \leq i \leq n-1$.

Now assume that a_{nj} is even for some j, $1 \leq j \leq 3$. If not all a_{kj}'s are odd for $1 \leq k \leq n-1$, then $M_J \to D_{|J|}$ as above. Hence if one of a_{nj} is even for $j = 1, 2, 3$, then we may assume that the a_{kj}'s are all odd for $k = 1, 2, \ldots, n-1$. If two of the a_{nj}'s are even, then $l \to D_2 \perp I_{m-2}$. Therefore, without loss of generality, we may assume that a_{n3} is odd and a_{k3} are all even for $k = 1, 2, \ldots, n-1$. For a fixed s, $s = 1$ or 2, if the number of a_{ks}'s which are odd is less than $n-1$ for $k = 1, 2, \ldots, n$, then $l \to D_{|J|} \perp I_{m-|J|}$, where J is the set containing s and the $(m-k+1)$'s for which the a_{ks}'s are odd. In the remaining case, it is easy to see that $l \to D_k \perp I_{m-k}$, where k is 2 or 3 or 4. $\square$

Theorem 2.3. *The lattice $E_7 \perp I_6$ is 6-universal. In particular, $U_{\mathbb{Z}}(n) = 13$.*

Proof. First observe that $gen(I_8 \perp A_1) = \{I_8 \perp A_1, E_7 \perp I_2\}$. Hence it suffices to show that every sublattice l of $A_1 \perp I_8$ of rank 6 is represented by $E_7 \perp I_6$ by Lemma 2.1. By Lemma 2.2 (1), $l(I_8) \to D_{5-i} \perp I_{3+i}$, for some $i = 0, 1, \ldots, 4$. If $i \neq 4$, then we have

$$l \to A_1 \perp I_8 \to A_1 \perp D_{5-i} \perp I_{i+3} \to E_7 \perp I_6.$$

If $i = 4$, then $l' = l(I_8) \to D_1 \perp I_7$. We apply Lemma 2.2 (2) to $l'(I_7)$. By similar reasoning as above, we need only consider the case when $l'(I_7) \to D_7$. This indeed implies $l \to A_1 \perp D_1 \perp D_7$ and $d(l(D_7)) \equiv 7 \pmod 8$. For all prime p (including ∞), since $l(D_7)_p$ is represented by $(E_7 \perp A_1)_p$ and the class number of $E_7 \perp A_1$ is 1 [19], we have $l(D_7) \to E_7 \perp A_1$, which proves the theorem. $\square$

In order to prove that $E_6 \perp I_7$ is the other 6-universal lattice of rank 13, we need the following lemma.

Lemma 2.4. *If a $\mathbb{Z}$-lattice l of rank 6 is not represented by a sum of squares, then $l \to E_6 \perp I_5$.*

Proof. We may assume that $l \to E_7 \perp I_2$. By [8], we may also assume that $d(l(E_7))$ is an odd determinant. Since the class number of $E_6 \perp A_2$ is 1, it can easily be checked that $l(E_7) \to E_6 \perp A_2$, and hence $l \to E_6 \perp I_5$ if $d(l(E_7)) \not\equiv 1 \pmod 3$. So we assume that $d(l(E_7)) \equiv 1 \pmod 6$. By considering local conditions for representation, we can conclude that $l(E_7) \to gen(E_6 \perp I_2)$ and consequently $l \to E_6 \perp I_5$ from the fact that $gen(E_6 \perp I_2) = \{E_6 \perp I_2, \langle 3\rangle \perp I_7\}$. $\square$

Remark. Ko conjectured [11] that if l is of rank 6 and represented by a sum of squares, then $l \to I_9$, and if l is of rank 6 and not represented by a sum of squares, then $l \to E_6 \perp I_3$ and $l(E_6) = E_6$. But both conjectures are false because $l = A_2 \perp A_2 \perp A_1 10[1\frac{1}{2}]$ is represented by I_{10} but not by I_9 for the former conjecture (see [8], [9] for further results) and $l = D_5 124[1\frac{1}{4}]$, which is not represented by a sum of squares, is represented by $E_6 \perp I_3$ but does not satisfy $l(E_6) = E_6$.

