

**FAMILIES OF IRREDUCIBLE PRINCIPALLY POLARIZED
ABELIAN VARIETIES ISOMORPHIC TO A PRODUCT
OF ELLIPTIC CURVES**

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ABSTRACT. For each n greater than or equal to two, we give a family of n -dimensional, irreducible principally polarized abelian varieties isomorphic to a product of elliptic curves. This family corresponds to the modular curve $X_0(n+1)$.

1. INTRODUCTION

The Jacobian variety associated to a compact Riemann surface is an irreducible principally polarized abelian variety. An interesting problem, then, is to find families of Jacobians that are isomorphic to products of sub-tori. Examples of this kind of families are given in [3] and in [4].

A natural generalization is to ask for families of irreducible principally polarized abelian varieties that are isomorphic (and not only isogenous) to products of sub-tori.

In this short paper we prove that, for each dimension n , there is a family of irreducible principally polarized abelian varieties that are isomorphic to a product of n elliptic curves. These families arise from the standard action of the symmetric group and are parametrized by the modular curves $X_0(n+1)$.

The special case of dimension three appears in [8] and in [10]. The more general case for Weyl groups will appear elsewhere.

2. NOTATION

Let S_{n+1} be the symmetric group on $n+1$ letters. We let α and γ denote, respectively, the permutations $\alpha = (1\ 2\ 3\ \dots\ n+1)$ and $\gamma = (1\ 2)$.

Let \mathbb{H}_n denote the Siegel upper half-space of $n \times n$ complex symmetric matrices whose imaginary part is positive definite, and let $\mathrm{Sp}(2n, \mathbb{Z})$ denote the corresponding symplectic group with the action given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} * Z = (AZ + B)(CZ + D)^{-1} .$$

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We let \mathcal{A}_n denote the quotient $\mathbb{H}_n/\mathrm{Sp}(2n, \mathbb{Z})$; that is, the moduli space of principally polarized abelian varieties of dimension n . Then an element of \mathcal{A}_n is (the class of) a pair (A, H) , where A is a complex torus of dimension n and H is a non-degenerate principal Riemann form on A . In the sequel, we abbreviate principally polarized abelian variety by p.p.a.v.

An *automorphism* of (A, H) is an automorphism of the complex torus A which preserves the Riemann form H .

3. RIEMANN MATRICES

Let M_n denote the \mathbb{Z} -module $\bigoplus_1^n \mathbb{Z}$. Then the map $\phi : M_n \rightarrow M_{n+1}$ given by $\phi(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n, -\sum_{i=1}^n x_i)$ is \mathbb{Z} -linear and injective; its image is the hyperplane

$$U = \{(y_1, y_2, \dots, y_n, y_{n+1}) \in M_{n+1} : \sum_{i=1}^{n+1} y_i = 0\}.$$

By permutation of coordinates, each permutation in S_{n+1} induces a \mathbb{Z} -linear map over U . If these maps are transported by ϕ , then we obtain the standard representation (of degree n) of S_{n+1} , denoted $\tilde{\theta} : S_{n+1} \rightarrow \mathrm{GL}_{\mathbb{Z}}(M_n)$.

The following result is clear.

Lemma 3.1. *The direct sum of the representation $\tilde{\theta}$ and its dual induces a faithful symplectic representation of the symmetric group*

$$\theta : S_{n+1} \longrightarrow \mathrm{Sp}(2n, \mathbb{Z}).$$

In our next lemma we characterize the subvariety of \mathbb{H}_n , which is fixed pointwise by $\theta(S_{n+1})$.

Lemma 3.2. *Let θ be the given symplectic representation of S_{n+1} . Then the subvariety $\mathbb{H}_n(\theta)$ of \mathbb{H}_n fixed pointwise under $\theta(S_{n+1})$ is of dimension one and its elements are given by the Riemann matrices of the form*

$$Z_\tau = \tau \begin{pmatrix} n & -1 & \dots & -1 \\ -1 & n & & \vdots \\ & & \ddots & -1 \\ -1 & \dots & -1 & n \end{pmatrix},$$

with $\Im \tau > 0$.

