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THE RESIDUES OF THE RESOLVENT ON DAMEK-RICCI SPACES

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Abstract. We determine the poles and residues of the resolvent kernel of the Laplacian on a Damek-Ricci space S. We show that all poles are simple and the residues define convolution operators of finite rank. This generalizes a result of Guillopé-Zworski for the real hyperbolic n-space. If S corresponds to a symmetric space of negative curvature G/K, the image of each residue is a $\mathfrak{g}_c$-module with a specific highest weight. We compute the dimension by the Weyl dimension formula.

1. Preliminaries

In this section we will recall some basic notions on H-type groups and their canonical solvable extensions, following mainly [2] (see also [1]).

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra endowed with an inner product $\langle\, ,\, \rangle$ such that $\mathfrak{n}$ has an orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$. If $\mathfrak{n}$ is abelian, we shall use the convention that $\mathfrak{v} = 0$ and $\mathfrak{n} = \mathfrak{z}$.

Define a linear mapping $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ by

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle \tag{1}$$

(note that J_Z is skew-symmetric). Now $\mathfrak{n}$ is said to be an *H-type algebra* if for any $Z_1, Z_2 \in \mathfrak{z}$,

$$J_{Z_1} J_{Z_2} + J_{Z_2} J_{Z_1} = -2\langle Z_1, Z_2\rangle. \tag{2}$$

The corresponding H-type group is the simply connected Lie group N with Lie algebra $\mathfrak{n}$, endowed with the left-invariant metric induced by the inner product $\langle\, ,\, \rangle$ on $\mathfrak{n}$.

Consider the solvable extension, $S = AN$, the semidirect product of $A = \mathbf{R}^+$ and N, where each $t \in A$ acts on N by $(x, z) \to (t^{\frac{1}{2}} x, tz)$.

Let $\mathfrak{s}$, $\mathfrak{a}$, denote respectively the Lie algebras of S, A. Then $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{a} = \mathbf{R}H$, where $ad\, H$ is the derivation of $\mathfrak{n}$ such that $ad\, H|_{\mathfrak{v}} = \frac{1}{2}I$ and $ad\, H|_{\mathfrak{z}} = I$. Also, $\mathfrak{s}$ carries the inner product extending the one on $\mathfrak{n}$ such that $\|H\| = 1$, $\langle H, \mathfrak{n}\rangle = 0$; S carries the induced left-invariant riemannian structure. Furthermore, let $q = \dim \mathfrak{z}$, $p = \dim \mathfrak{v}$, $n = \dim \mathfrak{s} = p + q + 1$ and $Q = \frac{1}{2}(p + 2q)$.

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Using coordinates from $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathbf{R}^+$, the product on S is expressed as

$$(X, Z, a)(X', Z', a') = (X + a^{\frac{1}{2}} X', Z + aZ' + \frac{1}{2} a^{\frac{1}{2}} [X, X'], aa').$$

The volume element of the induced left-invariant riemannian metric on S is the left Haar measure

$$dm = a^{-Q-1} dX dZ da.$$

We will use the fact that S can be realized as the unit ball in $\mathfrak{s}$:

$$B(\mathfrak{s}) = \{(X, Z, u) : |X|^2 + |Z|^2 + u^2 = 1\}$$

via a Cayley type transform $\tilde{C} : S \to B(\mathfrak{s})$ (see [2], Section 4).

In $B(\mathfrak{s})$ the geodesics through the origin are the diameters and the geodesic distance to the origin $r = d(\tilde{p}, 0) = \log \frac{1+|\tilde{p}|}{1-|\tilde{p}|}$; thus $|\tilde{p}| = \tanh(r/2)$, with $\tilde{p} = \tilde{C}(p)$, if $p \in S$. Furthermore $\cosh\left(\frac{r}{2}\right)^{-2} = \frac{4a}{(1+a+\frac{1}{4}|X|^2)^2 + |Z|^2}$ and the image of the left Haar measure on S via $\tilde{C}^{-1}$ is $d\mu = J(r)\, d\sigma dr$, where r, σ are the radial coordinates on B, $r^2 = |X|^2 + |Z|^2 + u^2$ and $J(r) = 2^p \sinh(r/2)^p \sinh(r)^q$ (see [2], Section 4).

The symmetric spaces of negative curvature are a main subclass of the Damek-Ricci spaces. Let G be a connected, noncompact, semisimple Lie group of real rank one. Let K be a maximal compact subgroup of G and let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. If $G = NAK$ is an Iwasawa decomposition of G, then N is an H-type group and $S = NA \approx G/K$ is a solvable Lie group in the class introduced above. Indeed, if $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of A and N respectively, $\mathfrak{n}$ splits $\mathfrak{n} = \mathfrak{g}_{\alpha/2} \oplus \mathfrak{g}_{\alpha}$, where $\mathfrak{g}_{j\alpha}$, $j = 1/2, 1$, denote the $j\alpha$-root spaces of $\mathfrak{a}$. In the notation above we have $\mathfrak{n}_\alpha = \mathfrak{z}$, $\mathfrak{n}_{\alpha/2} = \mathfrak{v}$, $\mathfrak{a} = \mathbf{R}H_o$, with $H_o \in \mathfrak{a}$ such that $\alpha(H_o) = 1$. If on $S = NA$ we use the G-invariant metric induced by $2(p+4q)^{-1}B$ (B the Killing form of $\mathfrak{g}$), then S is isometric to a Damek-Ricci space. We note that, because of our convention, if $\mathfrak{n}$ is abelian, then $p = 0$, $q = \dim \mathfrak{n}$.

