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YANG INDEX OF THE DELETED PRODUCT

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ABSTRACT. For any $\kappa \geq 1$ a κ-dimensional polyhedron Y_κ is constructed such that the Yang index of its deleted product Y_κ^* equals 2κ. This answers a question of Izydorek and Jaworowski (1995).

For any $\kappa \geq 1$ a 2κ-dimensional closed manifold M with involution is constructed such that $\operatorname{index} M = 2\kappa$, but M can be mapped into a κ-dimensional polyhedron without antipodal coincidence.

The deleted product of Y is the space

$$Y^* = Y^2 \setminus \Delta,$$

where Δ is the diagonal of Y^2. There is a natural free involution $T(x, y) = (y, x)$ acting in Y^*.

Our goal is to compute the Yang index of the deleted product of some polyhedra (with respect to the involution T). In particular, we answer the question in [3] of whether there exists a κ-dimensional polyhedron Y_κ with $\operatorname{index} Y_\kappa^* = 2\kappa$. It is shown that the space $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ has $\operatorname{index} Y_\kappa^* = 2\kappa$.

In fact, we shall find in Y_κ^* a closed manifold $M_{2\kappa}$ with $\operatorname{index} M_{2\kappa} = 2\kappa$. Then the projection $p(x, y) = x$ is a map $p : M_{2\kappa} \to Y_\kappa$ without antipodal coincidence. Other examples of such manifolds (or even polyhedra) are not known to us. Let us note the theorem of Šchepin [4], which asserts that every map $f : S^{2\kappa} \to P_\kappa$ of the 2κ-sphere into a κ-dimensional polyhedron has an antipodal coincidence.

First some notation.

Δ^n is a standard n-simplex in $\mathbb{R}^n$ with center in the origin O.

Let P be a simplicial complex.

For $x \in P$, $[x]$ denotes the carrier of x, i.e. the (closed) simplex containing x in its interior.

If $a \in P$ is a vertex, its star $\operatorname{St}(a)$ is the union of all open simplexes with vertex a.

$[P]^\kappa$ denotes the κ-dimensional skeleton of P.

All maps are assumed to be continuous.

Now we shall list some properties of the Yang index that we shall make use of, and refer the reader to [7] for the definition and the whole index theory.

Let X be a compact metric space with a free involution $T : X \to X$. Then its Yang index is defined inductively by means of the equivariant homology groups with coefficients in $\mathbb{Z}_2$. We denote it here by $\operatorname{index} X$. An important property of

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the index is that if $\operatorname{index} X \geq n$, then every map $f : X \to \mathbb{R}^n$ has an antipodal coincidence: $f(Tx) = f(x)$.

Note also that $\operatorname{index} X \leq \dim X$.

The following useful proposition estimates the index of a manifold.

Proposition. *Let M_n be an n-dimensional closed manifold with a free involution $T : M_n \to M_n$. Suppose that there exists an odd map $\varphi : M_n \to S^n$ (i.e. $\varphi(Tx) = -\varphi(x)$) with $\deg_2 \varphi = 1$, where $\deg_2$ is the degree $\bmod 2$. Then $\operatorname{index} M_n = n$.*

Proof. Let z^n be the invariant fundamental cycle $\bmod 2$ in M_n. Then $\varphi_*([z^n]) \neq 0$ (in the Cech homologies $\bmod 2$). We have $\nu([z^n]) = \nu(\varphi_*[z^n]) \neq 0$, as follows from the properties of the index homomorphism ν (cf. [7] for the definition of ν). But this means that $\operatorname{index} M_n \geq n$ by definition. The converse inequality follows from the fact that index $\leq$ dim.

I. The main theorems

Let M be a finite set in $\mathbb{R}^n$ and let $\sigma_1^\kappa, \sigma_2^\kappa$ be two κ-dimensional simplexes with vertices in M, without a common vertex. Suppose that every two such simplexes either do not intersect or have a single common point, interior to both σ_1^κ and σ_2^κ.