Theorem 2.5. *The lattice $E_6 \perp I_7$ is 6-universal.*

Proof. Let l be a $\mathbb{Z}$-lattice of rank 6. By the above lemma, we may assume that l is represented by a sum of squares, and hence by [8] we may assume that $l \to I_{10}$. This implies that $l \to D_{5-i} \perp I_{5+i}$ for some $i = 0, 1, \ldots, 4$ by Lemma 2.2 (1). If $i \neq 3, 4$, then $l \to D_{5-i} \perp I_{5+i} \to E_6 \perp I_7$. The desired conclusion for the case when $i = 3, 4$ can be deduced by applying Lemma 2.2 again if necessary. $\square$

3. DETERMINATION OF $U_{\mathbb{Z}}(n)$ FOR $7 \le n \le 10$

Theorem 3.1. *The lattice* $E_8 \perp I_8$ *is a unique* 8*-universal* $\mathbb{Z}$*-lattice of rank* 16*, and* $U_{\mathbb{Z}}(8) = 16$*.*

Proof. Note that the lattice $E_8 \perp I_8$ is the unique candidate of 8-universal $\mathbb{Z}$-lattice of rank 16, for E_8 is the unique additively indecomposable $\mathbb{Z}$-lattice of rank 8. Let l be a $\mathbb{Z}$-lattice of rank 8. Since $l \to gen(E_8 \perp I_3) = \{E_8 \perp I_3, I_{11}\}$, we may assume that $l \to I_{11}$. By Lemma 2.2, we may further assume that $l \to A_1 \perp A_1 \perp D_9$ and $d(l(D_9)) \equiv 1 \pmod 8$. Clearly, $l(D_9)$ is contained in one of the sublattices of I_9 of rank 9 with determinant 9. The following are all such sublattices of I_9:

$$\langle 9\rangle \perp I_8, A_1 18[1\frac{1}{2}] \perp I_7, A_2 \perp \langle 3\rangle \perp I_6, A_3 36[1\frac{1}{4}] \perp I_5, A_4 45[2\frac{1}{5}] \perp I_4,$$

$$A_5 6[3\frac{1}{2}] \perp I_3, A_6 63[3\frac{1}{7}] \perp I_2, A_7 72[3\frac{1}{8}] \perp I_1, \text{ and } A_8 9[3\frac{1}{3}].$$

One can easily check that if $l(D_9)$ is represented by one of these lattices except the first one, then $l \to E_8 \perp I_8$. So assume that $l(D_9) \to \langle 9\rangle \perp I_8$. Then $l(D_9)$ is represented by $\mathbb{Z}(e_1 - e_2) + \mathbb{Z}(e_2 - e_3) + \cdots + \mathbb{Z}(e_7 - e_8) + \mathbb{Z}(e_8 - 3e_9) + \mathbb{Z}(e_8 + 3e_9)$ and hence is represented by $A_8 \perp I_5$. Therefore l is represented by $E_8 \perp I_8$. ☐

Remark. In [5], Conway and Schneeberger proved the so-called 15-Theorem, i.e., every integral $\mathbb{Z}$-lattice which represents $1, 2, 3, 5, 6, 7, 10, 14, 15$ is 1-universal. An analogy for 8-universal $\mathbb{Z}$-lattices can be deduced from Theorem 3.1: Every $\mathbb{Z}$-lattice which represents both I_8 and E_8 is 8-universal.

