

## HILBERT COEFFICIENTS AND THE ASSOCIATED GRADED RINGS

HSIN-JU WANG

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ABSTRACT. Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . In this paper, we prove that if  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$  for some minimal reduction  $J$  of  $I$ , then  $\text{depth } G(I) \geq d - 2$ .

### 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen-Macaulay local ring with infinite residue field and  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . Let  $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$  be the associated graded ring of  $R$ . During the past years, many commutative algebraists tried to estimate the depth of  $G(I)$  for ideals  $I$  having good properties. In 1978, Valabrega and Valla obtained in [6] that  $G(I)$  is Cohen-Macaulay if and only if there exists a minimal reduction  $J$  of  $I$  such that  $I^n \cap J = I^{n-1}J$  for all  $n$ . Later on, Guerrieri studied the so called Valabrega-Valla module and made the following conjecture in her paper [1].

**Conjecture 1.** If  $\sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = t$  for some minimal reduction  $J$  of  $I$ , then  $\text{depth } G(I) \geq d - t$ .

On the other hand, Sally in [5] studied the depth of  $G(\mathfrak{m})$  by considering the classical bound of Abhyankar on the multiplicity  $e$  of  $R$ ; namely,  $e \geq \mu(\mathfrak{m}) - d + 1$ , where  $\mu(I)$  stands for the minimal number of a generating set of  $I$ . She first studied the case of rings with minimal multiplicity, i.e.,  $e = \mu(\mathfrak{m}) - d + 1$ , then the cases  $e - (\mu(\mathfrak{m}) - d + 1) = 1, 2$ . Recently, Huckaba and Marley showed in [4] that if one considers the first coefficient  $e_1(I)$ , then  $e_1(I)$  is bounded above by  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$  for any minimal reduction  $J$  of  $I$ , and later Huckaba and Vaz Pinto independently showed that  $\text{depth } G(I) \geq d - 1$  if  $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$  for some minimal reduction  $J$  of  $I$ .

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In a similar fashion to what Sally did with the Abhyankar’s bound, we can raise the following conjecture on the depth of  $G(I)$  by considering the difference of  $e_1(I)$  and  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$ .

**Conjecture 2.** *If  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = t$ , then  $\text{depth } G(I) \geq d - 1 - t$ .*

One can see in section 2 that Conjecture 1 holds if we can give an affirmative answer to Conjecture 2. In this paper, we are able to show, by using a method developed in [8] concerning the *Sally module* defined in [7], that if  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1$ , then  $\text{depth } G(I) \geq d - 2$ . Hence the Conjecture 1 holds if  $t \leq 2$ .

2. PRELIMINARIES

Throughout, let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and  $J$  a minimal reduction of  $I$ . Let  $G(I)$  be the associated graded ring of  $R$ . An element  $x \in I \setminus I^2$  is called *superficial* for  $I$  if  $(0 :_{G(I)} x^*)_n = 0$  for all  $n$  sufficiently large. Here,  $x^*$  denotes the image of  $x$  in  $I/I^2 \subseteq G(I)$ . A sequence  $x_1, \dots, x_k$  is called *superficial sequence* for  $I$  if  $x_1$  is superficial for  $I$  and  $x_i$  is superficial for  $I/(x_1, \dots, x_{i-1})$ . In [3], Huckaba proved that if  $\dim R = 1$ , then  $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J)$  for any minimal reduction  $J$  of  $I$ ; therefore it is easy to see the following:

**Lemma 2.1.** *If  $\dim R = d$  and  $x_1, \dots, x_{d-1} \in J$  is a superficial sequence for  $I$ , then*

$$e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x_1, \dots, x_{d-1}))).$$

In [4], Huckaba and Marley gave in Lemma 2.2 a sufficient conditions for  $G(I)$  having positive depth. We restate it here in the following special form.

**Lemma 2.2.** *Let  $x \in J$  be a superficial element for  $I$ . If  $\text{depth } G(I/(x)) > 0$ , then  $\text{depth } G(I) > 0$ .*

**Corollary 2.3.** *Let  $(R, \mathfrak{m})$  be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and  $J$  be a minimal reduction of  $I$ . Suppose that*

$$\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) = 1.$$

*Let  $x \in J$  be a superficial element for  $I$ . If  $\sum_{n=1}^{\infty} \lambda(I^n/(I^{n-1}J + I^n \cap (x))) = e_1(I)$ , then  $\text{depth } G(I) > 0$ .*

*Proof.* The conclusion follows from Lemma 2.2 and the fact ([3, Theorem 3.1]) that if  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) = e_1(I)$ , then  $\text{depth } G(I) \geq \dim R - 1$ . □

The following two lemmas are easy to derive; we leave the proofs to the reader.

