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ON A POLYNOMIAL INEQUALITY OF E. J. REMEZ

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Abstract. We prove a result which extends a well-known polynomial inequality of E. J. Remez and another one due to W. A. Markov.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n with real or complex coefficients. Polynomials in $\mathcal{P}_n$ whose coefficients are all real will form the sub-class $\mathcal{P}_{n,\mathbb{R}}$. As usual we shall denote by T_n the nth Chebyshev polynomial of the first kind, which is given by $\cos n \arccos x$ for $-1 \leq x \leq 1$. In particular, $|T_n(x)| \leq 1$ for $-1 \leq x \leq 1$ and $T_n(\cos((n-j)\pi/n)) = (-1)^{n-j}$ for $j = 0, 1, \dots, n$. All its zeros are real and lie in the open interval $(-1, 1)$. It was observed by P.L. Chebyshev (see [7] or [9]) that if $f \in \mathcal{P}_n$ and $|f(x)| \leq 1$ for $-1 \leq x \leq 1$, then

$$|f(x)| \leq |T_n(x)| \quad \text{for all} \quad x \in \mathbb{R} \setminus [-1, 1]\,. \tag{1}$$

Subsequently, it was shown by W.A. Markov [5] that under the same condition on f, we have

$$\left|f^{(k)}(x)\right| \leq \left|T_n^{(k)}(x)\right| \quad \text{for all} \quad x \in \mathbb{R} \setminus [-1, 1] \quad \text{and} \quad 1 \leq k \leq n\,. \tag{2}$$

Now we must introduce a couple of additional notations. We shall write $\mu(\mathfrak{S})$ for the measure of a Lebesgue measurable subset $\mathfrak{S}$ of $\mathbb{R}$. For *any* polynomial f and any subinterval $\mathbb{I}$ of $\mathbb{R}$ we denote the set $\{x \in \mathbb{I} : |f(x)| \leq 1\}$ by $\mathfrak{E}(f; \mathbb{I})$.

The following generalization of Chebyshev's inequality (1) is due to E.J. Remez (see [1], [2], [3, Lemma 7.3], [8]). The proof in [1] is the simplest.

Theorem A. *For all $g \in \mathcal{P}_n$, the following inequality holds:*

$$\max_{-1 \leq x \leq 1} |g(x)| \leq T_n\left(\frac{4}{\mu(\mathfrak{E}(g\,;\,[-1,1]))} - 1\right). \tag{3}$$

An equivalent formulation of this result stated below as Theorem A′ shows clearly why it contains (1).

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For each $R \geq 1$, let

$$\pi_n(R) := \{f \in \mathcal{P}_n : \mu(\mathfrak{E}(f\,;\,[-1,R])) \geq 2\}\,,$$

$$\pi_{n,\mathbb{R}}(R) := \{f \in \mathcal{P}_{n,\mathbb{R}} : \mu(\mathfrak{E}(f\,;\,[-1,R])) \geq 2\}\,.$$

If $f \in \pi_n(R)$ for some $R \geq 1$ and $g(x) := f((R+1)x/2 + (R-1)/2)$, then

$$\mu(\mathfrak{E}(g;[-1,1])) = \frac{2}{R+1}\mu(\mathfrak{E}(f;[-1,R])) \geq \frac{4}{R+1}\,.$$

Hence Theorem A implies that *if* $f \in \pi_n(R)$, *then*

$$\max_{-1\leq x\leq R} |f(x)| \;=\; \max_{-1\leq x\leq 1} |g(x)| \;\leq\; T_n(R)\,. \tag{4}$$

Conversely, let (4) hold for all $f \in \pi_n(R)$. If g is any polynomial of degree at most n and $f(x) := g((2x - R + 1)/(R + 1))$, then

$$\mu(\mathfrak{E}(f\,;\,[-1,R])) = \frac{R+1}{2}\,\mu(\mathfrak{E}(g\,;\,[-1,1])) \geq 2$$

if $R \geq 4/\mu(\mathfrak{E}(g\,;\,[-1,1])) - 1$. Hence, $f \in \pi_n(R)$ for all such values of R and

$$\max_{-1\leq x\leq 1} |g(x)| \;=\; \max_{-1\leq x\leq R} |f(x)| \;\leq\; T_n(R)\,,$$

that is, (3) holds. Thus, Theorem A may be reformulated as follows.

