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ON p-HYPONORMAL OPERATORS

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ABSTRACT. In this paper we show that p-hyponormal operators with $0 \notin \sigma(|T|_r^{\frac{1}{2}})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

1. INTRODUCTION

Let H and K be separable complex Hilbert spaces and let $\mathcal{L}(H,K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H,K)$.

An operator $T \in \mathcal{L}(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where T^* is the adjoint of T. If $p = 1$, T is called hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [Xi]), and p-hyponormal operators for a general p, $0 < p < 1$, have been studied by Aluthge. Any p-hyponormal operators are q-hyponormal if $q \leq p$. But there are examples to show that the converse of the above statement is not true (see [Al]).

A bounded linear operator S on H is called scalar of order m if it has a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \longrightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where as usual z stands for the identity function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to a closed invariant subspace. We now define the weaker form of a subscalar operator. An operator $T \in \mathcal{L}(H)$ is quasi-subscalar if there exists a one-to-one $V \in \mathcal{L}(H,K)$ such that $VT = SV$ where S $(= \Phi(z)$ in the above definition) is a scalar operator.

This paper has been divided into three sections. Section 2 deals with some preliminary facts. In section 3, we show that p-hyponormal operators with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$ are subscalar. As a corollary, we get that such operators with rich spectra have non-trivial invariant subspaces.

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2. Preliminaries

Let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let H be a complex separable Hilbert space, and let D be a bounded open disc in $\mathbf{C}$. We shall denote by $L^2(D,H)$ the Hilbert space of measurable functions $f : D \to H$, such that

$$\|f\|_{2,D} = (\int_D \|f(z)\|^2 d\mu(z))^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(D,H)$ which are analytic functions in D (i.e., $\bar{\partial} f = 0$) is defined by

$$A^2(D,H) = L^2(D,H) \cap \mathcal{O}(D,H)$$

where $\mathcal{O}(D,H)$ denotes the Fréchet space of H-valued analytic functions on D with respect to uniform topology. $A^2(D,H)$ is called the Bergman space for D. Note that $A^2(D,H)$ is a Hilbert space. The operator $T - z$ on the space $\mathcal{O}(D,H)$ has property (β), which means by definition that $T - z$ is one-to-one and has closed range for every disc D.

Let us define now a Sobolev type space, called $W^2(D,H)$ where D is a bounded disc in $\mathbf{C}$. $W^2(D,H)$ will be the space of those functions $f \in L^2(D,H)$ whose derivatives $\bar{\partial} f, \bar{\partial}^2 f$ in the sense of distributions still belong to $L^2(D,H)$. Endowed with the norm $\|f\|^2_{W^2} = \sum_{i=0}^{2} \|\bar{\partial}^i f\|^2_{2,D}$, $W^2(D,H)$ becomes a Hilbert space contained continuously in $L^2(D,H)$.

Now for $f \in C_0^2(\mathbf{C})$, let M_f denote the operator on $W^2(D,H)$ given by multiplication by f. This has a spectral distribution of order 2, defined by the functional calculus

$$\Phi_M : C_0^2(\mathbf{C}) \longrightarrow \mathcal{L}(W^2(D,H)), \quad \Phi_M(f) = M_f.$$

Therefore M_z is a scalar operator of order 2. In fact, it can be shown [Pu] that M_z is subnormal.

3. Subscalarity

This section deals with the characterization for some p-hyponormal operators. Recall that an operator $T \in \mathcal{L}(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$ where T^* is the adjoint of T.

We need the following lemmas to give a proof of the main theorem.

Lemma 1 ([Xi], Lemma 2.1)**.** *Let $T = U|T|_r$ be the polar decomposition of T, $Q = |T|_r - |T|_l$, $z = \rho e^{i\theta}, 0 < \rho$, and $|e^{i\theta}| = 1$ where $|T|_r = (T^*T)^{\frac{1}{2}}$ and $|T|_l = (TT^*)^{\frac{1}{2}}$. Then*

$$\|(T-z)f\|^2_{2,D} = \|(|T|_r - \rho)f\|^2_{2,D} + \rho\||T|_r^{\frac{1}{2}}(U - e^{i\theta})^* f\|_{2,D} + \rho\langle Qf, f\rangle$$

for all $f \in L^2(D,H)$.