**Lemma 2.4.** *Let  $(R, \mathfrak{m})$  be a 3-dimensional Cohen-Macaulay local ring with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and  $J = (x_1, x_2, x_3)$  be a minimal reduction of  $I$ . Let  $N \geq 2$ . Suppose that  $I^n \cap (x_i, x_j) \subseteq I^{n-1}J \ \forall n < N$  and  $\forall i, j \in \{1, 2, 3\}$ . Then  $\forall n < N$  and  $\forall m \geq 1$ ,*

- (1)  $I^n : x_i = I^{n-1} \ \forall i$ .
- (2)  $I^n J^m : x_i = I^n J^{m-1} \ \forall i$ .
- (3)  $I^n : x_2 = I^n : x_3 = I^{n-1} \pmod{x_1}$ .
- (4)  $I^n J^m : x_2 = I^n J^m : x_3 = I^n J^{m-1} \pmod{x_1}$ .
- (5) Let  $\lambda_0, \dots, \lambda_t$  be either unit or 0 but not all 0. Let  $s \in R$  be such that  $s(\sum_{i=0}^t \lambda_i x_1^{n-i} x_2^i) \in I^n J^m$ . Then  $s \in I^n J^{m-t}$  if  $m \geq t$  or  $s \in I^{n-t+m}$  if  $m < t$ .

If, moreover,  $I^N \cap (x_1) \subseteq I^{N-1}J$ , then  $I^N : x_1 = I^{N-1}$ .

**Lemma 2.5.** *Let  $N \geq 2$ . If  $a_1, \dots, a_n \in I^N$  not all in  $I^{N-1}J$ , then there are only finite number of units  $\lambda$  such that  $\sum_{i=1}^n a_i \lambda^{i-1} \in I^{N-1}J$ .*

The following proposition presents a relation between the two conjectures stated in the previous section.

**Proposition 2.6.** *If Conjecture 2 has a positive answer, then so does Conjecture 1.*

Suppose that Conjecture 2 holds. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and let  $J$  be a minimal reduction of  $I$ . Let  $t = \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap J}{I^{n-1}J})$ . Then, by Lemma 2.1, for any superficial sequence  $x_1, \dots, x_{d-1} \in J$  for  $I$ ,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) &= \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap (x_1, \dots, x_{d-1}) + I^{n-1}J}{I^{n-1}J}) \\ &\leq \sum_{n=1}^{\infty} \lambda(\frac{I^n \cap J}{I^{n-1}J}) = t. \end{aligned}$$

Let  $k$  be the least integer such that  $\lambda(\frac{I^k \cap J}{I^{k-1}J}) \neq 0$ . Then, by [2, Lemma 3.1],  $\lambda(\frac{I^k \cap (x_1, \dots, x_{d-1}) + I^{k-1}J}{I^{k-1}J}) < \lambda(\frac{I^k \cap J}{I^{k-1}J})$ , so that  $\sum_{n=1}^{\infty} \lambda(I^n/I^{n-1}J) - e_1(I) \leq t - 1$ . Therefore,  $\text{depth } G(I) \geq d - t$  by assumption. This shows that Conjecture 1 holds.

### 3. MAIN THEORY

The goal of this section is to prove the following:

**Theorem 3.1.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring of dimension  $d \geq 2$  with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . Suppose that there is a minimal reduction  $J$  of  $I$  such that  $\sum_{n=0}^{\infty} \lambda(I^{n+1}/I^n J) - e_1(I) = 1$ . Then  $\text{depth } G(I) \geq d - 2$ .*

By [5], it suffices to consider the case  $d = 3$ , so we assume in the following that  $d = 3$ . We also assume now that Theorem 3.1 doesn't hold. We shall reach a contradiction later.

Let  $x_1, x_2 \in J$  be a superficial sequence of  $I$ ; then, by Corollary 2.3, we have for  $i = 1, 2$ ,

$$(1) \quad \sum_{n=1}^{\infty} \lambda(I^n / (I^{n-1}J + I^n \cap (x_i))) - e_1(I) = 1.$$

Moreover, by Lemma 2.1, we have

$$(2) \quad \sum_{n=1}^{\infty} \lambda(I^n / (I^{n-1}J + I^n \cap (x_1, x_2))) - e_1(I) = 0.$$

Let  $\{x, y, z\}$  be a minimal generating set of  $J$ . Consider the exact sequence:

$$0 \longrightarrow T_{k,n} \longrightarrow \bigoplus^{\binom{n+2}{2}} I^k / I^{k-1}J \xrightarrow{\phi_n} S_{k,n} = I^k J^n / I^{k-1}J^{n+1} \longrightarrow 0,$$

where  $\phi_n = (x^n, x^{n-1}y, x^{n-1}z, \dots, z^n)$  and  $T_{k,n} = \ker(\phi_n)$ . From the proof of [8, Theorem 2.4], we see that there is an unique integer  $N \geq 2$  such that  $T_{N,n} \neq 0$  for some positive integer  $n$ . Notice that  $N$  is independent of the choice of a generating set of  $J$  since  $S_{k,n}$  and  $I^k / I^{k-1}J$  are. As  $R/m$  is infinite, we may, after elementary transformation of  $x, y$  and  $z$ , require that  $\{x, y, z\}$  satisfies the following conditions.

**Proposition 3.2.** *There is a generating set  $\{x, y, z\}$  of  $J$  satisfying the following conditions:*

- (i)  $\{x, y\}, \{x, z\}, \{y, z\}$  and  $\{z\}$  are all superficial sequences for  $I$ .
- (ii)  $I^n \cap (x), I^n \cap (y)$  and  $I^n \cap (z)$  are all contained in  $I^{n-1}J \forall n$ .
- (iii)  $I^n \cap (x, y), I^n \cap (x, z)$  and  $I^n \cap (y, z)$  are all contained in  $I^{n-1}J \forall n \neq N$ .