Theorem A′. *If* $f \in \pi_n(R)$ *for some* $R \geq 1$, *then*

$$\max_{-1\leq x\leq R} |f(x)| \leq T_n(R)\,. \tag{5}$$

It was noted by B.D. Bojanov that if $m(t) := \sup\{|f(t)|\,:\, f \in \pi_n(R)\}$, then

$$m(t) \;\leq\; m(R)$$

for all $t \in (-1, R)$. His argument goes roughly as follows.

Take an arbitrary $t \in (-1, R)$ and any $f \in \pi_n(R)$. If

$$q_1(t) := \frac{\mu(\mathfrak{E}(f\,;\,[-1,t]))}{1+t} \quad \text{and} \quad q_2(t) := \frac{\mu(\mathfrak{E}(f\,;\,[t,R]))}{R-t}\,,$$

then

$$\max\{q_1(t), q_2(t)\} \geq \frac{2}{1+R}\,,$$

since otherwise, we would have

$$\mu(\mathfrak{E}(f\,;\,[-1,R])) = \mu(\mathfrak{E}(f\,;\,[-1,t])) + \mu(\mathfrak{E}(f\,;\,[t,R])) < 2\,.$$

Now consider the linear transformations

$$\alpha_1(x) := \frac{(1+t)x - R + t}{1+R} \quad \text{and} \quad \alpha_2(x) := \frac{(t-R)x + R^2 + t}{1+R}\,.$$

It is to be noted that as x increases from -1 to R, the number $\alpha_1(x)$ increases from -1 to t whereas $\alpha_2(x)$ decreases from R to t. Under the first transformation every subinterval of $[-1, R]$ shrinks by the factor $(1+t)/(1+R)$; under the second, they all shrink by the factor $(R-t)/(1+R)$. This means that if $\mathbb{I}$ is an interval contained either in $[-1\,,\,t]$ or in $[t\,,\,R]$, then $\mu(\{x \in [-1\,,\,R] : \alpha_1(x) \in \mathbb{I}\})$ is equal to $((1+R)/(1+t))\,\mu(\mathbb{I})$ in the first case and $\mu(\{x \in [-1\,,\,R] : \alpha_2(x) \in \mathbb{I}\})$ is equal to $((1+R)/(R-t))\,\mu(\mathbb{I})$ in the second. Hence, choosing $\kappa \in \{1,2\}$ such that $q_\kappa(t) \geq 2/(1+R)$ we obtain

$$\mu\left(\mathfrak{E}\left(f\left(\alpha_\kappa(\cdot)\right)\,;\,[-1,R]\right)\right) \geq 2\,.$$

In other words, $f\left(\alpha_\kappa(\cdot)\right) \in \pi_n(R)$. Since $t = \alpha_\kappa(R)$, we conclude that

$$|f(t)| = \left|f\left(\alpha_\kappa(R)\right)\right| \leq m(R)\,.$$

In view of this fact, Theorem A$'$ may be stated as follows.

Theorem A$''$. *Let $R \geq 1$. If $f \in \pi_n(R)$, then for all $x \geq R$,*

$$|f(x)| \leq T_n(x)\,. \tag{6}$$

We prove

Theorem 1. *Let $R \geq 1$. If $f \in \pi_{n,\mathbb{R}}(R)$, then for all $z \in \mathcal{H}_R := \{z \in \mathbb{C} : Re(z) \geq R\}$ we have*

$$\left|f^{(k)}(z)\right| \leq \left|T_n^{(k)}(z)\right| \qquad (k = 0, 1, \ldots, n)\,. \tag{7}$$

If k belongs to $\{1,\ldots,n\}$, then equality holds in (7) for any $z \in \mathcal{H}_R$ if and only if $f(z) = \pm T_n(z)$. The same can be said when $k = 0$ if $z \in \mathcal{H}_R \setminus \{1\}$.