For reference, we quote Lemma 2 from [Pu].

Lemma 2 ([Pu], Proposition 2.1)**.** *For every bounded disk D in $\mathbf{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D,H)$ we have*

$$\|(I-P)f\|_{2,D} \leq C_D(\|(T-z)^*\bar{\partial} f\|_{2,D} + \|(T-z)^*\bar{\partial}^2 f\|_{2,D})$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$.

For p-hyponormal operator $T = U|T|$, Aluthge ([Al]) introduced the operator $\tilde{T} = |T|_r^{\frac{1}{2}} U |T|_r^{\frac{1}{2}}$ and showed very interesting results on $\tilde{T}$.

Lemma 3 ([Al])**.** *Let $T = U|T|_r$ be a p-hyponormal operator, $0 < p < 1$, and U unitary. Then the operator $\tilde{T} = |T|_r^{\frac{1}{2}} U |T|_r^{\frac{1}{2}}$ is hyponormal if $\frac{1}{2} \le p < 1$, and $(p + \frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$.*

Lemma 4. *Let $T = U|T|_r$ be semi-hyponormal and let U be unitary. Let D be a bounded disk which containes $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $Vh = \widetilde{1 \otimes h}$ $(\equiv 1 \otimes h + \overline{(T - z)W^2(D, H)})$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.*

Proof. Let $h_n \in H$ and $f_n \in W^2(D, H)$ be sequences such that

$$\lim_{n\to\infty} \|(T - z)f_n + 1 \otimes h_n\|_{W^2} = 0. \tag{1}$$

Then by the definition of the norm of Sobolev space (1) implies

$$\lim_{n\to\infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{2}$$

for $i = 1, 2$. Since T is a semi-hyponormal operator, Lemma 1 and equation (2) imply

$$\begin{cases} \lim_{n\to\infty} \|(|T|_r - \rho)\bar{\partial}^i f_n\|_{2,D} = 0, \\ \lim_{n\to\infty} \rho \||T|_r^{\frac{1}{2}} (U - e^{i\theta})^* \bar{\partial}^i f_n\|_{2,D} = 0, \quad \text{and} \\ \lim_{n\to\infty} \rho \langle Q\bar{\partial}^i f_n, \bar{\partial}^i f_n\rangle = 0. \end{cases} \tag{3}$$

We note that for $i = 1, 2$

$$\begin{aligned} (T - z)^* \bar{\partial}^i f_n = {} & |T|_r^{\frac{1}{2}} [|T|_r^{\frac{1}{2}} (U - e^{i\theta})^* \bar{\partial}^i f_n] \\ & + e^{-i\theta} [(|T|_r - \rho)\bar{\partial}^i f_n]. \end{aligned} \tag{4}$$

By equations (3) and (4), we get

$$\lim_{n\to\infty} \|(T - z)^* \bar{\partial}^i f_n\|_{2,D} = 0. \tag{5}$$

Lemma 2 and equation (5) imply

$$\lim_{n\to\infty} \|(I - P)f_n\|_{2,D} = 0, \tag{6}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto $A^2(D, H)$. Then by (1)

$$\lim_{n\to\infty} \|(T - z)Pf_n + 1 \otimes h_n\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} \|Pf_n(z) + (T - z)^{-1}(1 \otimes h_n)\| = 0$$

uniformly. Hence

$$\lim_{n\to\infty} \|\frac{1}{2\pi i} \int_\Gamma Pf_n(z)dz + h_n\| = 0.$$

But by Cauchy's theorem,

$$\int_\Gamma Pf_n(z)dz = 0.$$

Hence $\lim_{n\to\infty} h_n = 0$. Thus V is one-to-one and has closed range. □

Proposition 5. *Let $T = U|T|_r$ be a p-hyponormal operator with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $0 < p < 1$, and U unitary. Let D be a bounded disk which contains $\sigma(T)$. Then the map $V : H \to H(D)$ defined by $Vh = \widetilde{1 \otimes h}$ $(\equiv 1 \otimes h + \overline{(T-z)W^2(D,H)})$ is one-to-one and has closed range, where $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.*