Proof. For every permutation $g \in S_{n+1}$, let m_g denote the corresponding $n \times n$ unitary integral matrix $\tilde{\theta}(g)$. Then $\theta(g) = m_g \oplus {}^t m_g^{-1}$, where t denotes the transpose of the given matrix.

Hence the action of $\theta(g)$ on \mathbb{H}_n is given by $Z \rightarrow m_g Z {}^t m_g$ and the result follows (for instance, by calculating with the generators α and γ). \square

Passing to the quotient \mathcal{A}_n , the pencil of Lemma 3.2 defines a one-dimensional family of p.p.a.v.'s and we obtain the following result.

Proposition 3.3. *For each n greater than or equal to two, there exists a pencil $\mathcal{A}_n(\theta)$ of p.p.a.v.'s of dimension n such that, for each A in $\mathcal{A}_n(\theta)$, $\mathrm{Aut}(A)$ contains $\theta(S_{n+1})$.*

Also, we can now prove the following two results.

Corollary 3.4. *For each n greater than or equal to five, there are no Jacobians in $\mathcal{A}_n(\theta)$.*

Proof. Let C be a curve of genus n at least five, and let $J(C)$ denote its Jacobian. If we assume that $J(C)$ belongs to $\mathcal{A}_n(\theta)$, then $\text{Aut}(J(C))$ contains a subgroup G isomorphic to S_{n+1} . But, by Torelli's theorem, $\text{Aut}(J(C))$ is either isomorphic to a $\mathbb{Z}/2\mathbb{Z}$ -extension of $\text{Aut}(C)$ or to $\text{Aut}(C)$, which is a contradiction by the Hurwitz bound for $\text{Aut}(C)$. \square

Corollary 3.5. *For n equal to four, the only curve whose Jacobian is contained in $\mathcal{A}_4(\theta)$ is Bring's curve.*

Proof. We begin by recalling that Bring's curve, which we denote by B , is the unique curve of genus four such that its automorphism group contains (in fact, it is isomorphic to) a group isomorphic to S_5 .

Following [9], a Riemann matrix for $J(B)$ may be taken as

$$Z_B = \tau_0 \begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & 1 & -1 \\ -1 & 1 & 4 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix}$$

where $\tau_0 \in \mathbb{H}_1$ is such that $j(\tau_0) = \frac{-25}{2}$ and that $j(5\tau_0) = \frac{-5 \times 29^3}{2^5}$.

If $A = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, then, for $E = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}$ in $\text{Sp}(8, \mathbb{Z})$, we have that $E * Z_B = Z_{\tau_0}$ is in $\mathbb{H}_4(\theta)$. \square

Remark 3.6. The cases $n = 3$ and $n = 2$ are discussed later on. See Remark 4.4 and Section 6.

4. DECOMPOSITION AND IRREDUCIBILITY

In this section we will study the different kinds of decompositions that the p.p.a.v.'s of $\mathcal{A}_n(\theta)$ admit.

Since there are different kinds of equivalences between Riemann matrices (or abelian varieties), and so as to avoid ambiguities, we begin by recalling the definitions involved in our results.

The two p.p.a.v.'s (A_1, H_1) and (A_2, H_2) are called *isomorphic* if there exists a surjective homomorphism $h : A_1 \rightarrow A_2$ with trivial kernel. They will be called *isomorphic as principally polarized abelian varieties* if there exists an isomorphism h (as above) which further preserves the given polarizations H_1 and H_2 .

A p.p.a.v. is called *irreducible* if it is not isomorphic as a principally polarized abelian variety to the product of p.p.a.v.'s of smaller dimensions.

We can now prove the following result.