Corollary 3.2. *The lattice* $E_8 \perp I_7$ *is* 7*-universal and* $U_{\mathbb{Z}}(7) = 15$*.*

Proof. The 7-universality of $E_8 \perp I_7$ follows from the above theorem. Consider the only possible candidate for a 7-universal $\mathbb{Z}$-lattice of rank 14; namely, $E_7 \perp I_7$. But this cannot represent $A_6 77[2\frac{1}{7}]$, and the result follows. ☐

Theorem 3.3. *There are exactly three* 7*-universal* $\mathbb{Z}$*-lattices of rank* 15*. They are* $E_8 \perp I_7$*,* $E_7 \perp I_8$*, and* $E_7 6[1\frac{1}{2}] \perp I_7$*.*

Proof. Suppose that L is a 7-universal $\mathbb{Z}$-lattice of rank 15. Then $L = I_7 \perp L'$ and rank $(L') = 8$. Clearly, $E_7 \to L'$. If the lattice L' represents 1, then $L = I_8 \perp E_7$. So assume that L' does not represent 1. Since $A_6 77[2\frac{1}{7}] \to L$, either $D_7 \to L'$ or $A_6 77[2\frac{1}{7}] \to L'$. In the first case, L' must be E_8, for E_8 is the only lattice of rank 8 which represents E_7 and D_7 simultaneously. In the second case, since the minimum quadratic norm of the dual lattice $E_7^{\#}$ of E_7 is $\frac{3}{2}$, it can be easily deduced that L' must be $E_7 6[1\frac{1}{2}]$. Hence we have exactly three candidates $E_8 \perp I_7$, $E_7 \perp I_8$ and $E_7 6[1\frac{1}{2}] \perp I_7$ for 7-universal $\mathbb{Z}$-lattices of minimal rank, 15.

It suffices to show the 7-universality for the latter two. First, we show that $E_7 \perp I_8$ is 7-universal. Let l be any $\mathbb{Z}$-lattice of rank 7. Note that

$$l \to gen(E_8 \perp I_2) = \{E_8 \perp I_2, I_{10}\}.$$

If $l \to I_{10}$, it is easy to check that $l \to E_7 \perp I_8$ by Lemma 2.2(1). So assume that $l \to E_8 \perp I_2$. Note that $l(E_8)$ can be represented by one of the sublattices of E_8 with determinant 4; the only such sublattices are $E_7 \perp A_1$ and D_8. Therefore the 7-universality of $E_7 \perp I_8$ follows immediately.

Now we prove that $E_7 6[1\frac{1}{2}] \perp I_7$ is 7-universal. Note that for every $\mathbb{Z}$-lattice l of rank 7

$$l \to gen(E_7 6[1\frac{1}{2}] \perp I_2) = \{E_7 6[1\frac{1}{2}] \perp I_2, A_2 \perp I_8\}.$$

So we assume that $l \to A_2 \perp I_8$. Then $l(I_8)$ is contained in one such sublattice of I_8 of rank 8 with determinant 9. It is easy to check that $l \to E_7 6[1\frac{1}{2}] \perp I_7$ if $l(I_8)$ is contained in one such sublattice except $A_7 72[3\frac{1}{8}]$. Therefore, we may restrict ourselves to the case when

$$l(I_8) \to A_7 72[3\frac{1}{8}] = \{ \sum_{i=1}^{8} a_i e_i \mid \sum_{i=1}^{8} a_i \equiv 0 \pmod 3 \}.$$

Furthermore, we may assume that $d(l(I_8)) \equiv 2 \pmod 3$, for we may assume that $l(I_8)$ is not contained in any sublattice of I_8 of rank 8 with determinant 9 other than $A_7 72[3\frac{1}{8}]$. By Lemma 2.2, we obtain $l \to D_{8-i} \perp I_i \perp A_2$ for $i = 0, 1, \ldots, 7$. If $i \neq 0, 1$, then this implies $l \to E_7 6[1\frac{1}{2}] \perp I_7$, as desired. If $i = 0$, then

$$l(I_8) \to A_7 72[2\frac{1}{4}] = \{ \sum_{i=1}^{8} a_i e_i \mid \sum_{i=1}^{8} a_i \equiv 0 \pmod 6 \} \to E_7 6[1\frac{1}{2}] \perp I_3$$

and hence $l \to E_7 6[1\frac{1}{2}] \perp I_7$. If $i = 1$, then we may assume that $d(l(I_8)) \equiv 11 \pmod{12}$ by Lemma 2.2(2). Therefore

$$l(I_8) \to gen(A_2 \perp I_7) = \{A_2 \perp I_7, E_7 6[1\frac{1}{2}] \perp I_1\}.$$

Consequently, $l \to E_7 6[1\frac{1}{2}] \perp I_7$ as desired. □

Theorem 3.4. *The lattice* $E_8 \perp I_9 \perp D_{10} A_1[11]$ *is a* 9*-universal* $\mathbb{Z}$*-lattice and* $U_{\mathbb{Z}}(9) = 28$.