Proof. Let $h_n \in H$ and $f_n \in W^2(D,H)$ be sequences such that

$$\lim_{n\to\infty} \|(T-z)f_n + 1 \otimes h_n\|_{W^2} = 0. \tag{7}$$

Then equation (7) implies

$$\lim_{n\to\infty} \|(T-z)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{8}$$

for $i = 1, 2$.

(a) If $\frac{1}{2} \le p < 1$, then T is semi-hyponormal. Therefore, Proposition 5 follows from Lemma 4.

(b) Let $0 < p < \frac{1}{2}$. Since $T = U|T|_r$,

$$\lim_{n\to\infty} \||T|_r^{\frac{1}{2}}(U|T|_r - z)\bar{\partial}^i f_n\|_{2,D} = 0.$$

Since $\tilde{T} = |T|_r^{\frac{1}{2}} U |T|_r^{\frac{1}{2}}$, we have

$$\lim_{n\to\infty} \|(\tilde{T}-z)\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)\|_{2,D} = 0. \tag{9}$$

Since $\tilde{T}$ is $(p+\frac{1}{2})$-hyponormal by Lemma 3, $\tilde{T}$ is semi-hyponormal. Let $\tilde{T} = W|\tilde{T}|_r$ be the polar decomposition. Lemma 1 and equation (9) imply

$$\begin{cases} \lim_{n\to\infty} \|(|\tilde{T}|_r - \rho)\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)\|_{2,D} = 0, \\ \lim_{n\to\infty} \rho \||\tilde{T}|_r^{\frac{1}{2}}(W - e^{i\theta})^* \bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)\|_{2,D} = 0, \quad \text{and} \\ \lim_{n\to\infty} \rho \langle Q\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n), \bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)\rangle = 0. \end{cases} \tag{10}$$

Now we note that for $i = 1, 2$

$$\begin{aligned} (\tilde{T}-z)^*\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n) = |\tilde{T}|_r^{\frac{1}{2}}&[|\tilde{T}|_r^{\frac{1}{2}}(W-e^{i\theta})^*\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)] \\ &+ e^{-i\theta}[(|\tilde{T}|_r - \rho)\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)]. \end{aligned} \tag{11}$$

By (10) and (11), we get

$$\lim_{n\to\infty} \|(\tilde{T}-z)^*\bar{\partial}^i(|T|_r^{\frac{1}{2}} f_n)\|_{2,D} = 0. \tag{12}$$

Lemma 2 and equation (12) imply

$$\lim_{n\to\infty} \|(I-P)|T|_r^{\frac{1}{2}} f_n\|_{2,D} = 0. \tag{13}$$

Since $|T|_r^{\frac{1}{2}}(T-z) = (\tilde{T}-z)|T|_r^{\frac{1}{2}}$ and $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, it follows from (7) that $\sigma(T) = \sigma(\tilde{T})$ and

$$\lim_{n\to\infty} \|(\tilde{T}-z)|T|_r^{\frac{1}{2}} f_n + |T|_r^{\frac{1}{2}}(1 \otimes h_n)\|_{2,D} = 0. \tag{14}$$

By (13) and (14), we have

$$\lim_{n\to\infty} \|(\tilde{T}-z)P(|T|_r^{\frac{1}{2}} f_n) + |T|_r^{\frac{1}{2}}(1 \otimes h_n)\|_{2,D} = 0.$$