2. THE RESOLVENT OF THE LAPLACIAN ON S

Damek-Ricci spaces have strong similarities with symmetric spaces of negative curvature, in particular they are harmonic spaces. On S there is a radialization operator π which corresponds to the standard operator in the case of the ball model of S (see [2], p. 230). If $f \in C_c^\infty(S)$, $p \in S$ and $\tilde{p} = \tilde{C}(p)$ then

$$\pi f(p) := \int_{S^{p+q}} \tilde{f}(\|\tilde{p}\|\sigma) d\sigma,$$

where $\tilde{f} := f \circ \tilde{C}^{-1}$. In the symmetric case, if $f \in C^\infty(NA)$, then $\pi f(x) = \int_K \bar{f}(kx) dk$, where $\bar{f}$ denotes the right K-invariant extension of f.

If $\{Z_i\}$, $\{V_j\}$ are orthonormal bases of $\mathfrak{z}$ and $\mathfrak{v}$ respectively, the Laplace-Beltrami operator is given by $L = \sum_i Z_i^2 + \sum_j V_j^2 + H^2 - QH$; L generates the algebra of left-invariant differential operators on S which commute with π (see [2], Theorem 5.2).

If f is a smooth radial function on $S - \{e\}$, we will often abuse notation by writing $f(r) = f(x)$, where $r = d(x, e)$. The action of L on radial functions is given by

$$Lf(r) = \frac{d^2}{dr^2} f(r) + \frac{1}{2}(p \coth(r/2) + 2q \coth(r)) \frac{d}{dr} f(r). \tag{3}$$

In the symmetric case, if $\mathfrak{n}$ is not abelian and we set $r = 2t$, then $Lf(t)$ corresponds to $\frac{1}{4}Cf(a_t)$, C the Casimir element; [6], Section 1 (1). If $\mathfrak{n}$ is abelian, then L corresponds to C.

A *spherical function* ψ on S is a radial eigenfunction of L such that $\psi(e) = 1$. This generalizes the corresponding notion in the symmetric case and one has the following characterization ([2]).

Proposition 2.1. *Let $\nu \in \mathbf{C}$. The function $\phi_\nu = \pi(a^{\nu+Q/2})$ is a spherical function with eigenvalue $\lambda(\nu) = \nu^2 - Q^2/4$. Any spherical function on S is of this form.*

As in the symmetric case, we can express ϕ_ν by a hypergeometric function as follows. By letting $z = -\sinh(r/2)$, the equation

$$\left\{ \frac{d^2}{dr^2} + \frac{1}{2}(p\coth(r/2) + 2q\coth(r))\frac{d}{dr} - \lambda(\nu) \right\} f_\nu(r) = 0 \tag{4}$$

transforms into the hypergeometric equation with parameters $a = Q/2 - \nu$, $b = Q/2 + \nu$, and $c = n/2$. Since $\phi_\nu(e) = 1$, it follows that

$$\phi_\nu(r) = F\left(-\nu + Q/2, \nu + Q/2, \frac{n}{2}, -\sinh(r/2)^2\right). \tag{5}$$

Furthermore, if $\operatorname{Re}\nu > 0$, the asymptotic behavior of $\phi_\nu(r)$, as $r \to \infty$, is given by (see [2], p. 239)

$$\phi_\nu(r) \sim c(\nu)\, e^{r(\nu+Q/2)}, \quad \text{where} \quad c(\nu) = \frac{2^{-2\nu+Q}\Gamma(n/2)\Gamma(2\nu)}{\Gamma(\nu+Q/2)\Gamma(\nu+\frac{p+2}{4})}. \tag{6}$$

Here $c(\nu)$ coincides with Harish Chandra's c-function in the symmetric case. The Plancherel measure, $\mu(\nu) = (c(\nu)c(-\nu))^{-1}$, can be written $\mu(\nu) = c_o p(\nu) D(\nu)$, c_o a constant and $p(\nu)$ the polynomial given by

$$\begin{array}{ll} \displaystyle\prod_{j=0}^{\frac{p}{4}-1} \left(-\nu^2 + \left((2j+1)^2/4\right)\right) \prod_{j=0}^{\frac{Q}{2}-1} \left(-\nu^2 + \left(j^2/4\right)\right), & q, \tfrac{p}{2} \text{ even}, \\ \displaystyle -\prod_{j=1}^{p/4} (-\nu^2 + j^2)^2 \nu^3, & q = 1, \tfrac{p}{2} \text{ odd}, \\ \displaystyle -\prod_{j=0}^{\frac{p}{4}-1} \left(-\nu^2 + \left((2j+1)^2/4\right)\right) \prod_{j=0}^{\frac{Q}{2}-1} \left(-\nu^2 + \left((2j+1)^2/4\right)\right)\nu, & q \text{ odd}, \tfrac{p}{2} \text{ even}, \end{array}$$

and $D(\nu)$ equals respectively 1, $\cot(\pi\nu)$, and $\tan(\pi\nu)$ ([1]).