Proof. Suppose that L is a 9-universal $\mathbb{Z}$-lattice. Then L must decompose into $E_8 \perp I_9 \perp L'$. There exist exactly two additively indecomposable $\mathbb{Z}$-lattices of rank 9, namely, $A_8 63[4\frac{1}{9}]$ and $A_4 A_4 15[33\frac{1}{5}]$ (see [17]). Since L' must represent these lattices, the rank of L' is greater than 9. Suppose that the rank of L' is 10. Then $A_9 \to L'$, since $1 \notin Q(L')$. Furthermore, L' has a vector of norm 3, since $A_8 63[4\frac{1}{9}] \to L'$. The possible candidates for L' satisfying these properties are the following:

$$A_9 210[1\frac{1}{10}], A_9 35[2\frac{1}{5}], A_9 90[3\frac{1}{10}], A_9 15[4\frac{1}{5}], A_9 A_1[5\frac{1}{2}], \text{ and } A_9 \perp \langle 3 \rangle.$$

Among these lattices, only $A_9 15[4\frac{1}{5}]$ and $A_9 A_1[5\frac{1}{2}]$ can represent $A_8 63[4\frac{1}{9}]$ and $A_4 A_4 15[33\frac{1}{5}]$ simultaneously. But neither $E_8 \perp I_9 \perp A_9 15[4\frac{1}{5}]$ nor $E_8 \perp I_9 \perp A_9 A_1[5\frac{1}{2}]$ can represent $A_8 117[2\frac{1}{9}]$. Therefore the rank of L' is greater than 10. On the other hand, it can be easily checked by using Lemma 2.2 that every $\mathbb{Z}$-lattice l of rank 9, which is represented by $A_1 \perp I_{11}$, is represented by $E_8 \perp D_{10} A_1[11] \perp I_9$. Hence from

$$gen(A_1 \perp I_{11}) = \{A_1 \perp I_{11}, E_7 \perp I_5, D_{10} A_1[11] \perp I_1, E_8 \perp I_3 \perp A_1\}$$

we may conclude that $E_8 \perp D_{10} A_1[11] \perp I_9$ is 9-universal, and the result follows. □

Theorem 3.5. *The lattice* $E_8 \perp I_{10} \perp D_{12}[1]$ *is* 10*-universal and* $U_{\mathbb{Z}}(10) = 30$.

Proof. Suppose that L is a 10-universal $\mathbb{Z}$-lattice. The lattice L must decompose into $E_8 \perp I_{10} \perp L'$. The lattices $D_9 12[1\frac{1}{4}]$, $A_9 A_1[51]$ are additively indecomposable $\mathbb{Z}$-lattices of rank 10 (see [17]), so L' must represent these lattices. Suppose that the rank of L' is 11; then $A_{10} \to L'$, since $1 \notin Q(L')$. But there does not exist a lattice of rank 11 which represents the lattices A_{10} and $D_9 12[1\frac{1}{4}]$ simultaneously. Hence the rank of L' is greater than 11. On the other hand, since

$$gen(I_{13}) = \{I_{13}, E_8 \perp I_5, D_{12}[1] \perp I_1\},$$

it can be easily checked by applying Lemma 2.2 that $E_8 \perp I_{10} \perp D_{12}[1]$ is 10-universal. Therefore the result follows. □

Remark. It seems to be a very difficult problem to find the exact value of $U_{\mathbb{Z}}(n)$ for large n. For example, one can easily obtain $U_{\mathbb{Z}}(24) \geq 6673$ from a simple counting of all indecomposable unimodular $\mathbb{Z}$-lattices of rank less than or equal to 24 (see [2]).

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