

GEOMETRICAL SIGNIFICANCE OF THE LÖWNER-HEINZ INEQUALITY

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Dedicated to Mischa Cotlar, with affection and admiration, on his 86th anniversary

ABSTRACT. It is proven that the Löwner-Heinz inequality $\|A^t B^t\| \leq \|AB\|^t$, valid for all positive invertible operators A, B on the Hilbert space \mathcal{H} and $t \in [0, 1]$, has equivalent forms related to the Finsler structure of the space of positive invertible elements of $\mathcal{L}(\mathcal{H})$ or, more generally, of a unital C^* -algebra. In particular, the Löwner-Heinz inequality is equivalent to some type of “nonpositive curvature” property of that space.

0. INTRODUCTION

The space of positive definite matrices $M_n(\mathbb{C})^+$ is a well known Riemannian manifold with nonpositive curvature. It turns out that, for every unital C^* -algebra \mathcal{A} , the space \mathcal{A}^+ of all positive invertible elements of \mathcal{A} has a very rich Finsler (non-Riemannian) structure (see [14], [4], [5], [6], [15]). In this paper we show that the so-called Löwner-Heinz inequality

$$\|A^t B^t\| \leq \|AB\|^t \quad (A, B \in \mathcal{L}(\mathcal{H})^+, t \in [0, 1])$$

plays a central role in this geometrical study.

Recall that the inequality is a by-product of Löwner’s characterization of operator monotone functions [13] and it has many applications in spectral theory [7]. Recently, Fujii, Furuta and Nakamoto [8] proved that the Löwner-Heinz inequality is equivalent to a more technical one (see (2.4)) which appeared in [6] as a corollary of a geometrical property of \mathcal{A}^+ , namely the fact that the distance between two geodesics $\gamma(t), \delta(t)$ in \mathcal{A}^+ is a convex function of t .

We show that all these assertions are, indeed, equivalent. It is known that, for Riemannian manifolds, the convexity of the function $d(\gamma(t), \delta(t))$ for any pair of geodesics γ, δ is equivalent to the nonpositivity of the sectional curvature. Thus, the Löwner-Heinz inequality appears as the analytic form of this geometrical behaviour of \mathcal{A}^+ . In this sense, \mathcal{A}^+ offers a beautiful infinite dimensional example of what Gromov calls a “espace de longueur” [10, Ch. 1], and the Löwner-Heinz inequality is the expression of the “nonpositive curvature” of \mathcal{A}^+ .

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The paper contains also some results relating the geodesic distance to the work of Nussbaum [14], [15] on the part metric or Thompson distance on \mathcal{A}^+ , with its corresponding notion of geodesics.

1. EQUIVALENT INEQUALITIES

Theorem 1. *Let A be a unital C^* -algebra. Then each of the following properties holds and is equivalent to each other:*

- (1.1) $\|a^t b^t\| \leq \|ab\|^t$ for all $a, b \in \mathcal{A}^+$ and $t \in [0, 1]$;
- (1.2) $\|\log(a^{-t/2} b^t a^{-t/2})\| \leq t \|\log(a^{-1/2} b a^{-1/2})\|$ for all $a, b \in \mathcal{A}^+$ and $t \in [0, 1]$;
- (1.3) For all $a_1, a_2, b_1, b_2 \in \mathcal{A}^+$ and $t \in [0, 1]$,

$$\begin{aligned} & \| (a_1^{1/2} (a_1^{-1/2} b_1 a_1^{-1/2})^t a_1^{1/2})^{1/2} (a_2^{1/2} (a_2^{-1/2} b_2 a_2^{-1/2})^t a_2^{1/2})^{-1/2} \| \\ & \leq \| a_1^{1/2} a_2^{1/2} \|^{1-t} \| b_1^{1/2} b_2^{1/2} \|^t. \end{aligned}$$