Remark. We note that p is always even, since $\mathfrak{v}$ is a module over the Clifford algebra of $\mathfrak{z}$. If $p = 0$, then $X \approx H^{q+1}$, $G \simeq \mathbf{SO}(q+1, 1)$ and in this case $D(\nu)$ equals 1 or $\tan(\pi\nu)$ depending on whether q is even or odd.

In [6], the resolvent of the Laplacian $R(\lambda(\nu))$ was studied on symmetric (and locally symmetric spaces) of negative curvature. In the symmetric case, it is given for $\operatorname{Re}\nu > \rho$ by convolution with a smooth radial function Q_ν on $S - \{e\}$ which is an eigenfunction of L with eigenvalue $\lambda(\nu)$, and which has a meromorphic continuation to $\mathbf{C}$. As we shall now see, these properties remain valid for any S as above. Many arguments in [6] can be adapted, so we shall omit several proofs. On the other hand, we shall show how to obtain Q_ν by using a series solution. We thank N. Wallach for useful discussions on this point, which helped us to simplify the original argument.

If $b \in \mathbf{R}$ and $\delta > 0$, let $\mathcal{S}_{b,\delta} = \{\nu : \operatorname{Re}\nu > b,\ |\nu + j| > \delta\ \ \forall j \in -\mathbf{N} : b \leq j\}$. That is, $\mathcal{S}_{b,\delta} = \{\ \nu\ :\ \operatorname{Re}\nu > b\}$, if $b \geq 0$, and $\mathcal{S}_{b,\delta}$ is a half plane with finitely many discs removed, centered at $-1, -2, ..., -k$, with $-k \geq b$, if $b < 0$.

Theorem 2.2. *If $\nu \in \mathbf{C}$, $2\nu \notin -\mathbf{N}$, then there exists a radial function $Q_\nu \in C^\infty(S - \{e\})$ with the following properties:*

- (a) *$(L - \lambda(\nu))Q_\nu = 0$. For each $x \in S$, $Q_\nu(x)$ is holomorphic for $\nu \notin -\frac{1}{2}\mathbf{N}$ and in $\nu \in \frac{1}{2}\mathbf{N}$, $Q_\nu(s)$ has at most a simple pole. Furthermore, for any $b \in \mathbf{R}$, δ, $r_o > 0$, there exists $K = K(b, \delta, r_o)$ such that $|Q_\nu(r)| \leq K$ for any $r \geq r_o$, $\nu \in \mathcal{S}_{b,\delta}$.*
- (b) *Where defined, $\phi_\nu = c(-\nu)Q_\nu + c(\nu)Q_{-\nu}$.*
- (c) *As $r \mapsto 0$, $Q_\nu(r) \sim d(\nu) r^{-p-q+1} |\log r|^{\delta_{p+q,1}}$, for some meromorphic function $d(\nu)$ on $\mathbf{C}$, holomorphic if $2\nu \notin -\mathbf{N}$.*
- (d) $\lim_{r \mapsto 0^+} J(r) \frac{d}{dt} Q_\nu(r) = -2\nu c(\nu)$.
- (e) *If $f \in C_c^\infty(S)$ and $2\nu \notin -\mathbf{N}$, then*

$$\int_S Q_\nu(x^{-1}y)(L - \lambda(\nu)I)f(y)dy = -2\nu c(\nu) f(x). \tag{7}$$

Proof. We look for a solution of (4) of the form $q_\nu(r) = \sum_{j=0}^{\infty} a_j(\nu) e^{-(\nu + Q/2 + j)r}$. Substituting in (4) and using $\coth(r) = \frac{1+e^{-2r}}{1-e^{-2r}}$, we get that

$$\begin{aligned}\sum_{j \geq 0}(Q + j)(2\nu + Q + j)a_j(\nu)e^{-jr} \ &+\ p\sum_{j \geq 1}(\nu + Q/2 + j + 1)a_{j+1}(\nu)e^{-jr} \\ + \sum_{j \geq 2}(j + 2)(2\nu + j + 2)a_{j+2}(\nu)e^{-jr} \ &=\ 0.\end{aligned}$$

Thus, the coefficients $a_j(\nu)$ must satisfy the recurrence relations

$$a_1(\nu) = a_0(\nu) f_{-1}(\nu), \quad a_{j+2}(\nu) = a_{j+1}(\nu) f_j(\nu) + a_j(\nu) g_j(\nu), \tag{8}$$

where $f_j(\nu) = p\frac{\nu + Q/2 + j + 1}{(j+2)(2\nu + j + 2)}$ and $g_j(\nu) = \frac{(Q+j)(2\nu + Q + j)}{(j+2)(2\nu + j + 2)}$, for $j \geq 0$.

We thus set $q_\nu(r) = e^{-(\nu + Q/2)r} \sum_{j=0}^{\infty} a_j(\nu) e^{-jr}$, where $a_0 = 1$, and if $2\nu \notin -\mathbf{N}$, then the $a_j(\nu)$ are given by (8).