Theorem 4.1. *Each A_τ in $\mathcal{A}_n(\theta)$ that is induced by a Riemann matrix $Z_\tau \in \mathbb{H}_n(\theta)$ is isomorphic to the following product of n elliptic curves:*

$$E_{(n+1)\tau} \times E_{(n+1)\tau} \times \cdots \times E_{(n+1)\tau} \times E_\tau .$$

Proof. We consider the matrices C , D , and M in $\mathrm{GL}(n, \mathbb{Z})$ as well as the $2n \times 2n$ unimodular matrix N given by

$$C = \begin{pmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 1 & 1 \\ & & \dots & & \\ 1 & 1 & & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} n+1 & 0 & \dots & 0 & 0 \\ 0 & n+1 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & & n+1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} C & 0 \\ 0 & M \end{pmatrix}, \quad \text{respectively.}$$

Then we have

$$C(I_n \ Z_\tau) = (I_n \ \tau D)N,$$

for $Z_\tau \in \mathbb{H}_n(\theta)$, and the result follows. \square

The problem of finding Jacobians which are isomorphic (and not only isogenous) to the product of elliptic curves, for any value of the genus of the corresponding curve, is a difficult one (cf. [4]).

As far as the authors have been able to learn from the literature, it was known to be possible up to genus three (cf. [2]).

Our next result gives an example for genus four.

Corollary 4.2. *Let B be the Bring's curve, as in Corollary 3.5. Then the Jacobian of B is isomorphic to the product of elliptic curves.*

It is well known that each p.p.a.v. (A, H) of dimension n is isomorphic as a principally polarized abelian variety to a unique product of irreducible p.p.a.v.'s $\prod_{i=1}^s (A_i, H_i)$ (cf. [1]); this is the decomposition of the theta divisor associated to the polarization H of A .

We now prove that, for n at least three, each $A \in \mathbb{H}_n(\theta)$ is irreducible as a principally polarized abelian variety. Remark that the irreducibility is well known for the Jacobian of a curve [5]. Notwithstanding, we already know that: our pencil does not contain Jacobians for $n \geq 5$, and, for $n = 4$, it only contains one; now, for $2 \leq n \leq 3$, we only know a priori that the generic points of $\mathbb{H}_2(\theta)$ and $\mathbb{H}_3(\theta)$ are Jacobians.

Theorem 4.3. *For each n greater than or equal to three, every A in $\mathcal{A}_n(\theta)$ is irreducible as a principally polarized abelian variety.*

Proof. Let (A, H) be in $\mathcal{A}_n(\theta)$ and assume that (A, H) is reducible as a principally polarized abelian variety. Then it is isomorphic to a product $(A_1, H_1) \times (A_2, H_2) \times \dots \times (A_s, H_s)$, where each (A_i, H_i) is an irreducible p.p.a.v. of dimension n_i , with $n_i < n$ and $n_1 + n_2 + \dots + n_s = n$.

Since the group S_{n+1} acts on the tangent space at the origin of A and since the standard representation is irreducible, the action must permute transitively the set of components $\{(A_1, H_1), \dots, (A_s, H_s)\}$.

If we call F the isotropy subgroup of one component (A_k, H_k) , then F is a proper subgroup of index γ in S_{n+1} , with $2 \leq \gamma \leq n$, and the action of S_{n+1} on the tangent space is induced by F .

We first assume that $n \geq 4$. Then F is isomorphic to the alternating group A_{n+1} , hence we obtain a contradiction.

For n equal to three, the group F may be isomorphic to A_4 or to D_4 (the dihedral group of order 8) and, again, these subgroups cannot induce the given representation. \square

Remark 4.4. While writing this proof, the authors learned that it had been shown (cf. [8]) that the p.p.a.v.'s of dimension three with standard action of S_4 are all Jacobians, which gives another proof of the irreducibility in the particular case $n = 3$.

If we consider n equal to two, then the standard representation of S_3 is induced by the cyclic subgroup of order three and A could be isomorphic as a principally polarized abelian variety to the product $E_\omega \times E_\omega$, where E_ω is the elliptic curve such that $j(\omega) = 0$. We will see that this case occurs for special values of τ , but, in order to give the result in a precise form, we will postpone the analysis until Section 6.

5. UNIFORMIZATION FOR $A_n(\theta)$

In this section we will discuss a geometric description of the pencil $A_n(\theta)$ of p.p.a.v.'s that admit the standard action of S_{n+1} .

Let $N_{\text{Sp}}(\theta)$ denote the normalizer of the subgroup $\theta(S_{n+1})$ in $\text{Sp}(2n, \mathbb{Z})$, and let $\text{Aut}_{\text{Sp}}(\theta)$ denote the group of automorphisms of S_{n+1} that are realized inside $\text{Sp}(2n, \mathbb{Z})$. Note that all automorphisms of $\text{Sp}(2n, \mathbb{Z})$ are inner (cf. [7]).