We shall denote by $\#_\kappa(M)$ the number of intersections of pairs $\{\sigma_1^\kappa, \sigma_2^\kappa\}$ as above. Such an intersection will be called a κ-intersection.

For example, if $M = \{5 \text{ points lying on a circle}\}$, then $\#_1(M) = 5$.

Lemma 1. *There exists in $\mathbb{R}^{2\kappa}$ a set M of $2\kappa + 3$ points such that $\#_\kappa(M) = 1$.*

The proof is given in Section III. For example, in the case $\kappa = 1$ it suffices to take 5 points in $\mathbb{R}^2$ in general position, whose convex hull is a triangle.

Consider now the complex

$$Y_\kappa = [\Delta^{2\kappa+2}]^\kappa.$$

We shall prove that $\operatorname{index} Y_\kappa^* = 2\kappa$. Set

$$M_{2\kappa} = \{(x, y) \in Y_\kappa^2 | [x] \cap [y] = \varnothing\}.$$

Clearly, $M_{2\kappa}$ is an invariant compact subset of Y_κ^*.

Lemma 2. *$M_{2\kappa}$ is a closed manifold.*

The proof of this interesting proposition is given in Section II. Notice that $M_{2\kappa}$ has a structure of a cell complex. It is also easy to show that there is a deformation of Y_κ^* on $M_{2\kappa}$, so $M_{2\kappa}$ contains all the information about Y_κ^*.

Theorem 1. *Let $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$. Then*

$$\operatorname{index} Y_\kappa^* = 2\kappa.$$

Proof. It suffices to prove that $\operatorname{index} M_{2\kappa} = 2\kappa$. As follows from Lemma 1, there exists in $\mathbb{R}^{2\kappa}$ a set

$$M = \{a_1, a_2, \ldots, a_{2\kappa+3}\}$$

of $2\kappa + 3$ points such that $\#_\kappa(M) = 1$. Let the single κ-intersection arise between the simplexes $[a_1, \ldots, a_{\kappa+1}]$ and $[a_{\kappa+2}, \ldots, a_{2\kappa+2}]$. Consider in $\mathbb{R}^{2\kappa+1}$ the set

$$N = \{a_1 + l_{2\kappa+1}, a_2, \ldots, a_{2\kappa+3}\},$$

where $l_{2\kappa+1}$ is a unit vector orthogonal to $\mathbb{R}^{2\kappa}$. Clearly, $\#_\kappa(N) = 0$. Then $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ may be embedded in $\mathbb{R}^{2\kappa+1}$ with vertices in N. Let us define an odd map $\varphi : M_{2\kappa} \to S^{2\kappa}$ as follows:

$$\varphi(x, y) = \frac{y - x}{\|y - x\|}.$$

The preimage $\varphi^{-1}(-l_{2\kappa+1})$ contains a single point (x_0, y_0) corresponding to the single κ-intersection in M. Here $x_0 \in [a_1 + l_{2\kappa+1}, \dots, a_{\kappa+1}], y_0 \in [a_{\kappa+2}, \dots, a_{2\kappa+2}]$. We shall prove that $\deg_2 \varphi = 1 \pmod 2$. It is clear that there exists in $S^{2\kappa}$ a neighbourhood $W \ni -l_{2\kappa+1}$ such that the map φ restricted to $\varphi^{-1}(W)$ is a homeomorphism. Let us approximate φ with a smooth map $\varphi_0 : M_{2\kappa} \to S^{2\kappa}$ such that $\varphi_0(x) = \varphi(x)$ for $x \in \varphi^{-1}(W)$. Then $-l_{2\kappa+1}$ is a regular value of φ_0 and $\varphi_0^{-1}(-l_{2\kappa+1})$ contains a single point (x_0, y_0). But then for its degree mod 2 we have $\deg_2 \varphi_0 = 1$ (cf. [2]), and therefore $\deg_2 \varphi = 1$.

Then $\operatorname{index} M_{2\kappa} = 2\kappa$, as follows from the Proposition in the preliminary section.

The theorem is proved.