If $b \in \mathbf{R}$, $\delta > 0$ and $\nu \in \mathcal{S}_{b,\delta}$, we have

$$|f_j(\nu)| \leq \frac{p}{2j+4}\left(1 + \frac{Q+j}{|2\nu + j + 2|}\right) \leq \frac{p}{2j+4}\left(1 + \frac{Q+j}{(j + 2 - 2k)}\right),$$

$$|g_j(\nu)| \leq \frac{Q+j}{2+j}\left(1 + \frac{|Q-2|}{|2\nu + j + 2|}\right) \leq \frac{Q+j}{j+2}\left(1 + \frac{|Q-2|}{(j + 2 - 2k)}\right)$$

for $j + 2 > 2|k|$, where k is the first integer such that $k \leq b$. These estimates clearly imply that given $\varepsilon > 0$ there exist j_0 and $M = M(\varepsilon)$ such that $|f_j(\nu)| \leq \varepsilon, |g_j(\nu)| \leq 1 + \varepsilon$, if $j \geq j_0$, $|f_j(\nu)| \leq M$, $|g_j(\nu)| \leq M$, if $j < j_0$, uniformly for $\nu \in \mathcal{S}_{b,\delta}$. Using these estimates we see that if $\nu \in \mathcal{S}_{b,\delta}$, if $M' = M'(\varepsilon) = j_0 M^{j_0}$, then

$$|a_j(\nu)| \leq \begin{cases} jM^j & j \leq j_0, \\ M'(1 + 2\varepsilon)^{j - j_0 + 1} & j \geq j_0. \end{cases} \tag{9}$$

Now, by (9) $|q_\nu(r)| \leq e^{-(\operatorname{Re}\nu + \frac{Q}{2})r} M' \left(j_0 + \sum_{l \geq 0}(1 + 2\varepsilon)^{l+1} e^{-(l + j_0)r}\right)$; hence the series defining q_ν converges absolutely and uniformly for $\nu \in \mathcal{S}_{b,\delta}$ and $r > r_o$, for each $r_o > 0$. Since r_o is arbitrary, q_ν defines a uniformly bounded function for ν, r in this region. With a similar argument one proves the uniform convergence of the series of the derivatives, in each region $\mathcal{S}_{b,\delta}$, $r > r_o$; hence q_ν is smooth. If we now define $Q_\nu(x) = q_\nu(r)$ with $r = d(x,e)$, for $x \in S$, then $Q_\nu \in C^\infty(S - \{e\})$ is a radial eigenfunction of L of eigenvalue $\lambda(\nu)$ and has the properties stated in (a).

From now on we shall write $Q_\nu(r) = q_\nu(r)$, for simplicity. By the asymptotic behavior as $r \mapsto +\infty$, it follows that if $2\nu \notin \mathbf{Z}$, $Q_\nu(r), Q_{-\nu}(r)$ form a fundamental system of solutions of (4). Writing ϕ_ν in terms of Q_ν and $Q_{-\nu}$, the functional equation in (b) follows as in the symmetric case (see [6], p. 671).

We now prove (c). Equation (4) has a regular singular point at $r = 0$ and the corresponding indicial equation is $s(s-1) + (p+q)s = 0$, with roots $s = 0$, $s = 1 - p - q$. The solution $\phi_\nu(r)$ is associated to the root $s = 0$ and is continuous at $r = 0$. If $2\nu \notin -\mathbf{N}$, and if $p+q > 1$, Q_ν is a second linearly independent solution; hence $\lim_{r \mapsto 0^+} Q_\nu(r) r^{p+q-1} := d(\nu)$ exists and the meromorphy of Q_ν implies that of $d(\nu)$. Similarly, if $p + q = 1$, $Q_\nu(r) \sim d(\nu) \log r$ as $r \mapsto 0^+$. Thus (c) follows. The proof of (d) is similar to that of [6], Lemma 1.3, and will be omitted.

To see (e) we may assume that $x = e$. We have, for any $f \in C_c^\infty(S)$,

$$\begin{aligned} \int_S Q_\nu(y)(L - \lambda(\nu)I) f(y)\, d\mu(y) &= \int_{\partial B} \int_0^\infty \tilde{Q}_\nu(r\sigma)(L - \lambda(\nu)I)\tilde{f}(r\sigma) J(r) dr d\sigma \\ &= \int_0^\infty Q_\nu(r) J(r)(L - \lambda(\nu))\pi f(r)\, dr. \end{aligned}$$

Now we observe that for a radial function h on S, $J(r)^{1/2} Lh(r) = \frac{d^2}{dr^2} J^{1/2}(r)h(r) + J(r)^{1/2}\eta(r)h(r)$, where $\eta = \frac{(J')^2 - 2J''J}{4J^2}$. Hence we see that the above equals

$$\begin{aligned} &\int_0^\infty \frac{d^2}{dr^2}\Big(J(r)^{1/2}\pi f(r)\Big) J(r)^{1/2} Q_\nu(r) - J(r)^{1/2}\pi f(r) \frac{d^2}{dr^2}\Big(J(r)^{1/2} Q_\nu(r)\Big)\, dr \\ = \;& \int_0^\infty \frac{d}{dr}\left[\frac{d}{dr}(J(r)^{1/2}\pi f(r)) J(r)^{1/2} Q_\nu(r) - J(r)^{1/2}\pi f(r) \frac{d}{dr}(J(r)^{1/2} Q_\nu(r))\right] dr \\ = \;& -2\nu c(\nu) f(e) \end{aligned}$$

using (c) and (d). This gives (e); hence the theorem follows.

