

WEAK ERGODICITY OF STATIONARY PAIRWISE INDEPENDENT PROCESSES

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ABSTRACT. It is proven that a stationary process of pairwise independent random variables with values in a separable metric space is weakly ergodic, i.e. each random variable is independent of the system of invariant sets of the process. An example shows that a process of identically distributed pairwise independent random variables is in general, however, not weakly ergodic.

1. INTRODUCTION

Two famous results of probability theory for pairwise independent random variables, namely Etemadi's strong laws of large numbers (see [6], [7]) and the Borel-Cantelli Lemma (see [3], Th. 4.2.5), led to an increasing interest in the study of pairwise independent random variables. Unfortunately, however, it turned out that pairwise independence performs poorly as a substitute for independence: It was shown that neither the central limit theorem nor the law of the iterated logarithm nor Kolmogoroff's zero-one law hold for pairwise independent instead of independent random variables (see [1], [5], [8], [9]).

Cuesta and Matrán constructed a stationary process of pairwise independent random variables which simultaneously violates the just cited three famous laws of probability theory (see [5]). Their nice and tricky example also shows that for a stationary process of pairwise independent random variables the zero-one law does not hold for the invariant sets; i.e. such a process is not ergodic in general. In Theorem 2.1 we prove that such a process is, however, weakly ergodic; a result which is not true any more for identically distributed pairwise independent random variables (see Example 2.2).

2. THE RESULTS

Let (Ω, \mathcal{A}, P) be a probability space and (M, δ) a separable metric space with Borel σ -field \mathcal{B} . Let $X_n : \Omega \rightarrow M, n \in \mathbb{N}$, be random variables, i.e. \mathcal{A}, \mathcal{B} -measurable functions. $X_n, n \in \mathbb{N}$, is called a stationary process if the distribution of (X_k, X_{k+1}, \dots) is the same for all $k \in \mathbb{N}$. We denote by $\mathcal{I}(X_n, n \in \mathbb{N})$ the σ -field of all invariant sets of the process $X_n, n \in \mathbb{N}$; i.e. $I \in \mathcal{I}(X_n, n \in \mathbb{N})$ iff $I = (X_n)_{n \in \mathbb{N}}^{-1}(B)$ where $B \subset M^{\mathbb{N}}$ is an invariant Borel set, that means $B \in \mathcal{B}^{\mathbb{N}}$ and $(x_1, x_2, \dots) \in B \iff$

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$(x_2, x_3, \dots) \in \mathcal{B}$. The process $X_n, n \in \mathbb{N}$, is called ergodic, if $P(I) \in \{0, 1\}$ for all $I \in \mathcal{I}(X_n, n \in \mathbb{N})$ (see e.g. [2], pp. 118). $X_n, n \in \mathbb{N}$, is called weakly ergodic, if each X_k is independent of $\mathcal{I}(X_n, n \in \mathbb{N})$. Observe that each ergodic process is weakly ergodic, but not conversely. Ergodic processes have been widely investigated in the literature, and we hope that the new concept of weak ergodic processes will be similarly useful.

2.1 Theorem. *Let $X_n, n \in \mathbb{N}$, be a stationary process of pairwise independent random variables with values in a separable metric space M . Then $X_n, n \in \mathbb{N}$, is weakly ergodic.*

Proof. Since a separable metric space can be considered as a subspace of an uncountable, complete and separable metric space, we can, without loss of generality, assume that M is already an uncountable, complete and separable metric space. Hence there exists a Borel-isomorphism $\psi : M \rightarrow \mathbb{R}$ (see Theorem 8.3.6 of [4]). Then $Y_n := \psi \circ X_n, n \in \mathbb{N}$, is a stationary process of pairwise independent real-valued random variables. Since $\mathcal{I}(Y_n, n \in \mathbb{N}) = \mathcal{I}(X_n, n \in \mathbb{N})$ (see Lemma 2.3 (ii)), it suffices to show that $Y_n, n \in \mathbb{N}$, is weakly ergodic. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}$ be fixed. It suffices to show that

$$(1) \quad \{Y_k \leq x\} \text{ and } \mathcal{I}(Y_n, n \in \mathbb{N}) \text{ are independent.}$$

Let $\varphi : \mathbb{R} \rightarrow]0, 1[$ be injective and Borel-measurable. Then $\varphi \circ Y_n, n \in \mathbb{N}$, is a stationary process of pairwise independent random variables. Hence we have by the ergodic theorem (see Theorem 6.28 of [2])

$$(2) \quad \frac{1}{n} \sum_{i=1}^n \varphi \circ Y_i \rightarrow E(\varphi \circ Y_k | \mathcal{I}(\varphi \circ Y_n, n \in \mathbb{N})) \text{ a.e.}$$

Furthermore there holds by the theorem of Etemadi (see [6])

$$(3) \quad \frac{1}{n} \sum_{i=1}^n \varphi \circ Y_i \rightarrow E(\varphi \circ Y_k) \text{ a.e.}$$

By (2) and (3) we obtain

$$E(\varphi \circ Y_k) = E(\varphi \circ Y_k | \mathcal{I}(\varphi \circ Y_n, n \in \mathbb{N})) \text{ a.e.}$$

Since $\mathcal{I}(\varphi \circ Y_n, n \in \mathbb{N}) = \mathcal{I}(Y_n, n \in \mathbb{N}) =: \mathcal{I}$ by Lemma 2.3 (ii), we obtain that

$$(4) \quad E(\varphi \circ Y_k) = E(\varphi \circ Y_k | \mathcal{I}) \text{ a.e.}$$