Note that if P_κ is a contractible κ-dimensional polyhedron, then $\operatorname{index} P_\kappa^* \leq 2\kappa - 1$. This is established in [3].

Theorem 2. *For any $\kappa \geq 1$ there exists a closed manifold $M_{2\kappa}$ with a free involution T, such that* $\operatorname{index} M_{2\kappa} = 2\kappa$*, but there is a map*

$$f : M_{2\kappa} \to Y_\kappa$$

into a κ-dimensional polyhedron without antipodal coincidence: $f(Tx) \neq f(x)$ *for any $x \in M_{2\kappa}$.*

Proof. Let $M_{2\kappa}$ and Y_κ be as in Theorem 1. Set $f(x, y) = x$. Then $f : M_{2\kappa} \to Y_\kappa$ is a map without antipodal coincidence: $f(x, y) \neq f(y, x)$.

Let us note that not every manifold of index 2κ admits such a map. Ščhepin [4] has shown that every map $f : S^{2\kappa} \to P_\kappa$ of the sphere $S^{2\kappa}$ into a κ-dimensional polyhedron has an antipodal coincidence: $f(-x) = f(x)$.

The following is a simple but useful proposition that we shall refer to in the last two sections.

Proposition $(*)$**.** *Let $\Delta^n = [a_1, \dots, a_{n+1}]$ be the standard n-simplex in $\mathbb{R}^n$. Suppose that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$.*

Proof. Since $\sum a_i = 0$, we find $a_1 = -a_2 - \cdots - a_{n+1}$. Substitute in $\sum \lambda_i a_i = 0$ and make use of the fact that $a_2, \dots, a_{n+1}$ are independent.

II. Proof that $M_{2\kappa}$ is a manifold

In this section we shall prove Lemma 2. The key is Lemma 3, which is interesting for itself. Let

$$P_\kappa = O([\Delta^{2\kappa}]^{\kappa-1})$$

be the cone over $[\Delta^{2\kappa}]^{\kappa-1}$ with vertex the origin O. Consider the set

$$U = \{(x, y) \in P_\kappa^2 | [x] \cap [y] = \{O\}\}. \tag{1}$$

Lemma 3. *The map* $\varphi : U \to \mathbb{R}^{2\kappa}$ *defined by*

$$\varphi(x, y) = x - y$$

maps U *homeomorphically onto some open star-like subset of* $\mathbb{R}^{2\kappa}$ *with center* O.

Proof. Let $\Delta^{2\kappa} = [a_1, \ldots, a_{2\kappa+1}]$. If $(x, y) \in U$, then

$$x = \sum_{i \in I} \alpha_i a_i, \qquad y \in \sum_{j \in J} \beta_j a_j,$$

where $\alpha_i, \beta_j > 0$ and $|I| \le \kappa, |J| \le \kappa, I \cap J = \varnothing$. (The index sets are disjoint, since $[x] \cap [y] = \{O\}$ by definition.)

1) First we prove that φ is "mono".

Let $\varphi(x, y) = \varphi(u, v)$, i.e. $x - y = u - v$. We have as above

$$u = \sum_{r \in R} \gamma_r a_r, \qquad v = \sum_{s \in S} \delta_s a_s,$$

where $\gamma_r, \delta_s > 0$, $|R| \le \kappa$, $|S| \le \kappa$, $R \cap S = \varnothing$. Then

$$\sum_I \alpha_i a_i - \sum_J \beta_j a_j - \sum_R \gamma_r a_r + \sum_S \delta_s a_s = 0. \tag{2}$$

Let

$$M = \{1, 2, \ldots, 2\kappa + 1\}.$$

By Proposition $(*)$ all the coefficients in (2) are (after reduction) equal to some number c.

If $c \ge 0$, then from (2) $J \cup R$ is contained in $I \cup S$, so $I \supset R$ and $S \supset J$. Since $|S| + |I| \le 2\kappa < |M|$, this implies $I \cup J \cup R \cup S \ne M$, thus $c = 0$.