3. The residues of the resolvent

Let $\tilde{R}(\lambda(\nu))$ denote the kernel operator with kernel $K_\nu(x,y) = -\frac{Q_\nu(x^{-1}y)}{2\nu c(\nu)}$. If $\operatorname{Re}\nu > \rho$, then $\tilde{R}(\lambda(\nu)) = R(\lambda(\nu))$.

Theorem 3.1. *If p, q are both even, then $\tilde{R}(\lambda(\nu))$ is everywhere holomorphic. Otherwise, it has simple poles lying at $\nu_k = -Q/2 - k$ with $k \in \mathbf{N} \cup \{0\}$. If $\nu = \nu_k$, set $T_{\nu_k}(f) := \operatorname{Res}_{\nu = \nu_k} \tilde{R}(\lambda(\nu))(f)$. Then $T_{\nu_k}(f) = (2\pi\nu_k)^{-1} p(\nu_k)\, f * \check{\phi}_\nu$ and T_{ν_k} is a finite rank operator, for each value of k.*

Proof. The possible poles of $K_\nu(x,y)$ lie at $-\frac{1}{2}\mathbf{N}$ or at the zeros of $c(\nu)$. By using formula (6) one sees that $c(\nu)$ has no zeros in $\mathbf{C}$, if p and q are both even. Otherwise, q is odd and $c(\nu)$ has simple zeros at $\nu_k = -Q/2 - k$, for any $k \in \mathbf{N} \cup \{0\}$, and possibly simple poles at $\nu \in -\frac{1}{2}\mathbf{N}$.

Since $\nu = 0$ is a simple pole of $c(\nu)$, and Q_ν is holomorphic at 0, $\frac{Q_\nu}{2\nu c(\nu)}$ is holomorphic at $\nu = 0$.

On the other hand $c(-\nu)$ and $Q_{-\nu}$ are holomorphic and nonvanishing on $\mathbf{R}^{<0}$, ϕ_ν is everywhere holomorphic and $\phi_\nu(1) = 1$. So Theorem 2.2 (b) implies that a pole of Q_ν must be compensated by a pole of $c(\nu)$ and a zero of $c(\nu)$ cannot be a zero of Q_ν.

Therefore, $\frac{Q_\nu}{2\nu c(\nu)}$ has a pole at ν if and only if ν is a zero of $c(\nu)$, that is, $\nu = \nu_k = -Q/2 - k$, $k \in \mathbf{N} \cup \{0\}$. On the other hand, $\frac{Q_{-\nu}}{2\nu c(-\nu)}$ is analytic at $\nu = \nu_k$. Thus, if $f \in C_c^\infty(S)$ and using that $-\frac{Q_\nu}{2\nu c(\nu)} = \frac{Q_{-\nu}}{2\nu c(-\nu)} - \frac{\mu(\nu)\phi_\nu}{2\nu}$, we have

$$T_{\nu_k}(f) = \mathrm{Res}_{\nu=\nu_k} \tilde{R}(\lambda(\nu))(f) = \frac{p(\nu_k)}{2\pi\nu_k} f * \check{\phi}_{\nu_k}. \tag{10}$$

From (5) and the expression for $\cosh(\frac{r}{2})$ in the Preliminaries, we have that

$$\phi_\nu(X, Z, a) = \sum_{i \geq 0} \frac{(Q/2-\nu)_i (Q/2+\nu)_i}{i!\,(n/2)_i} \left[\frac{(a + \frac{1}{4}|X|^2)^2 + |Z|^2}{4a}\right]^i, \tag{11}$$

where $(u)_i = \displaystyle\prod_{l=0}^{i-1} u + l$ for $u \in \mathbf{C}$. Hence we see that the coefficients in the expansion (11) are zero for $i \geq k+1$, for the special values $\nu_k = -Q/2 - k$. Fix $\{V_i\}$ and $\{W_j\}$, orthonormal bases of $\mathfrak{v}$ and $\mathfrak{z}$ respectively, and write $X = \displaystyle\sum_{i=1}^{p} x_i V_i$ and $Z = \displaystyle\sum_{j=1}^{q} z_j W_j$. If $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$, set $X^I = \prod x_j^{i_j}$, $Z^J = \prod z_l^{j_l}$, $|I| = \displaystyle\sum_{1}^{p} i_j$ and similarly for $|J|$. Let $\mathcal{F}_k$ be the linear span of the functions $a^i X^{2I} Z^J \ : \ i \in \mathbf{Z}$, $|i| \leq k$, $|I|, |J| \leq 2k$. Clearly $\phi_{\nu_k} \in \mathcal{F}_k$. If $t = (Y, U, b)$ with $Y = \displaystyle\sum_{i=1}^{p} y_i V_i \in \mathfrak{v}$, $U = \displaystyle\sum_{i=1}^{q} u_i W_i \in \mathfrak{z}$, $b \in A$ and $s = (X, Z, a) \in S$, then

