

## ON SUBDIAGONAL ALGEBRAS FOR SUBFACTORS

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ABSTRACT. We show that if  $N \subseteq M$  are type  $II_1$  factors with finite index (and common identity) and  $E : M \rightarrow N$  is the trace preserving conditional expectation, then there are no subdiagonal algebras in  $M$  with respect to  $E$  unless  $M = N$ .

Let  $N \subseteq M$  be finite factors (with common identity) and let  $E : M \rightarrow N$  be the trace preserving conditional expectation. A  $\sigma$ -weakly closed algebra  $A$  is (maximal) subdiagonal in  $M$  with respect to  $E$  if the following conditions are satisfied [1]:

- [S1]  $A \cap A^* = N$ .
- [S2]  $A + A^*$  is  $\sigma$ -weakly dense in  $M$ .
- [S3]  $E(xy) = E(x)E(y)$  for any  $x, y \in A$ .

Saito and Watatani showed in [4] that if  $M$  is a crossed product for an outer action of a discrete group  $G$  on  $N$ , then maximal (inclusionwise) subdiagonal algebras are in bijective correspondence with left invariant total orders on  $G$ . A totally ordered group is torsion free and, thus, infinite. On the basis of this result Saito and Watatani conjectured [4] that if  $N$  is a proper subfactor of  $M$  with finite index, then there are no subdiagonal algebras in  $M$  with respect to  $E$ . They have verified this in the case when either  $M$  is finite dimensional or the index does not exceed 4 [4, 5]. The purpose of this paper is to prove the conjecture in full generality.

**Theorem 1.** *Let  $N \subseteq M$  be an inclusion of type  $II_1$  factors with common identity and finite index, and let  $E : M \rightarrow N$  be the trace preserving conditional expectation. Then there are no subdiagonal algebras in  $M$  with respect to  $E$  unless  $M = N$ .*

*Proof.* Suppose that  $A$  is subdiagonal in  $M$  with respect to  $E$  and denote  $A_0 = A \cap \ker E$ . Let  $\tau$  be the trace on  $M$ ,  $\Omega$  the canonical trace vector in  $\mathcal{H} = L^2(M, \tau)$ , and  $e_N$  the Jones projection. We let  $M_1 = \langle M, e_N \rangle$  be the Jones extension of  $M$  by  $N$ . Let  $f, g$  be the orthogonal projections of  $\mathcal{H}$  onto  $\overline{A_0 \mathcal{H}}$  and  $\overline{A_0^* \mathcal{H}}$ , respectively. Then  $I = e_N + f + g$  and  $e_N, f, g \in N' \cap M_1$  [4]. We define  $E_{A_0} : M \rightarrow M$  by

$$E_{A_0}(x) = [M : N]E_M(fxe_N),$$

where  $E_M : M_1 \rightarrow M$  is the trace preserving conditional expectation. We claim that  $E_{A_0}$  is an  $N$ - $N$ -bimodule projection of  $M$  onto  $A_0$ . Indeed, if  $n \in N$ ,  $a \in A_0$ ,

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and  $x \in M$ , then

$$\begin{aligned} fne_Nx\Omega &= fnE(x)\Omega = 0, \\ fa^*e_Nx\Omega &= f(E(x^*)a)^*\Omega = 0, \\ fae_Nx\Omega &= faE(x)\Omega = aE(x)\Omega = ae_Nx\Omega, \end{aligned}$$

since  $A_0$  is an  $N$ - $N$ -bimodule. Consequently,  $E_{A_0}(N) = 0$ ,  $E_{A_0}(A_0^*) = 0$ , and

$$E_{A_0}(a) = [M : N]E_M(fae_N) = [M : N]E_M(ae_N) = a$$

for any  $a \in A_0$ . Hence, the claim follows from [S1] and [S2]. Likewise,

$$E_{A_0^*}(x) = [M : N]E_M(gxe_N)$$

gives an  $N$ - $N$ -bimodule projection of  $M$  onto  $A_0^*$ .