If $c \le 0$, then in the same way it follows that $R \supset I$, $J \supset S$ and $I \cup J \cup R \cup S \ne M$, hence $c = 0$.

The single possibility for this is $I = R, J = S$. But then (2) implies that

$$\sum_I \alpha_i a_i = \sum_R \gamma_r a_r, \quad \sum_J \beta_j a_j = \sum_S \delta_s a_s.$$

So, $x = u, y = v$, i.e. φ is "mono".

2) We shall show that $\varphi(U)$ contains O in its interior.

Let $w \in \mathbb{R}^{2\kappa}$ be a vector with a small norm. Then it may be written in the form

$$w = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{2\kappa} a_{2\kappa},$$

where we assume that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2\kappa}$.

Clearly,

$$a_\kappa = -a_1 - \cdots - a_{\kappa-1} - a_{\kappa+1} - \cdots - a_{2\kappa+1}$$

and, substituting above,

$$\begin{aligned} w = {} & (\lambda_1 - \lambda_\kappa) a_1 + \cdots + (\lambda_{\kappa-1} - \lambda_\kappa) a_{\kappa-1} \\ & + (\lambda_{\kappa+1} - \lambda_\kappa) a_{\kappa+1} + \cdots + (\lambda_{2\kappa} - \lambda_\kappa) a_{2\kappa} - \lambda_\kappa a_{2\kappa+1}. \end{aligned}$$

Set

$$x = (\lambda_{\kappa+1} - \lambda_\kappa) a_{\kappa+1} + \cdots + (\lambda_{2\kappa} - \lambda_\kappa) a_{2\kappa},$$
$$y = (\lambda_\kappa - \lambda_1) a_1 + \cdots + (\lambda_\kappa - \lambda_{\kappa-1}) a_{\kappa-1} + \lambda_\kappa a_{2\kappa+1}.$$

Then $w = x - y$, where $x, y \in O([\Delta^{2\kappa}]^{\kappa-1})$, since x and y are of small norm and the coefficients are nonnegative. Note also that $[x] \cap [y] = \{O\}$, thus $(x, y) \in U$.

So $w = \varphi(x, y)$, hence $w \in \varphi(U)$. We proved that $\varphi(U)$ contains O in its interior. Note finally, that $\varphi(U)$ is star-like, as follows immediately from the definition of φ.

The lemma is proved.

Proof of Lemma 2. Here we prove that $M_{2\kappa}$ is a manifold. Recall that

$$M_{2\kappa} = \{(x, y) \in Y_\kappa^2 | [x] \cap [y] \neq \varnothing\},$$

where $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$. It is clear that $M_{2\kappa}$ is subdivided in a natural way to prisms of the form $\sigma^i \times \sigma^j$, where $\sigma^i, \sigma^j \subset Y_\kappa$ are nonintersecting simplexes. We shall prove that the star of each vertex of $M_{2\kappa}$ is homeomorphic to $\mathbb{R}^{2\kappa}$. It suffices to check it for an arbitrary vertex (a_1, a_2). Clearly,

$$\mathrm{St}(a_1, a_2) = \{(x, y) \in Y_\kappa^2 | [x] \ni a_1, [y] \ni a_2, [x] \cap [y] = \varnothing\}.$$

We shall embed Y_κ in $\mathbb{R}^{2\kappa+1}$ as follows: Let $\{a_3, \ldots, a_{2\kappa+3}\}$ be the vertices of the standard simplex $\Delta^{2\kappa}$ in $\mathbb{R}^{2\kappa}$ and $a_1 = l_{2\kappa+1}, a_2 = -l_{2\kappa+1}$, where $l_{2\kappa+1}$ is a unit vector orthogonal to $\mathbb{R}^{2\kappa}$. The points $\{a_1, \ldots, a_{2\kappa+3}\}$ are in general position in $\mathbb{R}^{2\kappa+1}$, consequently we may embed there $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ with vertices a_i. Let $p : \mathbb{R}^{2\kappa+1} \to \mathbb{R}^{2\kappa}$ denote the orthogonal projection. We have

$$p(Y_\kappa) = O([\Delta^{2\kappa}]^{\kappa-1}).$$

It is easy to see that $(x, y) \in \mathrm{St}(a_1, a_2)$ if and only if $(p(x), p(y)) \in U$, where U is defined by (1). Then the map $P(x, y) = (p(x), p(y))$ is a homeomorphism and according to Lemma 3, U is homeomorphic to $\mathbb{R}^{2\kappa}$. Therefore $\mathrm{St}(a_1, a_2) \approx \mathbb{R}^{2\kappa}$.