Moreover,

$$\begin{aligned} \lambda\left(\frac{I^N \cap (x, y) + I^{N-1}J}{I^{N-1}J}\right) &= \lambda\left(\frac{I^N \cap (x, z) + I^{N-1}J}{I^{N-1}J}\right) \\ &= \lambda\left(\frac{I^N \cap (y, z) + I^{N-1}J}{I^{N-1}J}\right) = 1. \end{aligned}$$

*Proof.* (Sketch.) Notice that (ii) follows from (i) and (1); therefore we need only to show (i) and (iii).

Let  $\{x, y, z\}$  be a generating set of  $J$ . Let  $n$  be an integer such that  $T_{N,n} \neq 0$ . Then there are  $a_{ijk} \in I^N$  not all in  $I^{N-1}J$  such that  $\sum_{i+j+k=n} a_{ijk}x^i y^j z^k \in$

$I^{N-1}J^{n+1}$ . By Lemma 2.5, we may, after elementary transformation of  $x, y$  and  $z$ , assume that  $a_{n00}, a_{0n0}$  and  $a_{00n}$  are not in  $I^{N-1}J$ .

Next, we can use prime avoidance and Corollary 2.3 to replace  $\{x, y, z\}$  by elements of the set  $\{x + \alpha y + \beta z\}$  so that the condition (i) holds without changing the condition that the coefficients of  $x^n, y^n$  and  $z^n$  are not in  $I^{N-1}J$ .

Since  $\sum_{i+j+k=n} a_{ijk}x^i y^j z^k \in (I^{N-1}J)J^n$ , there are  $a_1, a_2, a_3 \in I^{N-1}J$  such that  $a_{n00} - a_1 \in I^N \cap (y, z), a_{0n0} - a_2 \in I^N \cap (x, z)$  and  $a_{00n} - a_3 \in I^N \cap (x, y)$ . Thus the condition (iii) holds by condition (i) and (2). □

Let  $\{x, y, z\}$  be a generating set of  $J$  satisfying the conditions of Proposition 3.2. Let  $t$  be chosen least such that  $T_{N,n} \neq 0$  for  $n \geq t$ . Then there are elements  $\{a_{ijk} \mid i + j + k = t\}$  not all in  $I^{N-1}J$  such that  $\sum a_{ijk}x^i y^j z^k \in I^{N-1}J^{t+1}$ . If  $\{a_{0jk} \mid j + k = t\} \subseteq I^{N-1}J$ , then  $x(\sum a_{ijk}x^{i-1}y^j z^k) \in I^{N-1}J^{t+1}$ , so that by Lemma 2.4,  $\sum a_{ijk}x^{i-1}y^j z^k \in I^{N-1}J^t$ ; it follows that  $T_{N,t-1} \neq 0$ , which contradicts the choice of  $t$ . Therefore,  $\{a_{0jk} \mid j + k = t\}$  are not all in  $I^{N-1}J$ .

Let the overbars denote mod( $x$ ) in the following. Consider the exact sequence

$$0 \longrightarrow \bar{T}_{k,n} \longrightarrow \bigoplus_{i=0}^{n+1} \bar{I}^k / \bar{I}^{k-1} \bar{J} \xrightarrow{\bar{\phi}_n} \bar{I}^k \bar{J}^n / \bar{I}^{k-1} \bar{J}^{n+1} \longrightarrow 0,$$

where  $\bar{\phi}_n = (\bar{y}^n, \bar{y}^{n-1}\bar{z}, \dots, \bar{z}^n)$  and  $\bar{T}_{k,n} = \ker(\bar{\phi}_n)$ . Since  $\sum_{n=1}^{\infty} \lambda(\bar{I}^n / \bar{I}^{n-1} \bar{J}) - e_1(\bar{I}) = 1$ , there is a unique integer  $N'$  such that  $\bar{T}_{N',n} \neq 0$  for some  $n$ . However, by the following remark,  $\{\bar{a}_{0jk} \mid j+k = t\}$  are not all in  $\bar{I}^{N-1} \bar{J}$ . Since  $\sum \bar{a}_{0jk} \bar{y}^j \bar{z}^k \in \bar{I}^{N-1} \bar{J}^{t+1}$ , we see that  $N' = N$ .

*Remark 3.3.* Let  $b \in I^N \setminus I^{N-1}J$ ; then  $\bar{b} \notin \bar{I}^{N-1} \bar{J}$  by the fact that  $I^N : x = I^{N-1}$ .

In the sequel, let  $R^0$  denote the set  $\{\text{units of } R\} \cup \{0\}$  and  $R_n$  ( $n \geq 0$ ) denote the set  $\{f \mid f = \sum_{i=0}^n \lambda_i y^{n-i} z^i \text{ for some } \lambda_i \in R^0\}$ .

