

## CONORMAL VARIETIES AND CHARACTERISTIC VARIETIES

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ABSTRACT. We show that the conormal variety of a quasihomogeneous hypersurface in  $\mathbb{C}^n$ , for  $n \geq 4$ , whose link is a  $\mathbb{Q}$ -homology sphere is not the characteristic variety of any  $\mathcal{D}$ -module.

### 1. INTRODUCTION

Let  $X$  be a complex analytic manifold and  $\mathcal{D}_X$  its sheaf of differential operators. The most important geometric invariant of a  $\mathcal{D}_X$ -module is its characteristic variety. This is a conical subvariety of the cotangent bundle  $T^*X$ . It is well-known that the cotangent bundle is a symplectic manifold and that the characteristic variety of a  $\mathcal{D}_X$ -module is involutive with respect to the symplectic structure. Thus, there are two conditions that a subvariety of  $T^*X$  must satisfy in order to be a characteristic variety: it must be conical and involutive.

Conormal varieties are involutive and conical. In fact they are exactly the lagrangian conical subvarieties of  $T^*X$  (see, for example, [Ge, Proposition 3.1, p. 29]). Furthermore, an irreducible lagrangian variety is always a component of the characteristic variety of some  $\mathcal{D}_X$ -module (see [CL]).

As pointed out by Malgrange in [Ma2, p. 9], no necessary and sufficient conditions are known for a variety to be a characteristic variety. However, an unpublished result of Kashiwara states that the conormal variety of a quadratic cone in  $\mathbb{C}^n$ , for  $n \geq 5$  odd, is not a characteristic variety (see [CL] for a proof). In this paper we extend Kashiwara's result with the following theorem.

**1.1 Theorem.** *Let  $n \geq 4$  and let  $f \in \mathbb{C}[x_1, \dots, x_n]$  be a quasihomogeneous polynomial. If the hypersurface  $\mathcal{Z}(f)$  has only an isolated singularity at the origin and its link is a  $\mathbb{Q}$ -homology sphere, then its conormal variety is not the characteristic variety of any  $\mathcal{D}_{\mathbb{C}^n}$ -module.*

A simple specific example of a polynomial that satisfies these hypotheses is  $f = x_1^{a_1} + \dots + x_n^{a_n}$  with  $n \geq 4$ , and  $a_1, \dots, a_n$  pairwise coprime positive integers. See §4 for more details.

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2. PRELIMINARIES

Let  $f \in \mathbb{C}[x_1, \dots, x_n]$  be a polynomial. We say that  $f$  is *non-degenerate* if  $\mathcal{Z}(f)$  is singular but has only an isolated singularity at the origin. We call  $f$  *quasihomogeneous* if there exist positive integers  $w_1, \dots, w_n$  such that  $f$  is homogeneous when we set  $\deg x_i = w_i$ . Equivalently  $rf = \sum_{i=1}^n w_i x_i \partial f / \partial x_i$ , for some positive integer  $r$ , the weighted degree of  $f$ .

*Notation.* Throughout the rest of the paper let  $n \geq 4$  and let  $f \in \mathbb{C}[x_1, \dots, x_n]$  be a non-degenerate quasihomogeneous polynomial of weighted degree  $r$  with respect to the weights  $w_1, \dots, w_n$ . We will denote the corresponding Euler vector field by  $E = \sum_{i=1}^n w_i x_i \partial / \partial x_i$ .

Let  $V$  be the space  $\mathbb{C}^n$  with its analytic topology and  $C$  the hypersurface  $\mathcal{Z}(f)$ . Note that  $C' = C \setminus \{0\}$  is a smooth divisor on  $V' = V \setminus \{0\}$ . The *conormal variety* of  $C$ , denoted  $T_C^*V$ , is the closure in  $T^*V$  of the conormal bundle of  $C'$  in  $T^*V'$ . It is a well-known fact that  $T_C^*V$  is a lagrangian conical variety.

Note that since  $f$  is quasihomogeneous and non-degenerate, it is easy to calculate the fibres of  $T_C^*V$  everywhere, except at the origin. However, since  $T_C^*V$  is irreducible and lagrangian, it follows that it cannot contain the whole fibre at the origin, denoted by  $T_0^*V$ . This will eventually lead to the contradiction that will allow us to prove the theorem of §1.

Consider  $\mathcal{O}_V[f^{-1}]$ ; it is naturally a coherent  $\mathcal{D}_V$ -module. We will show that the characteristic variety of this module contains the fibre  $T_0^*V$ . To do this we will make use of a description of the characteristic variety due to Kashiwara. A more general result is proved in [LM] but we give a simpler approach to the special case we need.