III. Proof of Lemma 1

Here we shall construct in $\mathbb{R}^{2\kappa}$ a set M of $2\kappa + 3$ points such that $\#_\kappa(M) = 1$.

Let $\Delta^{2\kappa} = [a_1, \ldots, a_{2\kappa+1}]$ be the standard simplex in $\mathbb{R}^{2\kappa}$. Set

$$A = \lambda(a_1 + \cdots + a_\kappa) + \delta a_{\kappa+1},$$

where $0 < \delta < \lambda$ and $\kappa\lambda + \delta < 1$.

We shall show that the set

$$M = \{O, a_1, \ldots, a_{2\kappa+1}, A\}$$

meets the case. Note first that A lies in the interior of $\Delta^{2\kappa}$, since $\kappa\lambda + \delta < 1$. Consider two κ-dimensional simplexes σ_1 and σ_2 with vertices in M without a common vertex. If some of them contain neither O, nor A, then $\sigma_1 \cap \sigma_2 = \varnothing$. Indeed, suppose that $O, A \notin \sigma_1$. Then $\sigma_1 \subset \partial\Delta^{2\kappa}$ and $\sigma_1 \cap \sigma_2 = \sigma_1 \cap (\sigma_2 \cap \partial\Delta^{2\kappa}) = \varnothing$. The last equality holds, since $\sigma_2 \cap \partial\Delta^{2\kappa}$ is a simplex in $\partial\Delta^{2\kappa}$ without common vertices with σ_1.

So, we may suppose that

$$\sigma_1 = [O, a_{i_1}, \ldots, a_{i_\kappa}], \quad \sigma_2 = [A, a_{j_1}, \ldots, a_{j_\kappa}],$$

where all the vertices are different. Suppose that σ_1 and σ_2 intersect; then we have

$$\lambda_1 a_{i_1} + \cdots + \lambda_\kappa a_{i_\kappa} = \mu_0\lambda(a_1 + \cdots + a_\kappa) + \mu_0\delta a_{\kappa+1} + \mu_1 a_{j_1} + \cdots + \mu_\kappa a_{j_\kappa}, \tag{3}$$

where $\lambda_i \geq 0$, $\sum \lambda_i \leq 1$, $\mu_j \geq 0$, $\sum \mu_j = 1$.

Note that $\mu_0 > 0$, since σ_1 and σ_2 may intersect only in an interior point of $\Delta^{2\kappa}$.

Send all the members of (3) to the right-hand side, then each a_i appears (after reduction) with some coefficient $\nu(a_i)$. According to Proposition $(*)$ we should have

$$\nu(a_1) = \nu(a_2) = \cdots = \nu(a_{2\kappa+1}). \tag{4}$$

Let us note that exactly one a_i is not a vertex of either σ_1 or of σ_2. There are 3 possibilities about a_i.

1) $i \geq \kappa + 2$. Then $\nu(a_i) = 0$ and hence $\nu(a_j) = 0$ for any j. But there exists some $j \leq \kappa + 1$ such that a_j is a vertex of σ_2. Then the examination of (3) gives $\nu(a_j) > 0$, in contradiction with (4).