$$\begin{aligned} t^{-1}s &= \left(b^{-\frac{1}{2}}(X - Y), b^{-1}(Z - U + \frac{1}{2}[X, Y]), b^{-1}a\right) \\ &= \left(b^{-\frac{1}{2}}\sum(x_i - y_i)V_i, b^{-1}\sum(z_j - u_j)W_j + \frac{1}{2}\sum_l\sum_{i,j} x_i y_j a_{i,j}^l W_l, b^{-1}a\right), \end{aligned}$$

where $[X, Y] = \displaystyle\sum_l \sum_{i,j} x_i y_j a_{i,j}^l W_l$. Hence, by (11) $\phi_{\nu_k}(t^{-1}s)$ is a linear combination of functions of the form $a^{j_1} b^{j_2} X^{2I_1} Y^{2I_2} Z^{J_1} U^{J_2}$ with $j_i \in \mathbf{Z}$, $|j_i| \leq k$, $i = 1, 2$, and $|I_i|, |J_i| \leq 2k$ for $i = 1, 2$.

Therefore, if $f \in C_c^\infty(S)$, it follows that $f * \check{\phi}_{\nu_k}(t) = \int_S f(s)\phi_{\nu_k}(t^{-1}s)ds$ is a linear combination of expressions of the form

$$t \mapsto b^{j_2} Y^{2I_2} U^{J_2} \int_{\mathfrak{z}}\int_{\mathfrak{v}}\int_A f(X, Z, a) a^{j_1} X^{2I_1} Z^{J_1}\ a^{-Q-1} da\, dX\, dZ.$$

Therefore, $f * \check{\phi}_{\nu_k}$ belongs to $\mathcal{F}_k$, a finite dimensional space, as asserted.

4. The symmetric case

In the case when S is of symmetric type one can get more precise information on the operators T_{ν_k} by using representation theory.

The group of isometries G of S is a noncompact semisimple Lie group of real rank one. Let $\mathfrak{g}$, $\mathfrak{k}$, N, and A be as in Section 1, let M be the centralizer of A in K, let $P = MAN$ and let $\mathfrak{p}$ be the Lie algebra of P. Extend $\mathfrak{a}$ in the usual way to a Cartan subalgebra $\mathfrak{h}_c = \mathfrak{a}_c + \mathfrak{h}_c^-$ of $\mathfrak{g}$, where $\mathfrak{h}^-$ is a maximal abelian subalgebra of $\mathfrak{m}$, and introduce compatible orderings in the dual spaces of $\mathfrak{a}$ and $\mathfrak{a} + \sqrt{-1}\mathfrak{h}^-$. Let $\Sigma^+(\Delta^+)$ denote the corresponding set of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ (respectively $(\mathfrak{g}_c, \mathfrak{h}_c)$). Since $\mathfrak{g}$ has real rank one, there is only one real root $\tilde{\alpha} \in \Delta^+$. It satisfies $\tilde{\alpha}_{|\mathfrak{h}^-} = 0$ and $\tilde{\alpha}_{|\mathfrak{a}} = \alpha$.

For $\nu \in \mathbf{C}$, let (π_ν, H^ν) be the spherical principal series representation of G (see [7], Section 3.6). The zonal spherical function ϕ_ν is given by $\phi_\nu(g) = \langle \pi_\nu(g)1_\nu, 1_\nu \rangle$, where $1_\nu \in H^\nu$ is such that $1_\nu(nak) = a^{(\nu+\rho)\alpha}$, $n \in N, a \in A, k \in K$, and $\langle\, ,\, \rangle$ is the standard inner product on H^ν.

Theorem 4.1. *Let $S = G/K$ be a noncompact symmetric space of real rank one and let $\nu_k = -\rho - k$ with $k \in \mathbf{N} \cup \{0\}$. Then Im$(T_{\nu_k})$ is an irreducible $\mathfrak{g}_c$-module of highest weight $k\tilde{\alpha}$.*

Proof. By a result of Helgason (see [4], Ch. V, Theorem 4.1), the K-spherical finite dimensional representations of G can be characterized as the representations of $\mathfrak{g}_c$ of highest weight $\Lambda \in \mathfrak{h}_c^*$ such that: $\Lambda_{|\mathfrak{h}^-} = 0$ and $\langle \Lambda, \lambda\rangle / \langle \lambda, \lambda \rangle \in \mathbf{Z}^{\geq 0}$, for any $\lambda \in \Sigma^+$. Since in our case $\Sigma^+ = \{\alpha, \alpha/2\}$ or $\{\alpha\}$, this is equivalent to $\Lambda_{|\mathfrak{a}} = k\tilde{\alpha}$, with $k \in \mathbf{Z}^{\geq 0}$, and $\tilde{\alpha}$ the real root. We shall denote by $V_{k\tilde{\alpha}}$ the $\mathfrak{g}_c$-module with highest weight $k\tilde{\alpha}$.

Our claim is that 1_{ν_k} generates a finite dimensional $(\mathfrak{g}, K)$-submodule V_{ν_k} of H^{ν_k}, isomorphic to $V_{k\tilde{\alpha}}$.