Proof. The first one is the Löwner-Heinz inequality. A short proof of it can be found in [7] or [9]. The second equality has been proven in [5] as a consequence of (1.1). The third inequality has been proven in [6] and recently it has been shown in [8] that it is equivalent to (1.1). Thus, it suffices to prove that (1.2) implies (1.3). For this, take $a, b \in \mathcal{A}^+$. By a compactness argument, there exists $k > 0$ such that

$$a^t (kb)^{2t} a^t > 1 \quad \text{for all } t \in [0, 1].$$

Observe that $\|\log c\| = \max\{\log \|c\|, \log \|c^{-1}\|\}$ for all $c \in \mathcal{A}^+$ and that $\|\log c\| = \log \|c\|$ for $c > 1$. Then

$$\begin{aligned} \log \|a^t (kb)^{2t} a^t\| &= \|\log(a^t (kb)^{2t} a^t)\| \\ &\leq t \|\log(a(kb)^2 a)\| \\ &= t \log \|a(kb)^2 a\|, \end{aligned}$$

where we use inequality (1.2) replacing a, b by a^{-2}, b^2 , respectively. Thus,

$$\log \|a^t b^{2t} a^t\| + \log k^{2t} \leq t \log \|ab^2 a\| + t \log k^2$$

so that $\|a^t b^t\|^2 \leq \|ab\|^{2t}$, which is (1.1). □

2. THE GEOMETRICAL INTERPRETATION

We give now a geometrical interpretation of the theorem. For this, we present a short description of the differential geometry of \mathcal{A}^+ (see [4], [5], [6], [2] for a complete treatment). Let \mathcal{A} be a unital C^* -algebra, \mathcal{A}^\times the group of invertible elements of \mathcal{A} , \mathcal{A}_s the (real) subspace of Hermitian elements of \mathcal{A} . \mathcal{A}^+ is an open convex subset of \mathcal{A}_s so that it will be considered as an open submanifold of \mathcal{A}_s and the tangent spaces $(T\mathcal{A}^+)_a$ will be identified to \mathcal{A}_s for all $a \in \mathcal{A}^+$. There is a natural action of \mathcal{A}^\times over \mathcal{A}^+ given by $(g, a) \mapsto gag^*$ ($g \in \mathcal{A}^\times, a \in \mathcal{A}^+$). This is a transitive action: given $a, b \in \mathcal{A}^+$ $g = b^{1/2} a^{-1/2}$ verifies $gag^* = b$. For every $a \in \mathcal{A}^+$, the map $\tau_a : \mathcal{A}^\times \rightarrow \mathcal{A}^+$ $\tau_a(g) = gag^*$ is a principal fibre bundle with a natural connection.

The covariant derivative of a tangent field X along a curve γ in \mathcal{A}^+ is given by

$$\frac{DX}{dt} = \dot{X} = \frac{1}{2} (X\gamma^{-1}\dot{\gamma} + \dot{\gamma}\gamma^{-1}X).$$

The curve γ is a *geodesic* of the connection if $\frac{D\dot{\gamma}}{dt} = 0$, i.e., if $\ddot{\gamma} = \dot{\gamma}\gamma^{-1}\dot{\gamma}$.

It turns out that in this case $g\gamma g^*$ is also a geodesic for all $g \in \mathcal{A}^\times$. The unique geodesic γ such that $\gamma(0) = a$ and $\dot{\gamma}(0) = X \in \mathcal{A}_s$ is

$$\begin{aligned} \gamma(t) &= e^{(t/2)Xa^{-1}} \\ &= a^{1/2}e^{ta^{-1/2}Xa^{-1/2}}a^{1/2} \end{aligned}$$

and for every $a, b \in \mathcal{A}^+$ there is a unique geodesic $\gamma_{a,b}$ such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, namely,

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2} ,$$

corresponding to $X = a^{1/2} \log(a^{-1/2}ba^{-1/2})a^{1/2}$.