Let Γ be a curve in D surrounding $\sigma(T)$ $(= \sigma(\tilde{T}))$. Then for $z \in \Gamma$

$$\lim_{n\to\infty} \|P(|T|_r^{\frac{1}{2}} f_n(z)) + (\tilde{T} - z)^{-1}(|T|_r^{\frac{1}{2}}(1 \otimes h_n))\| = 0$$

uniformly. Hence

$$\lim_{n\to\infty} \|\frac{1}{2\pi i}\int_\Gamma P(|T|_r^{\frac{1}{2}} f_n(z))dz + |T|_r^{\frac{1}{2}} h_n\| = 0.$$

But by Cauchy's theorem,

$$\frac{1}{2\pi i}\int_\Gamma P(|T|_r^{\frac{1}{2}} f_n(z))dz = 0.$$

Therefore $\lim_{n\to\infty} |T|_r^{\frac{1}{2}} h_n = 0$. Since $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $|T|_r^{\frac{1}{2}}$ is bounded below. Hence $\lim_{n\to\infty} h_n = 0$. □

Theorem 6. *Let $T = U|T|_r$ be p-hyponormal, $0 < p < 1$, and U unitary. If $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded open disk D in the complex plane **C** and the quotient space

$$H(D) = W^2(D,H)/\overline{(T-z)W^2(D,H)}$$

endowed with the Hilbert space norm. The class of a vector f or an operator on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Let M be the operator of multiplication by z on $W^2(D,H)$. As noted at the end of section 2, M is a scalar of order 2 and has a spectral distribution Φ. Let $S \equiv \tilde{M}$. Since $\overline{(T-z)W^2(D,H)}$ is invariant under every operator $M_f, f \in C^2(D)$, we infer that S is a scalar operator of order 2 with spectral distribution $\tilde{\Phi}$.

Consider the natural map $V : H \longrightarrow H(D)$ defined by $Vh = \widetilde{1 \otimes h}$, for $h \in H$, where $1 \otimes h$ denotes the constant function identically equal to h. Note that $VT = SV$. In particular ran V is an invariant subspace for S. Since V is one-to-one and has closed range by Proposition 5, T is subscalar of order 2. □

Corollary 7. *Every invertible p-hyponormal operator is subscalar of order 2.*

Proof. Assume $T = U|T|_r$ is an invertible p-hyponormal operator where U is unitary. Then $|T|_r$ is invertible. By [Ru, Theorem 12.33], $|T|_r^{\frac{1}{2}}$ is invertible. Therefore, $0 \notin \sigma(|T|_r^{\frac{1}{2}})$. By Theorem 6, T is subscalar of order 2. □

Corollary 8. *Let $T = U|T|_r$ be a p-hyponormal operator with the property $0 \notin \sigma(|T|_r^{\frac{1}{2}})$, $0 < p < 1$, and U unitary. If $\sigma(T)$ has interior in the plane, then T has a non-trivial invariant subspace.*

Proof. The corollary follows from Theorem 6 and [Es]. □

Corollary 9. *Let T be as in Corollary 8. Then T has the property (β).*

Proof. Since every subscalar operator has the property (β), the corollary follows from Theorem 6. □

Recall that an X in $\mathcal{L}(H,K)$ is called a quasi-affinity if it has trivial kernel and dense range. An operator A in $\mathcal{L}(H)$ is said to be a quasi-affine transform of an operator T in $\mathcal{L}(K)$ if there is a quasi-affinity X in $\mathcal{L}(H,K)$ such that $XA = TX$ (notation: $A \prec T$).

Corollary 10. *Let T be as in Corollary 8. If A is any operator such that $A \prec T$, then $\sigma(T) \subseteq \sigma(A)$.*

Proof. This is clear from [Ko, Theorem 3.2] and Corollary 9. □

Corollary 11. *Under the same hypothesis as Corollary 10, $A \in \mathcal{L}(H)$ is quasi-subscalar.*

Proof. Let $X \in \mathcal{L}(H, K)$ be a quasi-affinity such that $XA = TX$. Since V (in the construction of V and S) and X are one-to-one, VX is one-to-one. Therefore VX implements the quasi-subscalar properties. Thus A is quasi-subscalar. □

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