In the notation of Lemma 3.8.2 in [7], we have that

$$\mathrm{Hom}_{\mathfrak{g},K}(V_{k\tilde{\alpha}}, H^{\nu_k}) \simeq \mathrm{Hom}_{\mathfrak{p},M}(V_{k\tilde{\alpha}}/\mathfrak{n}V_{k\tilde{\alpha}}, \mathbf{C}_{\nu_k}), \tag{12}$$

where $\mathbf{C}_{\nu_k}$ denotes the MAN-module $\mathbf{C}$, with MN acting trivially and $a \in A$ acting by multiplication by $a^{(\nu_k+\rho)\alpha}$. To prove our claim it will thus be sufficient to show that there exists a nontrivial $(\mathfrak{p}, M)$-morphism $f : V_{k\tilde{\alpha}}/\mathfrak{n}V_{k\tilde{\alpha}} \to \mathbf{C}_{\nu_k}$. We denote by Λ_o the lowest weight of $V_{k\alpha}$ and by v_o the corresponding lowest weight vector. Then $\Lambda_o = s_o\Lambda$, s_o the long element of the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$. Since $\Lambda' = -s_o\Lambda$ is the highest weight of the dual representation of $V_{k\tilde{\alpha}}$, which is also K-spherical, Λ' satisfies Helgason's conditions. This implies that $s_o\Lambda_{|\mathfrak{h}^-} = 0$ and $s_o\Lambda_{|\mathfrak{a}} = -k\alpha$. Arguing as in the proof of Theorem 4.1, Ch. V in [4], one shows that $\pi_\Lambda(M)v_o = v_o$.

Since $s_o\Lambda_{|\mathfrak{h}^-} = 0$, it follows that $V = \mathbf{C}v_o \oplus (\mathfrak{n} \oplus \mathfrak{m})V$. Now we can define a $(\mathfrak{p}, M)$-morphism $f : V/\mathfrak{n}V \to \mathbf{C}_\nu$ such that $f : [v_o] \mapsto 1$, where $[v_o]$ is the class of v_o and $f = 0$ on $\mathfrak{m}V$. Hence, by (12), there is a nonzero G-map of $V_{k\tilde{\alpha}}$ onto a subspace V_{ν_k} of H^{ν_k}, which must contain 1_{ν_k}.

Now we prove the statement in the theorem. If $f \in C_c^\infty(G/K)$, and $x \in G$, we have by (3.1)

$$T_{\nu_k}(f) = p_k\, f * \check{\phi}_{\nu_k}(x) = p_k\, \langle \pi(x^{-1})\, \pi(f) 1_{\nu_k}, 1_{\nu_k} \rangle,$$

where $p_k = -\frac{p(\nu_k)}{\pi \nu_k} \neq 0$, for all k (see the formula of $p(\nu)$ in Section 2).

By irreducibility, as f varies, $\pi(f)1_{\nu_k}$ fills $V_{\nu_k} \simeq V_{k\alpha}$. Hence, the image of T_{ν_k} coincides with the image of the G-morphism $T_k : V_{\nu_k} \mapsto C^\infty(G/K)$ given by $T_k(v)(x) = \langle \pi_{\nu_k}(x^{-1})\, v, 1_\nu \rangle$, for $v \in V_{\nu_k}$. This proves the theorem.

Remark 4.2. We will now use the Weyl dimension formula to calculate the dimension of the $\mathfrak{g}_c$-module $V_{k\tilde{\alpha}}$ in each case. The real roots $\tilde{\alpha}$ can be read from the Satake diagram of $\mathfrak{g}$. They are listed, for each rank one group, in [5] (for instance). We shall thus use the notation in [5].

(i) $\mathfrak{g} = \mathfrak{so}(\mathbf{n}, \mathbf{1})$ (n even). In this case, the real root is $\tilde{\alpha} = \epsilon_1$, the first fundamental weight. The corresponding $\mathfrak{g}_c$-module $V_{k\tilde{\alpha}}$ is isomorphic to the representation of G on $\mathcal{H}_k$, the space of homogeneous harmonic polynomials of degree k in $n+1$ variables, which has dimension $\frac{(k+n-2)!(2k+n-1)}{k!(n-1)!}$. This can easily be computed by the Weyl dimension formula.

(ii) $\mathfrak{g} = \mathfrak{su}(\mathbf{n}, \mathbf{1})$. Here, the real root is $\tilde{\alpha} = \epsilon_1 - \epsilon_{n+1}$, and the positive roots are $\epsilon_i - \epsilon_j$, $i < j$ and $2\rho = \sum_{j=1}^{n+1} (n - 2j + 2)\epsilon_j$. Thus

$$\dim(V_{k\tilde{\alpha}}) = \prod_{1 \le i < j \le n+1} \frac{\langle k(\epsilon_1 - \epsilon_{n+1}) + \rho, \epsilon_i - \epsilon_j \rangle}{\langle \rho, \epsilon_i - \epsilon_j \rangle}$$

$$= \prod_{2 \le j \le n} \frac{k+j-1}{j-1} \prod_{2 \le i \le n} \frac{k+n+1-i}{n+1-i} \frac{2k+n}{n} = \binom{k+n-1}{k}^2 \frac{2k+n}{n}.$$