It is easy to see that there exists a sequence of injective and Borel-measurable functions $\varphi_n : \mathbb{R} \rightarrow]0, 1[$ with $\varphi_n(t) \xrightarrow{n \rightarrow \infty} 1_{] -\infty, x]}(t)$ for all $t \in \mathbb{R}$. Hence we obtain by (4) for all $n \in \mathbb{N}$

$$(5) \quad E(\varphi_n \circ Y_k) = E(\varphi_n \circ Y_k | \mathcal{I}) \text{ a.e.}$$

By Lebesgue's dominated convergence theorem (5) yields

$$E(1_{] -\infty, x]} \circ Y_k) = E(1_{] -\infty, x]} \circ Y_k | \mathcal{I}) \text{ a.e.,}$$

which implies (1). □

The following example is a special case of example 2.3 of [5] adapted to our special need.

2.2 Example. There exists a sequence of real-valued pairwise independent and identically distributed random variables $X_n, n \in \mathbb{N}$, such that $X_n, n \in \mathbb{N}$, is not weakly ergodic.

Proof. Let $X_{2n-1}, n \in \mathbb{N}$, be independent $\{0, 1\}$ -valued random variables with $P(X_{2n-1} = 0) = P(X_{2n-1} = 1) = 1/2$. Put

$$X_{2n} := X_{2n+1} \oplus X_1$$

where \oplus means sum mod 2. Then it is easy to see that $X_n, n \in \mathbb{N}$, are pairwise independent and identically distributed. We show that there exists $I \in \mathcal{I}(X_n, n \in \mathbb{N})$ with

$$(1) \quad P(\{X_1 = 0\} \Delta I) = 0,$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Since $P(\{X_1 = 0\}) = 1/2$, (1) implies that I is not independent of $\{X_1 = 0\}$; hence $X_n, n \in \mathbb{N}$, is not weakly ergodic. To prove (1) put

$$(2) \quad I := \{\overline{\lim} \frac{1}{n} \sum_{i=2}^{n+1} 1_{\{X_i = X_{i+1}\}} > 1/2\}.$$

If $X_1(\omega) = 0$, then $X_{2k}(\omega) = X_{2k+1}(\omega)$ and $X_{2k-1}(\omega) = X_{2k}(\omega) \iff X_{2k-1}(\omega) = X_{2k+1}(\omega)$. Hence for $X_1(\omega) = 0$ we have

$$\begin{aligned} Y_n &:= \frac{1}{n} \sum_{i=2}^{n+1} 1_{\{X_i = X_{i+1}\}} \\ &= \frac{1}{n} \#\{k : 2 \leq 2k \leq n+1\} + \frac{1}{n} \sum_{2 \leq 2k-1 \leq n+1} 1_{\{X_{2k-1} = X_{2k+1}\}}. \end{aligned}$$

As $\{X_{2k-1} = X_{2k+1}\}, k \in \mathbb{N}$, are independent with $P\{X_{2k-1} = X_{2k+1}\} = 1/2$, we obtain by the strong law of large numbers that $Y_n \rightarrow 3/4$ P -a.e. Therefore

$$(3) \quad \{X_1 = 0\} \subset I \quad P\text{-a.e.}$$

Since furthermore $X_1(\omega) = 1$ implies $X_{2k}(\omega) \neq X_{2k+1}(\omega)$, we obtain $\{X_1 = 1\} \subset \Omega \setminus I$. This and (3) imply (1). \square

2.3 Lemma. Let M, M_1 be separable metric spaces. Let $X_n, n \in \mathbb{N}$, be random variables with values in M .

(i) If $\psi : M \rightarrow M_1$ is Borel-measurable, then

$$\mathcal{I}(\psi \circ X_n, n \in \mathbb{N}) \subset \mathcal{I}(X_n, n \in \mathbb{N}).$$

(ii) If $\psi : M \rightarrow \mathbb{R}$ is injective and Borel-measurable and if furthermore M is complete, then

$$\mathcal{I}(\psi \circ X_n, n \in \mathbb{N}) = \mathcal{I}(X_n, n \in \mathbb{N}).$$

Proof. (i) Put $\Phi(x_1, x_2, \dots) := (\psi(x_1), \psi(x_2), \dots)$. Then $\Phi : M^{\mathbb{N}} \rightarrow M_1^{\mathbb{N}}$ is Borel-measurable. Let $I \in \mathcal{I}(\psi \circ X_n, n \in \mathbb{N})$. Then there exists an invariant Borel set $B \subset M_1^{\mathbb{N}}$ with

$$I = (X_n)_{n \in \mathbb{N}}^{-1}(\Phi^{-1}(B)).$$

As $\Phi^{-1}(B)$ is an invariant Borel set of $M^{\mathbb{N}}$, we obtain $I \in \mathcal{I}(X_n, n \in \mathbb{N})$.

(ii) According to (i) it suffices to show

$$\mathcal{I}(X_n, n \in \mathbb{N}) \subset \mathcal{I}(\psi \circ X_n, n \in \mathbb{N}).$$

This follows from (i) applied to $\psi \circ X_n$ instead of X_n and $\psi^{-1} : \psi(M) \rightarrow M$ instead of ψ , as ψ^{-1} is Borel-measurable (see Theorem 8.3.7 and Proposition 8.3.5 of [3]). \square

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