We begin by describing the centralizer of $\theta(S_{n+1})$.

For each $M \in N_{\text{Sp}}(\theta)$ we may define an automorphism $f_M : \theta(S_{n+1}) \rightarrow \theta(S_{n+1})$ by $f_M(X) = MXM^{-1}$. Hence we have the following exact sequence of groups:

$$\{e\} \rightarrow \ker \psi \rightarrow N_{\text{Sp}}(\theta) \xrightarrow{\psi} \text{Aut}_{\text{Sp}}(\theta) \rightarrow \{e\},$$

with $\psi(M) = f_M$ for each M in $N_{\text{Sp}}(\theta)$ and where the map $\ker \psi \rightarrow N_{\text{Sp}}(\theta)$ is the natural inclusion.

We can now describe the kernel of ψ (or, equivalently, the centralizer of $\theta(S_{n+1})$ in $\text{Sp}(2n, \mathbb{Z})$).

Lemma 5.1. *The subgroup $\ker \psi$ is isomorphic to the modular group $\Gamma_0(n+1)$.*

Proof. We consider $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\text{Sp}(2n, \mathbb{Z})$. If M is in $\ker \psi$, then

$$(5.1) \quad M \theta(g) M^{-1} = \theta(g), \text{ for every } g \in S_{n+1}.$$

Recalling the notation used in the proof of Lemma 3.2, $\theta(g) = m_g \oplus {}^t m_g^{-1}$, we obtain that (5.1) is equivalent to the equalities $A m_g = m_g A$, $B {}^t m_g^{-1} = m_g B$, $C m_g = {}^t m_g^{-1} C$ and $D {}^t m_g^{-1} = {}^t m_g^{-1} D$, for every $g \in S_{n+1}$.

Since the standard representation $\tilde{\theta}(g) = m_g$ given in Section 3 is irreducible, we obtain that $A = p I_n$, with $p \in \mathbb{Z}$, and that $D = s I_n$, with $s \in \mathbb{Z}$, where I_n is the $n \times n$ identity matrix.

An elementary calculation shows that $B = q U_n$ and $C = r V_n$, where q and r are integers and

$$U_n = \begin{pmatrix} n & -1 & \dots & -1 \\ -1 & n & \dots & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \dots & -1 & n \end{pmatrix}, \quad V_n = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}.$$

Note that $U_n V_n = (n + 1) I_n$.

Since M is symplectic, we obtain that $p s I_n - r q (n + 1) I_n = I_n$; therefore we may define the isomorphism $h : \Gamma_0(n + 1) \rightarrow \ker \psi$ by $h \begin{pmatrix} p & q \\ r(n + 1) & s \end{pmatrix} = \begin{pmatrix} p I_n & q U_n \\ r V_n & s I_n \end{pmatrix}$. □

Since the group S_{n+1} is complete for $n \neq 1$ and $n \neq 5$, we can identify $\text{Aut}_{\text{Sp}}(\theta)$ with $\theta(S_{n+1})$. Furthermore $\theta(S_{n+1})$ is centerless, therefore $\ker \psi \cap \theta(S_{n+1}) = \{e\}$. For $n = 1$, we have $\text{Aut}(S_2) = 1$, and for $n = 5$, a calculation shows that no external automorphism of S_6 is realized in $\text{Sp}(10, \mathbb{Z})$. Hence we have obtained our next result.

Theorem 5.2. *For n greater than or equal to two, $N_{\text{Sp}}(\theta)$ is isomorphic to the semidirect product $\Gamma_0(n + 1) \ltimes \theta(S_{n+1})$.*

Remark 5.3. The natural morphism $\mathbb{H}_n(\theta) \rightarrow \mathbb{H}_1$ which sends Z_τ to τ is equivariant under the action of $\ker \psi$ on $\mathbb{H}_n(\theta)$, hence there is an induced injective morphism

$$\mathbb{H}_n(\theta) / \ker \psi = A_n(\theta) \rightarrow \mathbb{H}_1 / \Gamma_0(n + 1) = X_0(n + 1).$$

We now use a beautiful geometric idea suggested to us by A. Beauville to show that this last morphism is also surjective.

