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NEW REPRESENTATIONS OF RAMANUJAN'S TAU FUNCTION

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ABSTRACT. Several formulas for Ramanujan's function τ, defined by

$$x\prod_1^\infty(1-x^n)^{24}=\sum_1^\infty \tau(n)x^n \quad (|x|<1),$$

are presented. We also present a congruence modulo 3 for some of the function values.

1. INTRODUCTION

Ramanujan's function τ is defined by the expansion

$$x\prod_1^\infty(1-x^n)^{24}=\sum_1^\infty \tau(n)x^n, \tag{1.1}$$

which is valid for each complex number x such that $|x|<1$. In this paper we present several formulas for τ, including one congruence involving function values modulo 3. Since these formulas involve several additional functions, we collect these in the following definition.

Definition 1.1. For $\mathbb{N} := \{0,1,2,\cdots\}$, put $\mathbb{P} := \mathbb{N}-\{0\}$. Then, for each $k\in\mathbb{P}$ and each $n\in\mathbb{N}$,

$$r_k(n) := |\{(x_1,x_2,\cdots,x_k)\in\mathbb{Z}^k | n = x_1^2+x_2^2+\cdots+x_k^2\}|.$$

(Of course, $\mathbb{Z} := \{0,\pm1,\pm2,\cdots\}$.)

For each $n\in\mathbb{P}, b(n)$ is the exponent of the exact power of 2 dividing n, and then $Od(n) := n2^{-b(n)}$ is the odd part of n.

For each $k\in\mathbb{N}$ and each $n\in\mathbb{P}, \sigma_k(n)$ is the sum of the kth powers of all of the positive divisors of n. For simplicity, $\sigma(n) := \sigma_1(n)$.

We are now prepared to state our main result.

Theorem 1.2. *For each $n\in\mathbb{N}$ and each $m\in\mathbb{P}$,*

(1.2)
$$\tau(4n+2) = -3\sum_{k=1}^{2n+1} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n+2-2k-j)r_8(j);$$

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$$\sum_{k=1}^{n} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(2n+1-2k-j)r_8(j)=0; \tag{1.3}$$

(1.4)

$$\tau(4m)=-2^{11}\tau(m)-3\sum_{k=1}^{2m} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4m-2k-j)r_8(j).$$

[*Here, the second sums of* (1.2), (1.3) *and* (1.4) *have respectively upper limits of summation* $4n+2-2k,\ 2n+1-2k$ *and* $4m-2k$.]

In section 2 we prove this theorem, and also present two immediate corollaries of the theorem.

In [6, pp.275-278] there is a good, though not exhaustive, list of references for the function τ. Here cited are almost all of the known properties of the function, including formulas, recurrences and congruences which some values satisfy. Perhaps the best known recursive determination of τ is that of Ramanujan [5, p. 152]. For certain analytic properties of τ, not cited in [6], the reader might consult the paper of Moreno [4].

2. Proof of Theorem 1.2

First of all, we state four identities which we require in our development.

$$\prod_{1}^{\infty}(1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1})=\sum_{-\infty}^{\infty}x^{n^2}t^n; \tag{2.1}$$

$$\prod_{1}^{\infty}(1+x^{n})(1-x^{2n-1})=1; \tag{2.2}$$

$$x\prod_{1}^{\infty}\frac{(1-x^{2n})^8}{(1-x^{2n-1})^8}=\sum_{1}^{\infty}2^{3b(n)}\sigma_3(Od(n))x^n; \tag{2.3}$$

$$\prod_{1}^{\infty}(1+x^{2n-1})^8=\prod_{1}^{\infty}(1-x^{2n-1})^8+16x\prod_{1}^{\infty}(1+x^{2n})^8. \tag{2.4}$$

Identity (2.1), the celebrated triple-product identity, is valid for each pair of complex numbers t, x such that $t \neq 0$ and $|x| < 1$. Each of the identities (2.2), (2.3) and (2.4) is valid for each complex number x such that $|x| < 1$. For proofs of (2.1) and (2.2) see [3, pp. 282-283 and p. 277]. For a proof of (2.3) see [1, pp. 1291-1292]; and, for a proof of (2.4) see [2, pp. 421-422]. Actually, we need only two special cases of (2.1) corresponding to the substitutions $t \to 1$ and $t \to -1$. Under the former substitution we observe that the kth power of the right side of the resulting identity generates the sequence $r_k(n), n \in \mathbb{N}$. Similarly, under the latter substitution the sequence $(-1)^n r_k(n), n \in \mathbb{N}$, is generated. We begin our argument by multiplying both sides of (2.4) by the infinite product $\prod(1-x^{2n})^8$ to get

$$-16x\prod_{1}^{\infty}(1-x^{4n})^8=\prod_{1}^{\infty}(1-x^{n})^8-\prod_{1}^{\infty}(1-(-x)^{n})^8.$$