**2.1 Proposition.** *The characteristic variety of  $\mathcal{O}_V[f^{-1}]$  is*

$$\{(v, \theta) \in T^*V : \theta \wedge df(v) = 0 \text{ and } f(v)\theta = 0\}.$$

*In particular, it contains  $T_0^*V$ , the fibre at the origin of  $T^*V$ .*

*Proof.* Define

$$W' = \{(v, \theta) \in T^*V : \theta = s(df/f)(v), \quad f(v) \neq 0 \text{ and } s \in \mathbb{C}^\times\},$$

and let  $W$  be the (Zariski) closure of  $W'$  in  $T^*V$ .

It is proved in [SKKO, Appendix] after [K, Proposition 6.2] (see also [Gi, Theorem 3.3], [Gy, §2.4]) that the characteristic variety of  $\mathcal{O}_V[f^{-1}]$  is equal to

$$\{(v, \theta) \in W : f(v)\theta = 0\}.$$

Let  $y_1, \dots, y_n$  be a basis of  $V$  and  $x_1, \dots, x_n$  the dual basis of  $V^*$ . Write  $S = S(V^*) = \mathbb{C}[x_1, \dots, x_n]$  and write  $J$  for the Jacobian ideal  $\sum_i \partial f / \partial x_i S$ . Let

$$\widetilde{W} = \{(v, \theta) \in T^*V : \theta \wedge df(v) = 0\}.$$

Now  $\widetilde{W} = \mathcal{Z}(I)$ , where  $I$  is the ideal of  $S(V^* \times V) = S[y_1, \dots, y_n]$  generated by

$$y_j \partial f / \partial x_i - y_i \partial f / \partial x_j.$$

Since  $\partial f / \partial x_1, \dots, \partial f / \partial x_n$  is a regular sequence, the map  $S[y_1, \dots, y_n] / I \rightarrow S_S(J)$  defined by  $y_i + I \mapsto \partial f / \partial x_i$  is an isomorphism. But the symmetric algebra is a domain by [Ei, Ex. 17.14]. Hence  $\widetilde{W}$  is a closed irreducible subset of  $T^*V$ .

Note that  $(v, \theta) \in \widetilde{W}$  if and only if  $df(v)$  and  $\theta$  are linearly dependent. As  $f$  is quasihomogeneous and non-degenerate, we have  $df(v) = 0$  if and only if  $v = 0$ . Thus,

$$\widetilde{W} = \{(v, \theta) \in T^*V : \theta = sdf(v), \text{ for some } s \in \mathbb{C}, \text{ or } v = 0\}.$$

It follows that

$$W' = \widetilde{W} \cap \{(v, \theta) : f(v)\theta \neq 0\}.$$

Therefore the Zariski closure of  $W'$  must be contained in  $\widetilde{W}$ ; since the latter is irreducible,  $W = \widetilde{W}$ . It is now clear that  $\text{Ch}(\mathcal{O}_V[f^{-1}])$  is as in the statement and contains  $T_0^*V$ .  $\square$

The module that plays the crucial rôle in the proof of the theorem is not  $\mathcal{O}_V[f^{-1}]$  itself but

$$(2.2) \quad \mathcal{H} = \mathcal{O}_V[f^{-1}]/\mathcal{O}_V.$$

This  $\mathcal{D}_V$ -module is isomorphic to  $\mathcal{H}_{[C]}^1(\mathcal{O}_V)$ , the first local cohomology sheaf with support in  $C$ . It is a regular holonomic  $\mathcal{D}_V$ -module, by [BK, Proposition 1.3(7)]. We next show that  $\text{Ch}(\mathcal{H})$  contains  $T_0^*V$ .

**2.3 Proposition.**  $\text{Ch}(\mathcal{H}) = T_C^*V \cup T_0^*V$ .

*Proof.* Let  $U = T^*V \setminus T_0^*V$ . Note that  $U \cap \text{Ch}(\mathcal{H}) = \text{Ch}(\mathcal{H}|_{V'}) = T_C^*V'$ . It follows at once that  $\text{Ch}(\mathcal{H}) \supseteq T_C^*V$ . Further, if there exists another irreducible component of  $\text{Ch}(\mathcal{H})$ , then it contains  $T_0^*V$ .

Recall that  $\mathcal{H} \cong \mathcal{H}_{[C]}^1(\mathcal{O}_V)$ . Now, (2.2) implies that

$$\text{Ch}(\mathcal{O}_V[f^{-1}]) = \text{Ch}(\mathcal{O}_V) \cup \text{Ch}(\mathcal{H}).$$

By Proposition 2.1, the fibre  $T_0^*V$  is contained in the left-hand side, but not in the characteristic variety of  $\mathcal{O}_V$ , which is the zero section of  $T^*V$ . Hence  $\text{Ch}(\mathcal{H})$  contains  $T_0^*V$ , and the proposition is proved.  $\square$

Recall that the *link* of  $\mathcal{Z}(f)$  is the intersection of  $C$  with a small sphere centred at the origin. Recall also the definition of the *Bernstein polynomial* [Co, p. 94]. This is the polynomial  $b(s)$  in an indeterminate  $s$  which has least degree amongst polynomials for which there exists a differential operator  $p$  with the property that  $pf^{s+1} = b(s)f^s$ . From the definition, it is clear that  $b(s) = (s+1)\tilde{b}(s)$ .