**Lemma 3.4.** *Let  $(R, \mathfrak{m})$  be a 2-dimensional Cohen-Macaulay local ring with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  and  $J = (y, z)$  be a minimal reduction of  $I$ . Let  $N \geq 2$ . Suppose that*

- (i)  $\lambda(\frac{I^N \cap (y) + I^{N-1}J}{I^{N-1}J}) = \lambda(\frac{I^N \cap (z) + I^{N-1}J}{I^{N-1}J}) = 1$ , and
- (ii)  $\forall n < N$  and  $\forall m \geq 1$ ,  $I^n : y = I^n : z = I^{n-1}$  and  $I^n J^m : y = I^n J^m : z = I^n J^{m-1}$ .

Consider the exact sequence

$$0 \longrightarrow T_{N,n} \longrightarrow \bigoplus_{i=0}^{n+1} I^i / I^{i-1} J \xrightarrow{\phi_n} I^N J^n / I^{N-1} J^{n+1} \longrightarrow 0,$$

where  $\phi_n = (y^n, y^{n-1}z, \dots, z^n)$  and  $T_{N,n} = \ker(\phi_n)$ .

Suppose that  $T_{N,n} \neq 0$  for some  $n$ . Let  $l$  be the least integer such that  $T_{N,n} \neq 0$  for all  $n \geq l$ . Let  $a_0, \dots, a_l \in I^N$  not all in  $I^{N-1}J$  such that  $\sum_{i=0}^l a_i y^{l-i} z^i \in I^{N-1} J^{l+1}$ .

Then the following hold:

- (1)  $a_0 \notin I^{N-1}J$  and  $\mathfrak{m}a_i \subseteq I^{N-1}J \forall i$ .
- (2) If  $\sum_{i=0}^n b_i y^{n-i} z^i \in I^{N-1} J^{n+1}$  for some  $b_i \in I^N$  and for some  $n$ , then  $\mathfrak{m}b_i \subseteq$

$$I^{N-1}J \forall i \text{ and there are } \lambda_0, \dots, \lambda_{n-l} \in R^0 \text{ such that } b_j - \sum_{i=0}^j a_i \lambda_{j-i} \in I^{N-1}J.$$

(Conventions:  $a_i = 0$  if  $i > l$  and  $\lambda_i = 0$  if  $i > n - l$ .)

*Proof.* By the choice of  $l$  and the fact that  $I^{N-1}J^m : z = I^{N-1}J^{m-1}$ , we obtain that  $a_0 \notin I^{N-1}J$ . Since  $a_0 \in I^N \cap (z) + I^{N-1}J$ , we have, by assumption,  $\mathfrak{m}a_0 \subseteq I^{N-1}J$ .

Furthermore, let  $w \in \mathfrak{m}$ ; then  $\sum_{i=1}^l (wa_i)y^{l-i}z^{i-1} \in I^{N-1}J^l$  by the assumption that  $I^{N-1}J^m : z = I^{N-1}J^{m-1}$ . Again, by the choice of  $l$ ,  $wa_i$  must belongs to  $I^{N-1}J$  for all  $i$ . This proves (1).

To see (2), we may assume that  $n \geq l$  and  $b_0 \in I^N \setminus I^{N-1}J$ . Then  $b_0 \in I^N \cap (z) + I^{N-1}J$ . Since  $a_0 \in I^N \cap (z) + I^{N-1}J$ ; there is a unit  $\lambda_0$  such that  $b_0 - \lambda_0 a_0 \in I^{N-1}J$ , therefore,  $\sum_{i=0}^{n-1} (b_{i+1} - \lambda_0 a_{i+1})y^{n-1-i}z^i \in I^{N-1}J^n$ .

If  $n = l$ , then, by the choice of  $l$ ,  $b_i - \lambda_0 a_i \in I^{N-1}J \forall i$ ; hence  $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$ . If  $n > l$ , then by induction  $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$  and there are  $\lambda_1, \dots, \lambda_{n-l} \in R^0$  such that  $\forall j \geq 1$   $b_j - \lambda_0 a_j = \sum_{i=0}^{j-1} a_i \lambda_{j-i}$ . This proves (2). □

Let  $\{x, y, z\}$  be a generating set of  $J$  satisfying the conditions of Proposition 3.2. Let the overbars denote mod( $x$ ) in the following. By Lemma 2.4, it is easy to check that  $\bar{R}, \bar{y}$  and  $\bar{z}$  satisfy all the assumptions of Lemma 3.4. Let  $l = \min\{n \mid \bar{T}_{N,n} \neq 0\}$ , where  $\bar{T}_{N,n}$  is defined as the above; then there are  $a_0, \dots, a_l \in I^N$  not all in  $I^{N-1}J$  such that  $\sum \bar{a}_i \bar{y}^{l-i} \bar{z}^i \in \bar{I}^{N-1} \bar{J}^{l+1}$ . Let  $u = \sum_{i=0}^l a_i y^{l-i} z^i$ ; then  $u$  has the following property.