Thus, the *exponential map* $\exp_a : (TA^+)_a \rightarrow \mathcal{A}^+$ is defined by

$$\exp_a(X) = e^{(1/2)Xa^{-1}}ae^{(1/2)a^{-1}X} = a^{1/2}e^{a^{-1/2}Xa^{-1/2}}a^{1/2}$$

and admits a global inverse given by

$$\log_a(c) = a^{1/2} \log(a^{-1/2}ca^{-1/2})a^{1/2} .$$

Even if \mathcal{A}^+ is not a Riemannian manifold, there is a natural Finsler structure on \mathcal{A}^+ defined as follows: for $X \in (TA^+)_a = \mathcal{A}_s$ define $\|X\|_a = \|\lambda^{-1/2}X\lambda^{-1/2}\|$. If \mathcal{A} is represented in a Hilbert space \mathcal{H} , $\|X\|_a$ is the norm of the symmetric sesquilinear form $B_X : \mathcal{H}_a \times \mathcal{H}_a \rightarrow \mathbb{C}$, $B_X(\xi, \eta) = \langle X\xi, \eta \rangle$, where \mathcal{H}_a is \mathcal{H} with the scalar product $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$. If $g \in \mathcal{A}^\times$, then $g : \mathcal{H}_{gag^*} \rightarrow \mathcal{H}_a$ is an isometry, so that

$$\|B_{gXg^*}\| = \|B_X\| \quad \text{and} \quad \|gXg^*\|_{gag^*} = \|X\|_a .$$

It can be shown that the geodesic $\gamma_{a,b}$ is shortest among all curves $\gamma : [0, 1] \rightarrow \mathcal{A}^+$ such that $\gamma(0) = a$ and $\gamma(1) = b$, where the length of γ is

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt .$$

Thus, if we define the *geodesic or rectifiable distance* d by

$$d(a, b) = \inf\{ \text{length } \gamma : \gamma \text{ is a smooth curve in } \mathcal{A}^+ \text{ joining } a \text{ and } b \} ,$$

then $d(a, b) = \text{length } \gamma_{a,b}$. Since $\|gXg^*\|_{gag^*} = \|X\|_a$ for all $g \in \mathcal{A}^\times$, $a \in \mathcal{A}^+$ and $X \in (TA^+)_a$, it follows that $d(gag^*, gbg^*) = \text{length } \gamma_{gag^*, gbg^*} = \text{length } \gamma_{a,b} = d(a, b)$; thus, \mathcal{A}^\times acts transitively and isometrically on \mathcal{A}^+ , and for computations one can usually transport the situation to $a = 1$. Observe also that the geodesics with origin 1 have the form $\gamma_{1,a}(t) = a^t$.

It is easy to verify that $d(a, b) = \|\log a^{-1/2}ba^{-1/2}\|$ so that, when a commutes with b , $d(a, b) = \|\log ba^{-1}\|$.

Remark 1. We shall prove later that the geodesic distance on \mathcal{A}^+ coincides with the so-called *Thompson or part metric* so that the results of Nussbaum [4] are still valid.

Remark 2. A relevant property of the geodesic distance on \mathcal{A}^+ is that the metric space (\mathcal{A}^+, d) is complete. A proof of this statement can be found in [14], where Nussbaum uses the Thompson metric. This fact, among many others, justifies the introduction of this distance. On the other hand, as observed by Vesentini [18], the

relevance of d is related to the fact that the Haar measure of the locally compact group $\mathbb{C}^+ = \{t \in \mathbb{R} : t > 0\}$ is determined, up to a constant, by

$$m(\alpha, \beta) = |\log \beta - \log \alpha| = \left| \log \frac{\beta}{\alpha} \right|$$

for any interval (α, β) .

The next result shows that the Löwner-Heinz inequality is equivalent to two geometrical properties which characterize Riemannian manifolds with nonpositive curvature. We shall make some comments on this after the proof of the theorem.