Theorem 1 is not only an extension of Theorem A$''$ but also of (2).

Note that if f is of degree n, then $\mathfrak{E}(f;(-\infty,\infty))$ consists of at most n disjoint closed though possibly degenerate intervals. To see this consider the polynomial $F(z) := f(z)\,\overline{f(\bar{z})}$. It is non-negative on the real axis and $\mathfrak{E}(F;(-\infty,\infty)) = \mathfrak{E}(f;(-\infty,\infty))$. Suppose that $\mathfrak{E}(f;(-\infty,\infty))$ consists of N disjoint closed intervals $[a_1,b_1],\ldots,[a_N,b_N]$,where $N \geq n+1$. It is geometrically evident that the derivative F' must vanish at least once in each of the $N-1$ open intervals $(b_1,a_2),\ldots,(b_{N-1},a_N)$ and also in each of the intervals $[a_1,b_1],\ldots,[a_N,b_N]$, even in the degenerate ones. Thus F' has at least $2N-1$ ($\geq 2n+1$) zeros, which is a contradiction since F' is of degree $2n-1$. It follows that $\mathfrak{E}\left(f;[-1,\xi]\right)$ consists of at most n disjoint, closed, possibly degenerate intervals for all $\xi \geq -1$.

Now let $x_j := \cos((n-j)\pi/n)$ for $j = 0,1,\ldots,n$ and let f be an arbitrary polynomial in $\pi_n(R)$, where $R \geq 1$. For $j = 0,1,\ldots,n$ let ξ_j be the infimum of all ξ such that $\mu\left(\mathfrak{E}\left(f;[-1,\xi]\right)\right) = 1 + x_j$. The numbers $\xi_0,\xi_1,\ldots,\xi_n$ are well defined and form an increasing sequence such that $\xi_{j+1}-\xi_j \geq x_{j+1}-x_j$ for $j = 0,\ldots,n-1$; in particular, $\xi_j \geq x_j$ for $j = 0,1,\ldots,n$. So, Theorem 1 is contained in the following result. This is what we shall really prove.

Theorem 1*. *Let*

$$x_j := \cos\frac{n-j}{n}\pi \quad \textit{for} \quad j = 0,1,\ldots,n\,.$$

Further, let $\xi_0,\xi_1,\ldots,\xi_n$ be another sequence of $n+1$ numbers in $[-1,\infty)$ such that

$$\xi_{j+1}-\xi_j \geq x_{j+1}-x_j \qquad (j = 0,1,\ldots,n-1), \tag{8}$$

and $\mathcal{H}_R$ as in Theorem 1. If f is a real polynomial of degree at most n such that

$$|f(\xi_j)| \leq 1 \qquad (j = 0,1,\ldots,n), \tag{9}$$

then for all $z \in \mathcal{H}_R$ with $R \geq \xi_n$ we have

$$\left|f^{(k)}(z)\right| \leq \left|T_n^{(k)}(z)\right| \qquad (k = 0,1,\ldots,n), \tag{10}$$

where $f^{(0)}(z) \equiv f(z)$ *. If* k *belongs to* $\{1, \ldots, n\}$*, then equality holds in (10) for any* $z \in \mathcal{H}_R$ *if and only if* $\xi_j = x_j$ *for all* j *and* $f(z) = \pm T_n(z)$ *. The same can be said when* $k = 0$ *if* $z \in \mathcal{H}_R \setminus \{1\}$ *.*

Remark 1. Let f be any polynomial (real or not) of degree at most n satisfying (9). For any $x_0 \geq \xi_n$ and any $k \in \{0, 1, \ldots, n\}$, let $f^{(k)}(x_0) = \left|f^{(k)}(x_0)\right| e^{i\gamma}$. Then $g(x) := Re\left(e^{-i\gamma} f(x)\right)$ is a real polynomial of degree at most n satisfying (9) and so

$$\left|f^{(k)}(x_0)\right| = Re\left(e^{-i\gamma} f^{(k)}(x_0)\right) = \left|g^{(k)}(x_0)\right| \leq T_n^{(k)}(x_0)$$

by (10). In other words, (10) holds for any polynomial f of degree at most n satisfying (9) if $z \in \mathcal{H}_R \cap \mathbb{R}$.