**Lemma 3.5.** *If  $\sum_{i=0}^n b_i y^{n-i} z^i \in (x) + I^{N-1}J^{n+1}$  for some  $b_i \in I^N$ , then*

- (1)  $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$ .
- (2) *There is an  $f \in R_{n-l}$  such that  $\sum_{i=0}^n b_i y^{n-i} z^i - fu \in I^{N-1}J^{n+1}$ .*

*Proof.* By Lemma 3.4,  $\mathfrak{m}\bar{a}_i \in \bar{I}^{N-1} \bar{J}$ ; hence  $\mathfrak{m}a_i \subseteq I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$ . Moreover, there are  $\lambda_0, \dots, \lambda_{n-l} \in R^0$  such that  $\bar{b}_j - \sum_{i=0}^j \bar{a}_i \lambda_{j-i} \in \bar{I}^{N-1} \bar{J}$ ; then

$b_j - \sum_{i=0}^j a_j \lambda_{j-i} \in I^{N-1}J + (x) \cap I^N \subseteq I^{N-1}J$ . Therefore  $\mathfrak{m}b_i \subseteq I^{N-1}J \forall i$ . Let  $f = \sum_{i=0}^{n-l} \lambda_i y^{n-l-i} z^i \in R_{n-l}$ . Then

$$\sum_{i=0}^n b_i y^{n-i} z^i - fu = \sum (b_j - \sum_{i=0}^j a_j \lambda_{j-i}) y^{n-j} z^j \in I^{N-1}J^{n+1}.$$

□

*Remark 3.6.* If, in Lemma 3.5, at least one of the  $b_i$  is not in  $I^{N-1}J$ , then  $n \geq l$  and we can choose  $f$  to be a nonzero element of  $R_{n-l}$ .

Since depth  $G(I) = 0$ ,  $I^n : J \neq I^{n-1}$  for some  $n$ . Let  $N' = \min\{n \mid I^n : J \neq I^{n-1}\}$ . Since  $I^n : x = I^{n-1} \forall n \leq N, N' > N$ . Let  $s \notin I^{N'-1}$  such that  $sJ \subseteq I^{N'}$ .

**Lemma 3.7.** *There exists an element  $s'$  with  $s - s' \in I^{N'-1}$  such that  $s'y \in I^N J^{N'-N}$ ,  $s'z \in I^N J^{N'-N}$  and  $s'x \in I^N(y, z)^{N'-N}$ .*

*Proof.* Suppose we have shown for some  $k > N$  that there is an element  $s'$  with  $s - s' \in I^{N'-1}$  such that  $s'y \in I^k J^{N'-k}$ ,  $s'z \in I^k J^{N'-k}$  and  $s'x \in I^k(y, z)^{N'-k}$ . Then there are  $a_i, b_{ijk} \in I^k$  such that  $s'x = \sum a_i y^{t-i} z^i$  and  $s'y = \sum b_{ijk} x^i y^j z^k$ , where  $t = N' - k$ . Therefore,

$$\sum a_i y^{t-i+1} z^i - \sum b_{ijk} x^{i+1} y^j z^k = 0 \in I^{k-1} J^{t+2}.$$

Since  $k \neq N$ ,  $T_{k,t+1} = 0$ ; therefore,  $a_i, b_{ijk}$  are in  $I^{k-1} J$ . It follows that  $s'x \in I^{k-1} J^{t+1}$  and  $s'y \in I^{k-1} J^{t+1}$ . Similarly, we can get  $s'z \in I^{k-1} J^{t+1}$ .

Finally, from the expression  $s'x \in I^{k-1} J^{t+1}$ , we see that there is a  $w \in I^{k-1} J^t \subseteq I^{N'-1}$  such that  $(s' - w)x \in I^{k-1}(y, z)^{t+1}$  and  $(s' - w)J \subseteq I^{k-1} J^{t+1}$ . This completes the proof.  $\square$

By Lemma 3.7 we may assume that  $s$  satisfies the conditions  $sJ \in I^N J^t$  and  $sx \in I^N(y, z)^t$ , where  $t = N' - N$ . Since  $s \notin I^{N'-1}$ ,  $sx \notin I^{N-1} J^{t+1}$  by Lemma 2.4. Therefore, by Remark 3.6 and the condition  $sx \in I^N(y, z)^t$ , we obtain that  $t \geq l$ . Hence by Lemma 3.5, there is a nonzero element  $f_0 \in R_{t-l}$  such that

$$(3) \quad sx - f_0 u \in I^{N-1} J^{t+1}.$$

In what follows, let  $\mathbf{k}$  be the residue field of  $R$ . If  $f = \sum_{i=0}^n \lambda_i y^{n-i} z^i \in R_n$ , then we associate to  $f$  a homogeneous polynomial  $T(f) = F = \sum_{i=0}^n \bar{\lambda}_i Y^{n-i} Z^i$  in  $\mathbf{k}[Y, Z]$ . (Here, the overbars denote mod( $\mathfrak{m}$ ).)

*Remark 3.8.* From (3) and Lemma 3.5, we obtain  $\mathfrak{m}sx \subseteq I^{N-1} J^{t+1}$ , and therefore  $\mathfrak{m}s \subseteq I^{N-1} J^t$ . Moreover, if  $f$  and  $g$  are two elements of  $R_n$  such that  $F = G$ , then  $f - g \in \mathfrak{m}R_n$ ; it follows that  $sf - sg \in I^{N-1} J^{n+t}$ .

From (3), Lemma 3.5 and Remark 3.6 we have the following corollary.

