

STRUCTURE OF CLOSED FINITELY STARSHAPED SETS

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ABSTRACT. A set S is finitely starshaped if any finite subset of S is totally visible from some point of S . It is well known that in a finite-dimensional linear space, a closed finitely starshaped set which is not starshaped must be unbounded. It is proved here that such a set must admit at least one direction of recession. This fact clarifies the structure of such sets and allows the study of properties of their visibility elements, well known in the case of starshaped sets. A characterization of planar finitely starshaped sets by means of its convex components is obtained. Some plausible conjectures are disproved by means of counterexamples.

1. STATEMENT OF THE PROBLEM

All the points and sets considered here are included in E^d , the d -dimensional Euclidean space, whose origin is denoted by θ . The interior, closure, boundary, complement, and convex hull of a set S are denoted by $\text{int } S$, $\text{cl } S$, ∂S , S^C , and $\text{conv } S$, respectively. The open segment joining x and y is (x, y) , while the substitution of one or both parentheses by square ones indicates the adjunction of the corresponding endpoints. The closed ray issuing from x and going through y is denoted by $R(x \rightarrow y)$, and $R(yx \rightarrow)$ is the closed ray issuing from x and having the opposite direction of $R(x \rightarrow y)$. In an analogous way, $L(x \rightarrow y)$ denotes the line that goes through x and y . The *unit sphere* is the set $\Omega_d = \{x \in E^d \mid \|x\| = 1\}$. A *direction of recession* of the set S is an element $v \in \Omega_d$ such that $\forall x \in S$ the ray $R(x \rightarrow x + v) \subset S$. The closed and open ball centered at x and having radius ε are denoted $B(x, \varepsilon)$ and $U(x, \varepsilon)$, respectively. If the context is clear we will simply write B_x and U_x , respectively. A *convex component* of S is a maximal convex subset of S . We say that x *sees* y via S if $[x, y] \subset S$. The *star of x in S* is the set $st(x, S)$ of all points of S that see x via S . A *star-center* of S is a point $x \in S$ such that $st(x, S) = S$. The *mirador* (or *convex kernel*) of S is the set $\text{mir } S$ of all the star-centers of S . S is a *convex set* if $\text{mir } S = S$, and S is *starshaped* if $\text{mir } S$ is nonvoid. Finally, we say that S is *finitely starshaped* if for any finite set $F \subset S$ there exists $p \in S$ such that $F \subset st(p, S)$.

The subject of finitely starshaped sets was introduced by B. Peterson [7]. M. Breen wrote some papers about nonclosed finitely starshaped sets (see [2] and [3]). It is well known (see Corollary 2) that compact finitely starshaped sets must be

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starshaped, but there are easy examples, even in the plane, of closed unbounded finitely starshaped sets that fail to be starshaped (see Section 6, Conjectures 1 and 5, below). It is our purpose to study here some relevant geometrical characteristics of such sets.

2. PREVIOUSLY KNOWN RESULTS

We quote here for future reference several well known combinatorial results concerning convex and starshaped sets. We begin with the classical Helly's Theorem [5] and an easy consequence of it.

Theorem 1. *Let \mathcal{F} be a family of compact convex subsets of E^d such that the intersection of each family of $d+1$ of these sets is nonempty. Then, the intersection of the whole family \mathcal{F} is nonempty.*

Corollary 1. *Let \mathcal{F} be a family of closed convex subsets of E^d such that at least one of the sets is compact and the intersection of each subfamily of $d+1$ of these sets is nonempty. Then, the intersection of the whole family \mathcal{F} is nonempty.*

The next two results provide descriptions of the mirador of a starshaped set. It is important to remark that these descriptions need no topological and/or dimensional conditions, either on the sets involved or on the space. The first one needs no proof since it is a different form of stating the definition of the mirador (or convex kernel). The second one is proved by Toranzos [9].

Theorem 2. *The set $\text{mir } S$ is the intersection of the stars of all the points of S .*

Theorem 3. *If \mathcal{K} is a covering family of convex components of S , then $\text{mir } S$ is the intersection of \mathcal{K} .*

The following result is easily obtained from Theorem 1 and Theorem 2. The reader can prove it.

Corollary 2. *Let $S \subset E^d$ be compact and finitely starshaped. Then S is starshaped.*

It is well known that an unbounded convex set includes rays issuing from each of its points. The following result is an extension of this property to unbounded starshaped sets. It is proved in [1].