2. Preparatory lemmas

The following auxiliary result is a simple principle of mechanics expressed in terms of complex numbers rather than vectors.

Lemma 1. *Let* $\varphi_1, \ldots, \varphi_n$ *and* $\psi_1, \ldots, \psi_n$ *be non-negative numbers with* $\varphi_k \leq \psi_k$ *for* $k = 1, \ldots, n$ *and* $\sum_{k=1}^{n} \psi_k < \pi/2$*. Besides, let* $\rho_0, \ldots, \rho_n$ *and* $R_0, \ldots, R_n$ *be two other sets of positive numbers such that* $\rho_k \leq R_k$ *for* $k = 0, 1, \ldots, n$*. Then*

$$\left|\rho_0 + \sum_{k=1}^{n} \rho_k \exp\bigl(-i \sum_{j=1}^{k} \psi_j\bigr)\right| \leq \left|R_0 + \sum_{k=1}^{n} R_k \exp\bigl(-i \sum_{j=1}^{k} \varphi_j\bigr)\right|, \tag{11}$$

where equality holds only if $\rho_k = R_k$ *for* $0 \leq k \leq n$ *and* $\psi_k = \varphi_k$ *for* $1 \leq k \leq n$ *.*

Proof. Clearly, $|\rho_0 + \sum_{k=1}^{n} \rho_k \exp\left(-i \sum_{j=1}^{k} \psi_j\right)|^2$ is equal to

$$\sum_{k=0}^{n} \rho_k^2 + 2\rho_0 \sum_{l=1}^{n} \rho_l \cos\bigl(\sum_{j=1}^{l} \psi_j\bigr) + \sum_{k=1}^{n-1} 2\rho_k \cos\bigl(\sum_{j=1}^{k} \psi_j\bigr)\bigl(\sum_{l=k+1}^{n} \rho_l \cos\bigl(\sum_{j=1}^{l} \psi_j\bigr)\bigr)$$
$$+ \sum_{k=1}^{n-1} 2\rho_k \sin\bigl(\sum_{j=1}^{k} \psi_j\bigr)\bigl(\sum_{l=k+1}^{n} \rho_l \sin\bigl(\sum_{j=1}^{l} \psi_j\bigr)\bigr),$$

which is in turn equal to

$$\sum_{k=0}^{n} \rho_k^2 + 2\rho_0 \sum_{l=1}^{n} \rho_l \cos\bigl(\sum_{j=1}^{l} \psi_j\bigr) + \sum_{k=1}^{n-1} 2\rho_k \sum_{l=k+1}^{n} \rho_l$$
$$\times \left\{\cos\bigl(\sum_{j=1}^{l} \psi_j\bigr) \cos\bigl(\sum_{j=1}^{k} \psi_j\bigr) + \sin\bigl(\sum_{j=1}^{l} \psi_j\bigr) \sin\bigl(\sum_{j=1}^{k} \psi_j\bigr)\right\}.$$

Thus, $|\rho_0 + \sum_{k=1}^{n} \rho_k \exp\left(-i \sum_{j=1}^{k} \psi_j\right)|^2$ can be written in the form

$$\sum_{k=0}^{n} \rho_k^2 + \sum_{k=0}^{n-1} 2\rho_k \sum_{l=k+1}^{n} \rho_l \cos\bigl(\sum_{j=k+1}^{l} \psi_j\bigr),$$

which, obviously, increases as any of the numbers $\rho_0, \rho_1, \ldots, \rho_n$ increases or as any of the numbers $\psi_1, \ldots, \psi_n$ decreases. Hence (11) holds, wherein equality holds only if $\rho_k = R_k$ for $0 \leq k \leq n$ and $\psi_k = \varphi_k$ for $1 \leq k \leq n$. □