**Corollary 3.9.** *Let  $B_0 = \sum b_i y^{n-i} z^i$  with  $b_i \in I^N$  not all in  $I^{N-1} J$  and  $B_j \in (y, z)^{n-j} I^N$ . Suppose that  $B_0 + B_1 x + \dots \in I^{N-1} J^{n+1}$ . Then there is a nonzero element  $h \in R_{n-l}$  such that  $B_0 + hu \in I^{N-1} J^{n+1}$  and  $sh - f_0(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l}$ .*

*Proof.* By Lemma 3.5 and Remark 3.6, there is a nonzero element  $h \in R_{n-l}$  such that  $B_0 + hu \in I^{N-1} J^{n+1}$ . By (3),  $shx - f_0 x(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l+1}$ . It follows by Lemma 2.4 that  $sh - f_0(B_1 + B_2 x + \dots) \in I^{N-1} J^{n+t-l}$ .  $\square$

Let  $Q_j$  ( $j \geq 0$ ) be the set of all integers  $n$  such that there exists a nonzero element  $f \in R_n$  with  $sf \in \sum_{i=0}^j x^i(y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$ . (For example, since  $sy \in I^N J^t$ ,  $1 \in Q_t$ .)

Let  $m = \min\{j \mid Q_j \neq \emptyset\}$ . Let  $k$  be the smallest integer in  $Q_m$ ; then there is a nonzero element  $f \in R_k$  with  $sf \in \sum_{i=0}^m x^i(y, z)^{t+k-1-i} I^N + I^{N-1} J^{t+k}$ , so that

there are  $A_i \in (y, z)^{t+k-1-i} I^N$  such that

$$(4) \quad sf - \sum_{i=0}^m A_i x^i \in I^{N-1} J^{t+k}.$$

**Lemma 3.10.** *Let  $C_i \in (y, z)^{n-i} I^N$  such that  $\sum_{i=0}^m C_i x^i \in I^{N-1} J^{n+1}$ . Then  $C_i \in I^{N-1} J^{n+1-i}$  and all the coefficients of  $C_i$  are all in  $I^{N-1} J$ .*

*Proof.* If  $m = 0$ , then  $C_0 \in I^{N-1} J^{n+1}$ . If the coefficients of  $C_0$  are not all in  $I^{N-1} J$ , then by Corollary 3.9 there is a nonzero element  $h \in R_{n-l}$  such that  $sh \in I^{N-1} J^{n+t-l}$ . This gives the contradiction  $s \in I^{N-1} J^t$  by Lemma 2.4.

Assume that  $m \geq 1$  and the assertion is false. Let  $j$  be the least integer such that the coefficients of  $C_j$  are not all in  $I^{N-1} J$ . Then, by Lemma 2.4,  $\sum_{i=j}^m C_i x^{i-j} \in I^{N-1} J^{n+1-j}$ ; hence by Corollary 3.9 there is a nonzero element  $h \in R_{n-j-l}$  such that  $sh - f_0(\sum_{i=j+1}^m C_i x^{i-j-1}) \in I^{N-1} J^{n+t-j-l}$ , which contradicts the choice of  $m$ . The assertion now follows. □

Let  $p$  be the maximal integer such that  $Z^p | F$ . Let  $F' = F/Z^p$ .

**Lemma 3.11.** *If  $g$  is a nonzero element of  $R_n$  such that  $sg \in \sum_{i=0}^m x^i (y, z)^{t+n-1-i} I^N + I^{N-1} J^{t+n}$ , then  $F | G$ .*

*Proof.* Let  $B_i \in (y, z)^{t+n-1-i} I^N$  such that

$$(5) \quad sg - \sum_{i=0}^m B_i x^i \in I^{N-1} J^{t+n}.$$

Write  $G = G'Z^q$  with  $(G', Z) = 1$ . Let  $g' \in R_{n-q}$  and  $f' \in R_{k-p}$  such that  $T(g') = G'$  and  $T(f') = F'$ . By Remark 3.8, we may assume that  $f = f'z^p$  and  $g = g'z^q$ .

Assume that  $q < p$ . Then from (4) and (5), we obtain that

$$(6) \quad \sum_{i=0}^m (f'z^{p-q} B_i - g' A_i) x^i \in I^{N-1} J^{t+k+n-q},$$

so that by Lemma 3.10 all the coefficients of  $f'z^{p-q} B_i - g' A_i$  are all in  $I^{N-1} J$ . Since  $(G', Z) = 1$ , there are  $A'_i \in (y, z)^{t+k-2-i} I^N$  such that  $A_i - z A'_i \in I^{N-1} J^{t+k-i}$ ; it follows from (4) that  $s(f'z^{p-1}) - \sum_{i=0}^m A'_i x^i \in I^{N-1} J^{t+k-1}$ , which contradicts the choice of  $k$ . Hence  $q \geq p$ .

Write  $G = G'Z^p$ . Then  $G' = F'Q + Z^{n-k+1}G''$  for some  $Q$  and  $G''$ . Suppose  $G'' \neq 0$ . Then  $\deg G'' = \deg F' - 1$ . Let  $g'' \in R_{k-p-1}$  such that  $T(g'') = G''$ ; then

$$(7) \quad sg''z^{n-k+1+p} - \sum_{i=0}^m C_i x^i \in I^{N-1} J^{t+n}$$

for some  $C_i \in (y, z)^{t+n-1-i} I^N$ .