We now consider a tunnel (downward basic construction - cf. [3])

$$\dots \subseteq M_{-i} \subseteq \dots \subseteq M_{-1} \subseteq M_0,$$

where  $M_0 = M$  and  $M_{-1} = N$ . We fix  $i$  and denote  $\tilde{N} = M'_{-i} \cap M_{-1}$ ,  $\tilde{M} = M'_{-i} \cap M_0$ ,  $\tilde{A} = M'_{-i} \cap A$ , and  $\tilde{E} = E|_{\tilde{M}} : \tilde{M} \rightarrow \tilde{N}$ . Note that  $\tilde{M}$  is finite dimensional [3]. We claim that  $\tilde{A}$  is a subdiagonal algebra in  $\tilde{M}$  with respect to  $\tilde{E}$ . Indeed, [S1] and [S3] are obviously satisfied. For any  $x \in \tilde{M}$  we have

$$x = [M : N]E_M((e_N + f + g)xe_N) = E(x) + E_{A_0}(x) + E_{A_0^*}(x).$$

Since  $E : \tilde{M} \rightarrow \tilde{N}$ ,  $E_{A_0} : \tilde{M} \rightarrow \tilde{A}$ , and  $E_{A_0^*} : \tilde{M} \rightarrow \tilde{A}^*$ , [S2] holds true as well, and the claim is proved.

Let  $p$  be a projection in  $\tilde{N}$ . We denote  $\tilde{N}_p = p\tilde{N}p$ ,  $\tilde{M}_p = p\tilde{M}p$ ,  $\tilde{A}_p = p\tilde{A}p$ , and  $\tilde{E}_p : \tilde{M}_p \rightarrow \tilde{N}_p$ , the restriction of  $\tilde{E}$  to  $\tilde{M}_p$ . Clearly,  $\tilde{A}_p$  is subdiagonal in  $\tilde{M}_p$  with respect to  $\tilde{E}_p$ . Suppose now that  $p$  is a minimal projection in  $\tilde{N}$ . Thus,  $\tilde{N}_p = Cp$  and there exists a faithful normalized trace  $tr$  on  $\tilde{M}_p$  such that  $\tilde{E}_p(x) = tr(x)p$ . Essentially the same argument as in Theorem 1 of [5] shows that we must have  $\tilde{N}_p = \tilde{M}_p$ . For the sake of completeness we give that reasoning below.

We denote  $(\tilde{A}_p)_0 = \{x \in \tilde{A}_p \mid tr(x) = 0\}$ . Let  $\tilde{M}_p \cong \bigoplus_{k=1}^r M_{n_k}(C)$  and let  $\lambda_k$  be the weight of  $tr$  corresponding to  $M_{n_k}(C)$ . Thus,  $\lambda_k > 0$  for each  $k = 1, \dots, r$  and  $\sum_k \lambda_k = 1$ . If  $x \in (\tilde{A}_p)_0$ , then there exist invertible  $v \in \tilde{M}_p$  such that  $vxv^{-1} = \bigoplus_{k=1}^r y_k$ , with each  $y_k$  lower triangular in  $M_{n_k}(C)$ . Let the diagonal entries of  $y_k$  be  $d_{k,1}, \dots, d_{k,n_k}$ . Since  $tr$  is multiplicative on  $\tilde{A}_p$ , for any natural  $m$  we have

$$0 = tr(x^m) = tr((vxv^{-1})^m) = \sum_{k=1}^r \sum_{j=1}^{n_k} \lambda_k d_{k,j}^m.$$

This implies that  $d_{k,j} = 0$  for all  $k, j$  and, hence,  $x$  is nilpotent. By the Corollary on page 13 of [2], for each  $k$  there exists a basis of  $C^{n_k}$  with respect to which all  $\{x_k \mid x = \bigoplus_{t=1}^r x_t, x \in (\tilde{A}_p)_0\}$  are strictly upper triangular. Consequently,

$$\sum_{k=1}^r n_k^2 = \dim \tilde{M}_p = 1 + 2 \dim(\tilde{A}_p)_0 \leq 1 + 2 \sum_{k=1}^r \left(\frac{n_k^2 - n_k}{2}\right).$$

Thus,  $r = 1$ ,  $n_1 = 1$  and, hence,  $\tilde{N}_p = \tilde{M}_p$ , as required.

By the above, any minimal projection in  $\tilde{N}$  is minimal in  $\tilde{M}$  as well. Applied to the case  $i = 1$ , this shows that  $M_{-1}$  is irreducible in  $M_0$ . Hence,  $M_{-2}$  is irreducible

in  $M_{-1}$  [3]. Considering now the case  $i = 2$ , we see that  $M'_{-2} \cap M_0 = CI$  and, hence,  $M = N$ , as required.  $\square$

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