Lemma 2. *Let x_j, ξ_j be as in Theorem 1^* and $z = x + iy$ where $x \geq \xi_n$, $y \geq 0$. Denote by A_j, B_j and P the points of the complex plane which correspond to x_j, ξ_j and z, respectively. If φ_j, ψ_j stand for the angles $\widehat{A_{j-1}PA_j}$, $\widehat{B_{j-1}PB_j}$, respectively, then*

$$\psi_j \;\geq\; \varphi_j \quad \text{for} \quad j = 1, \ldots, n, \tag{12}$$

where, in the case $y > 0$, equality holds for some j if and only if $\xi_j = x_j$ for all j.

Proof. There is nothing to prove when $y = 0$ since in that case ψ_j and φ_j are all zero. So we assume $y > 0$. For $j = 1, \ldots, n$ let δ_j, Δ_j denote the areas of the triangles $A_{j-1}PA_j, B_{j-1}PB_j$, respectively; then

$$\delta_j \;=\; \frac{1}{2}(x_j - x_{j-1})\,y, \quad \Delta_j \;=\; \frac{1}{2}(\xi_j - \xi_{j-1})\,y\,.$$

By assumption, $\xi_j - \xi_{j-1} \geq x_j - x_{j-1}$ and so

$$\Delta_j \geq \delta_j \quad \text{for} \quad 1 \leq j \leq n\,. \tag{13}$$

Using another well-known formula for the area of a triangle we write

$$\delta_j \;=\; \frac{1}{2}|z - x_j||z - x_{j-1}|\sin\varphi_j\;, \quad \Delta_j \;=\; \frac{1}{2}|z - \xi_j||z - \xi_{j-1}|\sin\psi_j\;,$$

from which, for $j = 1, \ldots, n$, we obtain

$$\sin\varphi_j \;=\; \frac{2\delta_j}{|z - x_j||z - x_{j-1}|}\;, \quad \sin\psi_j \;=\; \frac{2\Delta_j}{|z - \xi_j||z - \xi_{j-1}|}\,. \tag{14}$$

It is geometrically evident that $|z - \xi_j| \leq |z - x_j|$ for $j = 0, 1, \ldots, n$. Hence, (14) combined with (13) implies that

$$\sin\varphi_j \;\leq\; \sin\psi_j \qquad (\,1 \leq j \leq n\,)\,.$$

This is equivalent to the desired result since $0 < \varphi_j, \psi_j < \pi/2$. □

Lemma 3. *Let x_j, z and φ_j be as in Lemma 2. If*

$$G(z) \;:=\; \prod_{j=0}^{n}(z - x_j)\,, \quad G_k(z) \;:=\; \frac{G(z)}{z - x_k} \quad \text{for} \quad k = 0, 1, \ldots, n\,,$$

then with $\varphi_0 = 0$, we have

$$|T_n(z)| \;=\; \left|\sum_{k=0}^{n} \frac{1}{|G'(x_k)|}|G_k(z)|\,exp\big(-i\sum_{j=0}^{k}\varphi_j\big)\right|\,. \tag{15}$$

Proof. Note that

$$G'(x_k) \;=\; \prod_{j=0,\, j\neq k}^{n}(x_k - x_j) \;=\; (-1)^{n-k} \prod_{j=0,\, j\neq k}^{n}|x_k - x_j| \;=\; (-1)^{n-k}|G'(x_k)|$$

and

$$T_n(x_k) \;=\; (-1)^{n-k}\,.$$

Hence, by Lagrange interpolation in the points $x_0, x_1, \ldots, x_n$ we obtain

$$|T_n(z)| = \left|\sum_{k=0}^{n} T_n(x_k)\frac{1}{G'(x_k)}G_k(z)\right| = \left|\sum_{k=0}^{n}\frac{1}{|G'(x_k)|}G_k(z)\right|\,. \tag{16}$$