From (4) and (7), we obtain that

$$\sum_{i=0}^m (g''z^{n-k+1}A_i - f'C_i)x^i \in I^{N-1}J^{t+n+k-p};$$

therefore, by Lemma 3.10, all the coefficients of  $g''z^{n-k+1}A_i - f'C_i$  are in  $I^{N-1}J$ . Since  $(F', Z) = 1$ , there are  $C'_i \in (y, z)^{t+k-2-i}I^N$  such that  $C_i - z^{n-k+1}C'_i \in I^{N-1}J^{t+n-i}$ . Hence from (7),  $sg''z^p - \sum_{i=0}^m C'_i x^i \in I^{N-1}J^{t+k-1}$ , which contradicts the choice of  $k$ . Therefore,  $G'' = 0$  and  $F|G$ .  $\square$

In fact, Lemma 3.11 can be improved as follows.

**Lemma 3.12.** *Let  $m \leq m' \leq t$ . If  $g$  is a nonzero element of  $R_n$  such that  $sg \in \sum_{i=0}^{m'} x^i(y, z)^{t+n-1-i}I^N + I^{N-1}J^{t+n}$ , then  $F|G$ .*

*Proof.* We use induction on  $m'$ . If  $m' = m$ , then this is the content of Lemma 3.11.

Assume that  $m' > m$ . Let  $H_0 = \frac{F_0}{(F_0, F)}$  and let  $n_0 = \text{deg } H_0$ . Let  $h_0 \in R_{n_0}$  such that  $T(h_0) = H_0$ . Let  $B_i \in (y, z)^{t+n-1-i}I^N$  such that

$$(8) \quad sg - \sum_{i=0}^{m'} B_i x^i \in I^{N-1}J^{t+n}.$$

Let  $H = (F, G)$  and let  $k' = \text{deg } H$ . Then  $F = F'H$  and  $G = G'H$  for some  $F'$  and  $G'$ . Let  $f' \in R_{k-k'}$  and  $g' \in R_{n-k'}$  such that  $T(f') = F'$  and  $T(g') = G'$ . By Remark 3.8, we may assume that  $f = f'h$  and  $g = g'h$ . Set  $A_i = 0 \quad \forall i > m$ .

From (4) and (8), we obtain that  $\sum_{i=0}^{m'} (f'B_i - g'A_i)x^i \in I^{N-1}J^{t+n+k-k'}$ . Therefore,

by Lemma 3.5, there is an element  $h_1 \in R_{t+n+k-k'-l-1}$  such that  $f'B_0 - g'A_0 + h_1u \in I^{N-1}J^{t+n+k-k'}$ , and then, by (3),  $sh_1 - f_0(\sum_{i=0}^{m'-1} (f'B_{i+1} - g'A_{i+1})x^i) \in$

$I^{N-1}J^{2t+n+k-k'-l-1}$ . Hence, by induction,  $F|H_1$ . Moreover, since  $(F_0, F)$  is a common factor of  $H_1$  and  $F_0$ , there is an element  $g_1 \in R_{n_0+n+k-k'-1}$  such that

$sg_1 - h_0(\sum_{i=0}^{m'-1} (f'B_{i+1} - g'A_{i+1})x^i) \in I^{N-1}J^{t+n_0+n+k-k'-1}$ . Therefore, by induction,

$F|G_1$ . Hence there is an element  $g'_1 \in R_{n_0+n-k'-1}$  such that  $G_1 = G'_1F$ .

Suppose that  $m = 0$ . Since  $f'B_0 - g'A_0 + h_1u \in I^{N-1}J^{t+n+k-k'}$ , by Lemma 3.10,  $g'hA_0 - fA'_0 \in I^{N-1}J^{t+n+k}$  for some  $A'_0 \in (y, z)^{t+n-1}I^N$ . Therefore by (4),  $sf'g'h - fA'_0 \in I^{N-1}J^{t+n+k}$ , and then  $sg'h - A'_0 \in I^{N-1}J^{t+n}$  by Lemma 2.4, so that by Lemma 3.11,  $F|G'H$ . But  $(F, G') = 1$ ; we obtain  $F|H$ .

Assume now that  $m \geq 1$ . We claim: There are integers  $n_1, \dots, n_{m'-m}, d_1, \dots, d_{m'-m}$  and elements  $h_j \in R_{n_j}, g_j \in R_{d_j}$  and  $g'_j \in R_{d_j-k}$  satisfy the following conditions:

- (i)  $G_j = G'_jF$ .
- (ii)  $h_0^{j-1}(f'B_{j-1} - g'A_{j-1}) + h_0^{j-2}g'_1A_{j-2} - \dots - g'_{j-1}A_0 + h_ju \in I^{N-1}J^{n_j+l+1}$ .

$$(iii) \quad sg_j - \sum_{i=0}^{m'-j} (h_0^j(f'B_{i+j} - g'A_{i+j}) - h_0^{j-1}g'_1A_{i+j-1} - \cdots - h_0g'_{j-1}A_{i+1})x^i \in I^{N-1}J^{d_j+t}.$$