Then, we raise each side of this identity to the third power, and multiply the resulting identity by x to get

(2.5)

$$-2^{12}\sum_1^\infty \tau(n)x^{4n} = 2\sum_1^\infty \tau(2n)x^{2n} - 3x\prod_1^\infty (1-x^n)^{16}(1-(-x)^n)^8 + 3x\prod_1^\infty (1-x^n)^8(1-(-x)^n)^{16}.$$

Next,

$$\begin{aligned}
&-3x\prod_1^\infty (1-x^n)^{16}(1-(-x)^n)^8\\
&= 3(-x)\prod_1^\infty \frac{(1-x^{2n})^{16}(1-x^{2n-1})^{16}(1-x^{2n})^8(1+x^{2n-1})^{16}}{(1+x^{2n-1})^8}\\
&= 3(-x)\prod_1^\infty \frac{(1-x^{2n})^8}{(1+x^{2n-1})^8}\prod_1^\infty (1-x^{2n})^8(1-x^{2n-1})^{16}\prod_1^\infty (1-x^{2n})^8(1+x^{2n-1})^{16}\\
&= 3\sum_1^\infty (-1)^h 2^{3b(h)}\sigma_3(Od(h))x^h\cdot\sum_0^\infty (-1)^j r_8(j)x^j\cdot\sum_0^\infty r_8(k)x^k.
\end{aligned}$$

Similarly,

$$3x\prod_1^\infty (1-x^n)^8(1-(-x)^n)^{16} = 3\sum_1^\infty 2^{3b(h)}\sigma_3(Od(h))x^h\cdot\sum_0^\infty (-1)^j r_8(j)x^j\cdot\sum_0^\infty r_8(k)x^k.$$

Expanding the two products of three series, substituting the resulting expansions into (2.5), and cancelling a factor of 2, we get

$$\begin{aligned}
&-2^{11}\sum_1^\infty \tau(n)x^{4n}\\
= \quad &\sum_0^\infty \tau(4n+2)x^{4n+2} + \sum_1^\infty \tau(4n)x^{4n}\\
&+\sum_{n=0}^\infty x^{2n+1}3\sum_{k=1}^n 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(2n+1-2k-j)r_8(j)\\
&+\sum_{n=0}^\infty x^{4n+2}3\sum_{k+1}^{2n+1} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n+2-2k-j)r_8(j)\\
&+\sum_{n=1}^\infty x^{4n}3\sum_{k=1}^{2n} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n-2k-j)r_8(j).
\end{aligned}$$

Equating coefficients of like powers of x in the foregoing identity we thus prove our theorem.

Corollary 2.1. *For each $n \in \mathbb{N}$,*

$$\tau(2n+1) = \sum_{k=1}^{2n+1} 2^{3\{b(2k)-1\}} \sigma_3(Od(2k)) \sum_{j=0} (-1)^j r_8(4n+2-2k-j) r_8(j),$$

where the upper limit of summation of the second sum is $4n+2-2k$.

Proof. By the multiplicativity of $\tau, \tau(4n+2) = \tau(2(2n+1)) = \tau(2)\tau(2n+1)$. But, for $n = 0$, (1.2) yields $\tau(2) = -24$. Then, cancellation of -24 in (1.2) proves the corollary. □

Corollary 2.2. *For each $m \in \mathbb{P}$,*

$$\tau(4m) \equiv \tau(m) \pmod 3.$$

Proof. Part (1.4) of the theorem, and the obvious result $2 \equiv -1 \pmod 3$. □

Concluding remarks. How could one possibly use Theorem 1.2 to compute the values $\tau(n), n \in \mathbb{P}$? Well, first of all, we'd realize that for $n \in \mathbb{P}, r_8(n)$, like $\sigma_3(n)$, can also be expressed in terms of the positive divisors of n. As a matter of fact, for each $n \in \mathbb{P}$,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3.$$

For example, see [3, p. 314]. Then, we'd use Corollary 2.1 to compute $\tau(n)$ for odd values of n. For $n \equiv 2 \pmod 4$ we'd compute $\tau(n)$ by (1.2). And, for $n \equiv 0 \pmod 4$ we'd appeal to (1.4) and induction.

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