Theorem 2. *If \mathcal{A} is a unital C^* -algebra, then the following properties hold and are equivalent:*

- (2.1) $\|a^t b^t\| \leq \|ab\|^t$ for every $a, b \in \mathcal{A}^+$ and $t \in [0, 1]$;
- (2.2) $d(a^t, b^t) \leq td(a, b)$ for every $a, b \in \mathcal{A}^+$ and $t \in [0, 1]$;
- (2.3) for every pair of geodesics of \mathcal{A}^+ the real function

$$t \mapsto d(\gamma(t), \delta(t))$$

is convex.

Proof. Observe that (2.2) is the rewriting of inequality (1.2), using the explicit expression for d . On the other hand, (1.3) is an easy consequence of (2.3) (see [6], where there is also a proof of (2.3) by a completely different method). Thus, it suffices to show that (1.3) implies property (2.3). Observe that (2.3) is equivalent to the inequality

(2.4)

$$\|\log(\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2})\| \leq (1-t) \|\log(a_1^{-1/2} a_2 a_1^{-1/2})\| + t \|\log(b_1^{-1/2} b_2 b_1^{-1/2})\|$$

for every $t \in [0, 1]$, where γ joins a_1 and b_1 and δ joins a_2 and b_2 . The proof that (2.4) follows from (1.3) uses the same trick of Theorem 1. After changing γ by $k\gamma$ for a convenient $k > 0$ we may suppose that

$$\delta(t)^{-1} \gamma(t) \delta(t)^{-1/2} \geq 1$$

for every $t \in [0, 1]$, so that

$$\begin{aligned} \|\log(\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2})\| &= \log \|\delta(t)^{-1/2} \gamma(t) \delta(t)^{-1/2}\| \\ &\leq \log(\|a_1^{-1/2} a_2 a_1^{-1/2}\|^{1-t} \|b_1^{-1/2} b_2 b_1^{-1/2}\|^t) \\ &= (1-t) \log \|a_1^{-1/2} a_2 a_1^{-1/2}\| + t \log \|b_1^{-1/2} b_2 b_1^{-1/2}\| \\ &= (1-t) \|\log(a_1^{-1/2} a_2 a_1^{-1/2})\| + t \|\log(b_1^{-1/2} b_2 b_1^{-1/2})\| \end{aligned}$$

which ends the proof. \square

Remarks. In Gromov's lectures [1] one finds two characterizations of Riemannian manifold with nonpositive curvature. The first one rests on the local diffeomorphism property of the exponential map: if M is a Riemannian manifold and $r \in M$, the exponential map $\exp_r : (TM)_r \rightarrow M$ is a local diffeomorphism so that in the neighborhood of r there is a map defined by

$$\rho_t(m) = \exp_r(t \exp_r^{-1}(m)) \quad (t \in [0, 1]) ;$$

then M has nonpositive curvature if and only if for every $t \in [0, 1]$ and m_1, m_2 close to r

$$d(\rho_t(m_1), \rho_t(m_2)) \leq td(m_1, m_2) .$$

This expression reduces, in the case when M is the (non-Riemannian!) manifold \mathcal{A}^+ , to the expression

$$d(r^{1/2}(r^{-1/2}m_1r^{-1/2})^t r^{1/2}, r^{1/2}(r^{-1/2}m_2r^{-1/2})^t r^{1/2}) \leq td(m_1, m_2)$$

which is exactly (2.2), with $m_1 = a, m_2 = b$, due to the fact that \mathcal{A}^\times acts isometrically over \mathcal{A}^+ . Returning to Gromov’s lectures [1], one can find there a characterization of Riemannian manifolds with nonpositive curvature as those Riemannian manifolds so that the distance $d(\gamma(t), \delta(t))$, for two geodesics γ, δ , is a convex function of t . Thus, Theorem 2 shows that the Löwner-Heinz inequality means exactly that \mathcal{A}^+ has this type of “negatively curved space” behaviour. Recall that this is not the first time that a classical inequality appears as describing a geometrical behaviour of \mathcal{A}^+ : in [3] it has been shown that Segal’s inequality

$$\|e^{X+Y}\| \leq \|e^{X/2}e^Y e^{X/2}\|$$

valid for Hermitian operators X, Y [16] is equivalent to the property that \exp_a increases distances:

$$d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$$

for all $X, Y \in (T\mathcal{A}^+)_a$; in Riemannian geometry this happens, again, in nonpositive curvature manifolds. The reader may find a deduction of Segal’s inequality from Löwner-Heinz’s in [2]; see also [6].