Suppose that we have constructed, for some  $j \geq 1$ ,  $h_j$ ,  $g_j$  and  $g'_j$ . Then from (i), (iii) and (4), we see that the element

$$\begin{aligned} & h_0^j(f'B_j - g'A_j) - h_0^{j-1}g'_1A_{j-1} - \cdots - g'_jA_0 \\ & + \sum_{i=0}^{m'-j-1} (h_0^j(f'B_{i+j+1} - g'A_{i+j+1}) - h_0^{j-1}g'_1A_{i+j} - \cdots - g'_jA_{i+1})x^{i+1} \end{aligned}$$

is in  $I^{N-1}J^{d_j+t}$ . Therefore, by Lemma 3.5, there is an element  $h_{j+1} \in R_{n_{j+1}}$  for some  $n_{j+1}$  such that (ii) holds for  $j + 1$  and  $F|H_{j+1}$  (cf. the construction of  $h_1$ ) by induction. Moreover, from the construction of  $g_1$ , it is easy to see that there is an element  $g_{j+1} \in R_{d_{j+1}}$  for some  $d_{j+1}$  such that (iii) holds for  $j + 1$ . Since by induction  $F|G_{j+1}$ , there is an  $g'_{j+1} \in R_{d_{j+1}-k}$  such that (i) holds. This proves the claim.

Set  $j = m' - m$  in (iii) of the claim and compare with (4). We obtain that the element

$$\sum_{i=0}^m (h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i)x^i$$

is in  $I^{N-1}J^{d_{m'-m}+t}$ . However by Lemma 3.10,  $\forall i \leq m$ ,

$$(9) \quad h_0^{m'-m}(f'B_{i+m'-m} - g'A_{i+m'-m}) - h_0^{m'-m-1}g'_1A_{i+m'-m-1} - \cdots - g'_{m'-m}A_i \in I^{N-1}J^{d_{m'-m}+t-i}$$

From (ii) and (9), it is not hard to see that for  $0 \leq i \leq m$  there are  $A'_i \in (y, z)^{e_i}I^N$  for some  $e_i$  such that  $hh_0^i g'^{i+1}A_i - fA'_i \in I^{N-1}J^{e_i+k+1}$ . Therefore

by (4),  $shh_0^m g'^{m+1}f - f(\sum_{i=0}^m h_0^{m-i} g'^{m-i} A'_i x^i) \in I^{N-1}J^e$  for some  $e$ ; it follows by Lemma 2.4 that

$$shh_0^m g'^{m+1} - \sum_{i=0}^m h_0^{m-i} g'^{m-i} A'_i x^i \in I^{N-1}J^{e-k}.$$

Finally, by Lemma 3.11,  $F|G'^{m+1}HH_0^m$ . Since  $(F, G') = (F, H_0) = 1$ , we have  $F|H$ . This completes the proof.  $\square$

Now, choose a unit  $\lambda$  so that  $(Y + \lambda Z, F) = 1$ . Since  $s(y + \lambda z) \in \sum_{i=0}^t x^i(y, z)^{t-i}I^N + I^{N-1}J^{t+1}$ , by Lemma 3.12,  $F|Y + \lambda Z$ , which contradicts the choice of  $\lambda$ . This proves Theorem 3.1.

By Proposition 2.6 and Theorem 3.1, we have the following corollary.

**Corollary 3.13.** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring of dimension  $d \geq 2$  with infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . Suppose that there is a minimal reduction  $J$  of  $I$  such that  $\sum_{n=1}^{\infty} \lambda(I^n \cap J/I^{n-1}J) = 2$ . Then  $\text{depth } G(I) \geq d - 2$ .*

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## REFERENCES

1. A. Guerrieri, *On the depth of the associated graded ring of an  $m$ -primary ideal of a Cohen-Macaulay local ring*, J. Algebra. **167** (1994), 745–757. MR **95h**:13004
2. ———, *On the depth of the associated graded ring*, Proc. A.M.S. **123** (1995), 11–20. MR **95c**:13002
3. S. Huckaba, *A  $d$ -dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients*, Proc. Amer. Math. Soc. **124** (1996), 1393–1401. MR **96g**:13018
4. S. Huckaba and T. Marley, *Hilbert coefficients and the depth of associated graded rings*, J. London Math. soc.(2) **56** (1997), no. 1, 64–76. MR **98i**:13028
5. J. D. Sally, *Super-regular sequences*, Pacific J. Math. **84** (1979), 465–481. MR **81m**:13024
6. P. Valabrega and G. Valla, *Form rings and regular sequence*, Nagoya Math. J. **72** (1978), 93–101. MR **80d**:14010
7. W. V. Vasconcelos, *Hilbert functions, analytic spread, and Koszul homology*, Contemporary Math. **159** (1994), 401–422. MR **95a**:13006
8. H.-J Wang, *An interpretation of depth  $G(I)$  and  $e_1(I)$  via the Sally module*, Communications in Algebra **25**,(1) (1997), 303–309. MR **98a**:13035

DEPARTMENT OF MATHEMATICS, NATIONAL CHUNG CHENG UNIVERSITY, MINGHSIUNG, CHIAYI 621, TAIWAN

*E-mail address*: [hjwang@math.ccu.edu.tw](mailto:hjwang@math.ccu.edu.tw)