2) $i \leq \kappa$. Then $\nu(a_i) = \mu_0 \lambda > 0$. This implies that for $j \geq \kappa + 2$ all the a_j are in the right-hand side of (3), otherwise we would have $\nu(a_j) \leq 0$, in contradiction with (4). Then $a_{\kappa+1}$ takes part in the left-hand side with coefficient $\lambda_{\kappa+1} \geq 0$. Therefore

$$\nu(a_{\kappa+1}) = \mu_0 \delta - \lambda_{\kappa+1} = \nu(a_i) = \mu_0 \lambda,$$

so $\mu_0(\delta - \lambda) = \lambda_{\kappa+1} \geq 0$, which contradicts the condition $\delta < \lambda$.

3) $i = \kappa + 1$, i.e. $a_{\kappa+1}$ is not a vertex of $\sigma_1 \cup \sigma_2$. In this case we should have

$$\sigma_1 = [O, a_1, \ldots, a_\kappa], \qquad \sigma_2 = [A, a_{\kappa+2}, \ldots, a_{2\kappa+1}].$$

Really, if we suppose that $a_j \in \sigma_1$ for some $j \geq \kappa + 2$, then a_j takes part in the left-hand side of (3) with coefficient $\lambda_j \geq 0$, thus $\nu(a_j) = -\lambda_j$, though $\nu(a_{\kappa+1}) = \mu_0 \delta > 0$, in contradiction with (4).

We shall prove that σ_1 and σ_2 intersect in an interior point. We are looking for a solution of

$$\begin{aligned} \lambda_1 a_1 + \cdots + \lambda_\kappa a_\kappa = \mu_0 \lambda(a_1 + \cdots + a_\kappa) + \mu_0 \delta a_{\kappa+1} \\ + \mu_{\kappa+2} a_{\kappa+2} + \cdots + \mu_{2\kappa+1} a_{2\kappa+1} \end{aligned} \tag{5}$$

with $\lambda_i, \mu_j \geq 0$, $\sum \lambda_i \leq 1, \sum \mu_j = 1$.

It is straightforward to check that the numbers

$$\lambda_1 = \lambda_2 = \cdots = \lambda_\kappa = \frac{\lambda - \delta}{1 + \delta\kappa}, \qquad \mu_0 = \frac{1}{1 + \delta\kappa},$$
$$\mu_{\kappa+2} = \cdots = \mu_{2\kappa+1} = \frac{\delta}{1 + \delta\kappa}$$

satisfy (5), since it reduces to the identity $\sum a_i = 0$. This means that σ_1 and σ_2 intersect in an interior point. Therefore $\#_\kappa(M) = 1$.

IV. Concluding remarks

There is a natural generalization of the spaces Y_κ. Let as above $Y_\kappa = [\Delta^{2\kappa+2}]^\kappa$ and consider the join

$$X_\kappa = Y_{\kappa_1} * Y_{\kappa_2} * \cdots * Y_{\kappa_p}$$

where $\kappa = \kappa_1 + \cdots + \kappa_p + p - 1$. It is not difficult to see that the spaces X_κ also have index $X_\kappa^* = 2\kappa$. It is shown in [1] that X_κ is a κ-minimal complex, in the sense that it is not embeddable in $\mathbb{R}^{2\kappa}$ but each of its proper subcomplexes is embeddable in $\mathbb{R}^{2\kappa}$. Cohomological obstructions for embedding in $\mathbb{R}^{2\kappa}$ are discussed in [5] and [6].

For a graph Γ the question of computing index Γ^* is in fact solved in [6]:

We have index $\Gamma^* = 2$ if and only if Γ is nonplanar. Furthermore, index $\Gamma^* = 1$ if and only if Γ is planar, but not embeddable in $\mathbb{R}^1$. Otherwise index $\Gamma^* = 0$.

Another interesting phenomenon in the case of graphs is the following fact. Let

$$M(\Gamma) = \{(x, y) \in \Gamma^2 | [x] \cap [y] = \varnothing\}.$$

Then $M(\Gamma)$ is homeomorphic to a closed surface if and only if Γ is one of the two Kuratowski graphs K_5 and $K_{3,3}$. Moreover, $M(K_5)$ is a sphere with 6 handles and

$M(K_{3,3})$ is a sphere with 4 handles. This can be shown by computing the Euler characteristic of these spaces and by checking that they are both orientable.

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