The *part metric* (or *Thompson metric* [17]) of \mathcal{A}^+ can be defined as

$$d_p(a, b) = \max\{\log \inf\{\alpha > 0 : a \leq \alpha b\}, \log \inf\{\beta > 0 : b \leq \beta a\}\} .$$

Nussbaum [14] proves that

$$d_p(a, b) = \max\{\log \|a^{-1/2}ba^{-1/2}\|, \log \|b^{-1/2}ab^{-1/2}\|\} .$$

Let us show that d_p coincides with the distance d obtained by differential geometric methods.

Proposition. *For every $a, b \in \mathcal{A}^+$*

$$d_p(a, b) = \|\log a^{-1/2}ba^{-1/2}\| .$$

Proof. Recall that $\|\log c\| = \max\{\log \|c\|, \log \|c^{-1}\|\}$ for all $c \in \mathcal{A}^+$. Then

$$\begin{aligned} d(a, b) &= \|\log a^{-1/2}ba^{-1/2}\| \\ &= \max\{\log \|a^{-1/2}ba^{-1/2}\|, \log \|a^{1/2}b^{-1}a^{1/2}\|\} \end{aligned}$$

and it suffices to show that $\|a^{1/2}b^{-1}a^{1/2}\| = \|b^{-1/2}ab^{-1/2}\|$. But this is obvious, taking $x = a^{1/2}b^{-1/2}$ and using that $\|x^*x\| = \|xx^*\| = \|x\|^2$. \square

Remarks. 1) Nussbaum [14] proved that $\gamma_{ab}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$ curve which is shortest, for the part metric, among all curves joining a and b . On the other side, Corach, Porta and Recht [4, 5, 3] proved that $\gamma_{a,b}$ is a geodesic (in the sense of the differential geometry) and is a shortest curve joining a and b . The result above justifies the coincidence between both approaches. Thus, the convexity results obtained in the theorem above are valid also in Nussbaum’s context. It

should be noticed that Nussbaum [15] gets by a general method a whole family of shortest curves joining two fixed $a, b \in \mathcal{A}^+$. The geodesic $\gamma_{a,b}$ appears, thus, as a distinguished member of the family.

2) Liverani and Wojtkowski [12] got, by a different method, a similar expression for the distance d . There is also a previous paper by Vesentini [18] where the distance on \mathcal{A}^+ can be obtained as a Carathéodory type metric.

3) Given a metric space X and a map $\varphi : X \rightarrow X$, consider the *dilation index* of φ :

$$\text{dil}(\varphi) = \sup_{x_1 \neq x_2} \frac{d(\varphi(x_1), \varphi(x_2))}{d(x_1, x_2)},$$

which in analysis is usually called the best Lipschitz constant of φ . Gromov [10] uses, as a useful tool for the study of his “*espaces de longueur*”, the notion *short* and *strictly short* maps. The map φ is *short* if $\text{dil}(\varphi) \leq 1$ and is *strictly short* if $\text{dil}(\varphi) < 1$ (classically called contractions).

Inequality (2.2) shows that the map $\varphi_t : a \mapsto a^t$ (for $t \in [0, 1]$) verifies $\text{dil}(\varphi_t) \leq t$. Indeed, it can easily be shown that $\text{dil}(\varphi_t) = t$. Thus, φ_t is strictly short for $0 \leq t < 1$. More generally, using the isometric action of \mathcal{A}^\times over \mathcal{A}^+ , it follows that, for fixed a , the map $b \mapsto \gamma_{a,b}(t)$ has dilation index t .

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