**2.4 Proposition.** *The following are equivalent:*

- (1)  $\mathcal{H}$  is simple;
- (2) the link of  $\mathcal{Z}(f)$  is a  $\mathbb{Q}$ -homology sphere;
- (3)  $\tilde{b}(s)$  has no integer roots.

*Proof.* Let  $i : C' \rightarrow V'$  be the inclusion. Then  $\mathcal{H}|_{V'} \cong \mathcal{H}_{[C']}^1(\mathcal{O}_{V'})$  is simple since, by Kashiwara's equivalence (see [Bo, Theorem VI.7.13(ii), Theorem VI.7.4(i)(ii)(iii)]), one has that  $i_+(\mathcal{O}_{C'}) \cong \mathcal{H}_{[C']}^1(\mathcal{O}_{V'})$ . It follows from this (see [BK, Proposition 8.5]) that  $\mathcal{H}$  has a simple socle  $\mathcal{L}$  and that  $\mathcal{H}/\mathcal{L} \cong \mathcal{S}^t$  is a direct sum of  $t$  copies of the simple module  $\mathcal{S}$  supported only at the origin. Now consider the Riemann-Hilbert

solution functor  $S = R\mathcal{H}om_{\mathcal{D}_V}(\_, \mathcal{O}_V)$  which gives a (contravariant) equivalence between the category of regular holonomic  $\mathcal{D}_V$ -modules and the category of perverse sheaves on  $V$  [Bo, Theorem VIII.14.4]. One has  $S(\mathcal{H}) = \mathbb{C}_C[-1]$ , where  $\mathbb{C}_C$  is extended by zero to a sheaf on  $V$ , and likewise  $S(\mathcal{S}) = \mathbb{C}_{\{0\}}[-n]$ . Further, by [BK, Theorem 8.6], one has  $S(\mathcal{L}) = \mathcal{IC}_C[-1]$ , the (shifted) intersection cohomology complex [GM]. Thus, one has a short exact sequence:

$$0 \rightarrow \mathbb{C}_{\{0\}}[-n]^t \rightarrow \mathbb{C}_C[-1] \rightarrow \mathcal{IC}_C[-1] \rightarrow 0.$$

Since  $\mathcal{K} = \mathbb{C}_{\{0\}}[-n]^t$  is concentrated in degree  $n$ , and  $\mathbb{C}_C[-1]$  is concentrated in degree 1, we see that in the bounded derived category  $\mathcal{K} \cong \mathcal{H}^{n-2}(\mathcal{IC}_C)$ . On the other hand, using the Whitney stratification  $C = C' \cup \{0\}$  and Deligne's approach to  $\mathcal{IC}$  [GM, §3], we get

$$\mathcal{IC}_C \cong \tau_{\leq n-2} Rj_* \mathbb{C}_{C'},$$

where  $j : V' \rightarrow V$  is the inclusion and  $\tau$  is truncation. It follows that  $\mathcal{K}$  is the skyscraper sheaf at the origin with stalk

$$\lim_{\epsilon \rightarrow 0} H^{n-2}(C' \cap B_\epsilon, \mathbb{C}).$$

This limit is  $H^{n-2}(K, \mathbb{C})$ , where  $K$  is the link of  $C$ .

It remains to remark that  $K$  is a  $\mathbb{Q}$ -homology sphere if and only if  $H^{n-2}(K, \mathbb{C}) = 0$ . But  $K$  is a compact oriented  $(2n-3)$ -manifold which is  $(n-3)$ -connected, by [Mi, Theorem 5.2], and so by the Hurewicz theorem and Poincaré duality we have

$$H_0(K, \mathbb{C}) \cong H_{2n-3}(K, \mathbb{C})^* \cong \mathbb{C} \quad \text{and} \quad H_{n-2}(K, \mathbb{C}) \cong H_{n-1}(K, \mathbb{C})^*,$$

with all other homology groups equal to zero. The equivalence of (1) and (2) follows.