If $\alpha := Arg\, G_0(z)$ and $k \in \{0, 1, \ldots, n\}$, then

$$G_k(z) = G_0(z)\frac{z - x_0}{z - x_k} \tag{17}$$
$$= e^{i\alpha}|G_0(z)| \left|\frac{z - x_0}{z - x_k}\right| \, exp\left(-i\sum_{j=0}^{k}\varphi_j\right)$$
$$= e^{i\alpha}|G_k(z)| \, exp\left(-i\sum_{j=0}^{k}\varphi_j\right) .$$

Substituting this expression for $G_k(z)$ in (16) we obtain (15). □

Remark 2. Let ξ_j, z and ψ_j be as in Lemma 2. Further, let

$$H(z) := \prod_{j=0}^{n}(z - \xi_j) \,, \; H_k(z) := \frac{H(z)}{z - \xi_k} \;\text{ for }\; k = 0, 1, \ldots, n \,.$$

Arguing as for (17) we can show that if $\beta := Arg\, H_0(z)$, then with $\psi_0 = 0$, we have

$$H_k(z) = e^{i\beta}|H_k(z)| \, exp\left(-i\sum_{j=0}^{k}\psi_j\right) \;\text{ for }\; k = 0, 1, \ldots, n \,. \tag{18}$$

Lemma 4. *For $k = 0, 1, \ldots, n$ let $w_k = u_k + iv_k$, where $u_k > 0$, $v_k \leq 0$. If $-1 \leq t_k \leq 1$ for $k = 0, 1, \ldots, n$, then*

$$|\sum_{k=0}^{n} t_k w_k| \leq |\sum_{k=0}^{n} w_k| \,, \tag{19}$$

where equality holds if and only if the numbers t_k are all of the same sign and of modulus 1.

Proof. Since the numbers u_k are all of the same sign and so are the numbers v_k, we clearly have

$$|\sum_{k=0}^{n} t_k w_k|^2 = \left(\sum_{k=0}^{n} t_k u_k\right)^2 + \left(\sum_{k=0}^{n} t_k v_k\right)^2$$
$$\leq \left(\sum_{k=0}^{n} u_k\right)^2 + \left(\sum_{k=0}^{n} v_k\right)^2$$
$$= \left|\sum_{k=0}^{n} u_k + i\sum_{k=0}^{n} v_k\right|^2$$
$$= \left|\sum_{k=0}^{n} w_k\right|^2 .$$

□

3. Proof of Theorem 1*

First let $k=0$. For reasons of symmetry it is enough to prove that

$$|f(z)| \le |T_n(z)| \quad \text{for} \quad z=x+iy,\ x\ge \xi_n,\ y\ge 0\,. \tag{20}$$

By Lagrange interpolation in the points $\xi_0,\xi_1,\dots,\xi_n$ we have

$$f(z) = \sum_{k=0}^{n} f(\xi_k)\frac{1}{H'(\xi_k)}H_k(z)\,.$$

Noting that

$$H'(\xi_k) = \prod_{j=0,\, j\ne k}^{n} (\xi_k-\xi_j) = (-1)^{n-k} \prod_{j=0,\, j\ne k}^{n} |\xi_k-\xi_j| = (-1)^{n-k}\,|H'(\xi_k)|$$

and taking (18) into account we obtain

$$|f(z)| = \left| e^{i\beta}\sum_{k=0}^{n}(-1)^{n-k}f(\xi_k)\frac{|H_k(z)|}{|H'(\xi_k)|}exp\big(-i\sum_{j=0}^{k}\psi_j\big)\right|\,.$$

So by Lemma 4,

$$|f(z)| \le \left|\sum_{k=0}^{n}\frac{|H_k(z)|}{|H'(\xi_k)|}exp\big(-i\sum_{j=0}^{k}\psi_j\big)\right|\,.$$