Theorem 5.4. *Let n be greater than or equal to two. Then $A_n(\theta)$ is isomorphic to $X_0(n + 1)$.*

Proof. Let (E, G) be a point of the modular curve $X_0(n + 1)$. Then E is an elliptic curve and G is a cyclic subgroup of order $(n + 1)$ of E .

Let E^{n+1} be the abelian variety given by the product of $n + 1$ copies of E with the principal (product) polarization P . In E^{n+1} we consider the abelian subvariety A given by

$$A = \{(x_1, x_2, \dots, x_n, x_{n+1}) \in E^{n+1} : \sum_{i=1}^{n+1} x_i = 0\},$$

with the polarization P_A induced by the polarization P of E^{n+1} . Note that S_{n+1} acts on E^{n+1} by permutation of coordinates and that this action restricts to A .

If \widehat{A} denotes the dual abelian variety of A , then P_A induces an isogeny $P_A : A \rightarrow \widehat{A}$ with kernel given by the points of order $n + 1$ of E embedded in the diagonal of A . We denote this embedded group by N .

Furthermore, the image of the subgroup G in N (which we again denote by G) is totally isotropic for the Weil form. Then (cf. [6]) there exists a principal polarization Q on A/G whose inverse image is P_A .

It can also be verified (by a change of coordinates) that A/G is isomorphic to the product $E^{n-1} \times E/G$.

Since the construction is equivariant under the action of S_{n+1} , we have that $(A/G, Q)$ belongs to $A_n(\theta)$, and the result follows. \square

6. THE CASE $n = 2$

We now study the irreducibility of the members of $A_n(\theta)$ for the interesting special case $n = 2$.

Discussions with R. Salvati Manni were especially helpful to understand this case.

Proposition 6.1. *For n equal to two, each p.p.a.v. A_τ in $A_2(\theta)$ is irreducible as a p.p.a.v., for τ non-equivalent to $\tau_0 = \frac{-3 + i\sqrt{3}}{6} \bmod \Gamma_0(3)$.*

If τ is equivalent to $\tau_0 \bmod \Gamma_0(3)$, then A_τ is isomorphic as a principally polarized abelian variety to $E_\omega \times E_\omega$, where E_ω is the elliptic curve with $j(\omega) = 0$.

Proof. Let A_τ be in $A_2(\theta)$ and represented by the Riemann matrix $Z_\tau \in \mathbb{H}_2(\theta)$.

If we assume that A_τ is reducible, then it must be isomorphic as a principally polarized abelian variety to $E_\omega \times E_\omega$, where E_ω is an elliptic curve with $j(\omega) = 0$. Let $m \in \text{Sp}(4, \mathbb{Z})$ be a matrix inducing this isomorphism.

Now the p.p.a.v. $E_\omega \times E_\omega$ admits the automorphisms $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ and } R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}.$$

Furthermore, these automorphisms satisfy the relations

$$(6.1) \quad T^2 = I_4, \quad TR = RT, \quad \text{and } TS = -ST.$$

But then the automorphism group of A_τ must contain $n = mTm^{-1}$ and, by construction, $\theta(S_3)$. Of course, the relations (6.1) must be satisfied by the corresponding elements of $\text{Aut}(A_\tau)$.

These relations, together with the fact that n belongs to $\Gamma_2(2, 4)$, allow us to conclude that n must have the following form:

$$n = \begin{pmatrix} a & 2a & 0 & 2b \\ -2a & -a & -2b & 0 \\ 0 & -2c & a & -2a \\ 2c & 0 & 2a & -a \end{pmatrix},$$

with a, b, c integers and satisfying $4bc - 3a^2 = 1$.

Since n fixes Z_τ , a calculation shows that τ is equivalent to $\tau_0 = \frac{-3 + i\sqrt{3}}{6} \bmod \Gamma_0(3)$.

For an explicit example, we take

$$n = \begin{pmatrix} 1 & 2 & 0 & 2 \\ -2 & -1 & -2 & 0 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 2 & -1 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

□

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