(iii) $\mathfrak{g} = \mathfrak{sp}(\mathbf{n}, \mathbf{1})$. In this case, $\tilde{\alpha} = \epsilon_1 + \epsilon_2$, and the positive roots are $\epsilon_i \pm \epsilon_j$, $1 \le i < j \le n+1$, and $2\epsilon_i$, $1 \le i \le n+1$. Also, $\rho = \sum_{j=1}^{n+1} (n + 2 - j)\epsilon_j$. Hence

$$\dim(V_{k\tilde{\alpha}}) = \left(\prod_{i=1}^{2} \prod_{j=3}^{n+1} \frac{2n+4-j-i+k}{2n+4-j-i} . \frac{j-i+k}{j-i} \right) \tfrac{2n+1+2k}{2n+1} . \tfrac{n+k}{n} . \tfrac{n+k+1}{n+1}$$

$$= \tbinom{2n+k-1}{k}^2 \tfrac{2n+k}{(2n+1)(2n)} \tfrac{2n+2k+1}{k+1}.$$

(iv) $\mathfrak{g} = \mathfrak{f}_\mathbf{4}$. The real root is $\tilde{\alpha} = \lambda_4$ ($= \epsilon_1$), the fourth fundamental weight. The positive roots are ϵ_i, $\epsilon_i \pm \epsilon_j$, $1 \le i < j \le 4$, $\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ and $2\rho = 11\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + \epsilon_4$. Using the Weyl dimension formula we obtain in this case

$$\dim(V_{k\tilde{\alpha}}) = \frac{2k+11}{11} \prod_{j=1}^{j=10} \frac{k+j}{j} . \prod_{j=4}^{j=7} \frac{k+j}{j}.$$

4.1. **The real hyperbolic n-space.** If $S \approx H^n$, one can make the results in Theorem 2.2 more precise. In this case one can solve the recurrence in (8), obtaining an explicit series expression for Q_ν. Indeed, since $p = 0$, by (8) we see that $a_{2j+1} = 0$ for $j \ge 0$; hence $a_{2j} = a_{2j-2} \frac{(j-1+\rho)(j-1+\rho+\nu)}{j(\nu+j)}$, $j \ge 0$. Thus, if we set $c_j := a_{2j}$, for $j \ge 0$, and if $c_0 := 1$, we obtain for $j \ge 1$

$$c_j(\nu) = \frac{(\rho)_j}{j!} \frac{(\nu+\rho)_j}{(\nu+1)_j}. \tag{13}$$

Furthermore, $c(\nu) = \frac{2^{2\rho-1}\Gamma(n/2)}{\pi^{1/2}}\frac{\Gamma(\nu)}{\Gamma(\nu+\rho)}$ hence, using (13) and the duplication formula for the Gamma function, we obtain

$$\frac{Q_\nu(r)}{2\nu c(\nu)} = \frac{2^{-2n+3}}{(n-2)!}e^{-(\nu+\rho)r}\sum_{j=0}^{\infty}\frac{\Gamma(\rho+j)}{j!}\frac{\Gamma(\nu+\rho+j)}{\Gamma(\nu+j+1)}e^{-2jr}. \tag{14}$$

Now, if $S_{b,\delta}$ is as in Theorem 2.2, one sees, by using Stirling's estimates, that there exists a constant $K = K(b,\delta)$ such that the coefficients in (14) are bounded by $Kj^{\rho-1}\,|\nu+j|^{\rho-1}$, uniformly for ν in $S_{b,\delta}$. This gives an alternative proof of the convergence, as stated in Theorem 2.2 (a).

Regarding the poles, we see that if n is odd, since $\rho = \frac{n-1}{2} \in \mathbf{N}$, the coefficients in (14) are polynomial functions in ν; hence $\tilde{R}(\lambda(\nu))$ is everywhere holomorphic in this case.

If n is even, (14) implies that the kernel is meromorphic with poles at $\nu_k = -\rho-k,\ k \in \mathbf{N} \cup \{0\}$. Since $\Gamma(\nu+\rho+j)$ is holomorphic at $\nu = \nu_k$ for $j > k$, we get

$$\begin{aligned}\operatorname{Res}_{\nu=\nu_k}\tfrac{Q_\nu(r)}{2\nu c(\nu)} &= \tfrac{2^{-2n+3}}{(n-2)!}\sum_{j=0}^{k}\frac{\Gamma(\rho+j)(-1)^{k-j}}{j!(k-j)!\Gamma(-k-\rho+j+1)}e^{-(2j-k)r}\\ &= \tfrac{2^{-2n+4}}{(n-2)!}\sum_{j=0}^{[k/2]}\frac{\Gamma(\rho+j)(-1)^{k-j}}{j!(k-j)!\Gamma(-k-\rho+j+1)}\cosh(2j-k)r,\end{aligned}$$

since $\frac{\Gamma(\rho+j)\,(-1)^k}{\Gamma(-k-\rho+j+1)} = \frac{\Gamma(\rho+k-j)}{\Gamma(-\rho-j+1)}$, for $0 \le j \le k$.

Remark 4.3. We note that in [3], Section 2, Guillopé-Zworski consider the resolvent kernel for the real hyperbolic n-space, giving the location of the poles and showing that the residues define operators of finite rank.

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