From (8) it follows that $|\xi_k-\xi_j|\ge|x_k-x_j|$ and so

$$|H'(\xi_k)| = \prod_{j=0,\, j\ne k}^{n}|\xi_k-\xi_j| \ge \prod_{j=0,\, j\ne k}^{n}|x_k-x_j| = |G'(x_k)|\,.$$

Besides, it is geometrically evident that $|H_k(z)|\le|G_k(z)|$. Since $\psi_j\ge\varphi_j$ by Lemma 2, we may apply Lemma 1 to conclude that for $z=x+iy$ with $x\ge\xi_n$, $y>0$ we have

$$|f(z)| \le \left|\sum_{k=0}^{n}\frac{|G_k(z)|}{|G'(x_k)|}exp\big(-i\sum_{j=0}^{k}\varphi_j\big)\right| = |T_n(z)|$$

by Lemma 3. Since, in (11) equality holds if and only if $\rho_k=R_k$, $\psi_k=\varphi_k$ for $k=0,\dots,n$, it is easily seen from the above proof that $|f(z)|<|T_n(z)|$ for all $z\in\mathcal{H}_R$ with $R\ge\xi_n$ unless $f(z)\equiv\pm T_n(z)$.

Now let $1\le k\le n$. Assume that $\xi_j\ne x_j$ for some j. Then $\xi_n\ne x_n$ and $f(z)$ is not identically equal to $\pm T_n(z)$. Consequently, $|f(z)|<|\lambda T_n(z)|$ for all $z\in\mathcal{H}_R$ and all $\lambda\in\mathbb{C}$ such that $|\lambda|\ge 1$. This means that the polynomial $p(z):=f(z)-\lambda T_n(z)\ne 0$ in $\mathcal{H}_R$. Hence, there exists a positive number δ such that all the zeros of p lie in the half-plane $Re(z)\le\xi_n-\delta$. By the Gauss-Lucas theorem [4, p. 84] *all* the zeros of $p^{(k)}$, if any, also lie in the same half-plane. It follows that $f^{(k)}(z)-\lambda T_n^{(k)}(z)\ne 0$ in $\mathcal{H}_R$ for all $\lambda\in\mathbb{C}$ such that $|\lambda|\ge 1$. This is possible only if $|f^{(k)}(z)| < \left|T_n^{(k)}(z)\right|$ for all $z\in\mathcal{H}_R$. Indeed, if we had $|f^{(k)}(z_0)|\ge\left|T_n^{(k)}(z_0)\right|$ for some $z_0\in\mathcal{H}_R$, then with $\lambda_0=f^{(k)}(z_0)/T_n^{(k)}(z_0)$, which is of modulus ≥ 1, we would have

$$f^{(k)}(z_0) - \lambda_0 T_n^{(k)}(z_0) = 0$$

contradicting the fact that $f^{(k)}(z) - \lambda T_n^{(k)}(z) \neq 0$ in $\mathcal{H}_R$ if $|\lambda| \geq 1$.

Remark 3. Let x_j, ξ_j and $\mathcal{H}_R$ be as above. The argument used to prove Theorem 1^* shows that if f is *any* polynomial of degree at most n, with real or complex coefficients, such that $|f(\xi_j)| \leq 1$ for $j = 0, 1, \ldots, n$ and $z_0 \in \mathcal{H}_R$, then

$$|f(z_0)| \leq |T_{n,z_0}(z_0)| \,,$$

where T_{n,z_0} is the unique polynomial of degree n satisfying the interpolation condition

$$T_{n,z_0}(x_j) = (-1)^{n-j} \exp\left(i \arg(z_0 - x_j)\right) \qquad (j = 0, 1, \ldots, n)\,.$$

Note that the extremal polynomial $T_{n,z_0}(z)$ may change with z_0. But clearly, $T_{n,z_0}(z) \equiv T_n(z)$, when $z_0 \in \mathcal{H}_R \cap \mathbb{R}$.

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