Finally, let  $F$  denote the Milnor fibre,  $F = \mathcal{Z}(f-1)$ . The monodromy  $F \rightarrow F$ , given by  $v \mapsto e^{2\pi i/r} v$ , induces an operator on  $H_{n-1}(F, \mathbb{C})$ . Further, by [Di, Proposition 3.4.7],  $K$  is a  $\mathbb{Q}$ -homology sphere if and only if 1 is not an eigenvalue for the monodromy operator. However, by [Ma, Théorème 5.4], 1 is an eigenvalue of the monodromy operator if and only if  $\bar{b}$  has an integer root.  $\square$

### 3. PROOF OF THEOREM 1.1

Suppose that there exists a  $\mathcal{D}_V$ -module  $\mathcal{M}$  such that  $\text{Ch}(\mathcal{M}) = T_C^*V$  and let us aim at a contradiction. Since  $T_C^*V$  is irreducible, we may as well assume that  $\mathcal{M}$  is simple. Note that  $C'$  is homotopy equivalent to the link of  $C$  and so  $C'$  is simply connected.

It follows from Kashiwara's equivalence ([Bo, Theorem VI.7.11]) that  $\mathcal{M}|_{V'} = i_+(\mathcal{N})$ , for some simple  $\mathcal{D}_{C'}$ -module  $\mathcal{N}$  with characteristic variety  $T_{C'}^*C'$ . Since  $C'$  is simply connected,  $\mathcal{N} \cong \mathcal{O}_{C'}$ . Thus,  $\mathcal{M}|_{V'} \cong \mathcal{H}_{[C',1]}^1(\mathcal{O}_{V'})$ .

Now, looking at the characteristic variety, the support of  $\mathcal{M}$  is not concentrated at the origin. Thus, if  $j : V' \rightarrow V$  is the canonical embedding, there is a monomorphism  $k : \mathcal{M} \rightarrow \mathcal{G} := j_*(\mathcal{H}|_{V'})$ . Of course,  $k(\mathcal{M})|_{V'} = \mathcal{G}|_{V'}$ . Likewise,  $\mathcal{H}$  is a submodule of  $\mathcal{G}$  with  $\mathcal{H}|_{V'} = \mathcal{G}|_{V'}$ . Since the quotient  $\mathcal{G}/k(\mathcal{M})$  can only possibly be supported at the origin, we cannot have  $\mathcal{H} \cap k(\mathcal{M}) = 0$  and so  $\mathcal{H} = k(\mathcal{M}) \cong \mathcal{M}$ .

Of course, as  $T_C^*V$  is irreducible,  $\text{Ch}(\mathcal{M})$  doesn't contain the zero fibre  $T_0^*V$ . On the other hand, by Proposition 2.3, we have that  $\text{Ch}(\mathcal{H}) \supseteq T_0^*V$ . This contradiction completes the proof of the theorem.

## 4. EXAMPLES

It is now easy to give many examples: let  $f = x_1^{a_1} + \cdots + x_n^{a_n}$  with  $n \geq 4$ , and each  $a_i \geq 2$ . Consider the graph  $G$  with vertices  $1, \dots, n$  and an edge linking  $i$  and  $j$  if and only if  $hcf(a_i, a_j) > 1$ . Let  $G_{ev}$  denote the component of  $G$  containing all the  $i$  with  $a_i$  even. Suppose that either (a)  $G$  contains an isolated point, or (b)  $G_{ev}$  contains an odd number of vertices and  $hcf(a_i, a_j) = 2$ , for all distinct  $i, j$  in  $G_{ev}$ . After Brieskorn [Br],  $f$  satisfies the hypotheses of the theorem. Note that the example in the introduction is a special case of this one.

We complete the paper by showing that some conormal varieties of quasihomogeneous non-degenerate hypersurfaces are characteristic varieties.

**Example.** Let  $f = \sum_{i=1}^n x_i^2$  and suppose that  $n \geq 4$ . Then  $T_C^*V$  is a characteristic variety if and only if  $n$  is even.

*Proof.* The case when  $n$  is odd is covered by Theorem 1.1 and the previous example. So suppose that  $n$  is even and let  $p = \sum_{i=1}^n (\partial/\partial x_i)^2$ . Then,

$$pf^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right)f^s.$$

It follows that if  $s = -\frac{n}{2}$ , then  $p$  annihilates  $f^{s+1}$ . Thus, the symbol of  $p$  vanishes on  $\text{Ch}(\mathcal{G})$  where  $\mathcal{G}$  is the submodule  $(\mathcal{D}f^{s+1} + \mathcal{O})/\mathcal{O}$  of  $\mathcal{H}$ . Now,  $\text{Ch}(\mathcal{G}) \subseteq \text{Ch}(\mathcal{H})$  and hence either  $\text{Ch}(\mathcal{G}) = T_C^*V$  or  $\text{Ch}(\mathcal{G}) = T_C^*V \cup T_0^*V$ . Since the symbol of  $p$  doesn't vanish on  $T_0^*V$  we see that the second possibility does not occur.  $\square$

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