Theorem 4. *Let S be a closed, unbounded and starshaped subset of E^d and let $x \in \text{mir } S$. Then there exists a ray issuing from x and included in S .*

We finish this section with another well known result, usually labelled *Krasnoselsky's Lemma* since it is commonly used as a preparatory result for the classical Krasnoselsky's Theorem that characterizes compact starshaped sets in finite dimensional spaces. Its proof can be found in [12]. Let us remark that the compactness of the set involved in the original statement is unessential, although it is needed for the main Krasnoselsky's Theorem.

Theorem 5. *Let $S \subset E^d$ be a closed set, $x \in S$ and $y \in S$ such that x and y are mutually invisible via S . Then there exist a point $z \in \partial S$ and a hyperplane H that separates x from $st(z, S)$.*

3. FINITELY STARSHAPED SETS THAT ARE CLOSED AND NOT STARSHAPED

We define a *comet* as a closed, finitely starshaped subset S in E^d that is not starshaped. This paper intends to provide some basic geometrical facts about comets. Given a family \mathcal{F} of sets, we say that \mathcal{F} has the *finite intersection property* (abbrev. *fip*) if any finite subfamily of \mathcal{F} has nonempty intersection.

Lemma 1. *Let S be a comet. Then for every $x \in S$ there exists a closed ray issuing from x and included in $st(x, S)$.*

Proof. $\forall x \in S$ denote $K_x = cl\ conv\ st(x, S)$ and define $\mathcal{K} = \{K_x \mid x \in S\}$. The finite starshapedness of S implies that \mathcal{K} has the finite intersection property. From Theorems 2 and 5 it follows easily that $mir\ S$ is the intersection of the family \mathcal{K} . Since S is not starshaped, this intersection must be empty, but all the members of this family are closed and convex sets, whence Corollary 1 implies that none of these sets can be bounded. Consequently, $\forall x \in S, st(x, S)$ is closed and unbounded. Since each of these stars is itself a starshaped set, Theorem 4 implies the thesis. \square

The main theorem of this paper is the following refinement of the previous lemma.

Theorem 6. *Let $S \subset E^d$ be a comet. Then there exists $v_o \in \Omega_d$ such that $\forall x \in S, R(x \rightarrow x + v_o) \subset st(x, S) \subset S$.*

Proof. There exists a countable set $D = \{x_1 ; x_2 ; x_3 ; \dots\}$ dense in S . No point of S can see every point of D , since such a point would belong to $mir\ S$ that is an empty set. By the finite starshapedness (f.s.) of S we can pick a point $y_1 \in S$ that can see both x_1 and x_2 . Let n_1 be the first positive integer such that x_{n_1} is invisible from y_1 . Once more, the f.s. of S yields a new point $y_2 \in S$ that can see y_1 and every x_i with $i \leq n_1$. There must exist another integer $n_2 > n_1$ such that x_{n_2} is the first point of D invisible from y_2 . This inductive process can go on indefinitely yielding a sequence $Y = \{y_1 ; y_2 ; y_3 ; \dots\}$ such that its members not only can see progressively more points of D , but also each y_i can see every y_k with $k \leq i$. This means that every couple of points of Y can see each other via S . It is easy to see that the sequence Y cannot have points of accumulation, since otherwise such points would belong to $mir\ S$. Hence, Y must be unbounded. For each integer $k > 1$ define the point $v_k = \frac{(y_k - y_1)}{\|y_k - y_1\|} \in \Omega_d$. Since Ω_d is compact, the sequence $V = \{v_2 ; v_3 ; v_4 ; \dots\}$ must have a point of accumulation $v_o \in \Omega_d$. We claim that this point v_o is the common direction of the rays that we seek in the thesis. Denote by $\Upsilon(z)$ the statement “The ray $R(z \rightarrow z + v_o) \subset S$ ”. We proceed to prove our claim in three progressive steps.

(a) $\Upsilon(z)$ is **valid** $\forall z \in Y$. Assume the existence of $u = y_1 + \lambda v_o \in S^C$ with $\lambda > 0$. This would imply the existence of $\delta > 0$ such that $U(u, \delta) \subset S^C$. Since Y is unbounded, there would exist $y_k \in Y$ such that $\|y_k - y_1\| > \lambda$ and $\|v_k - v_o\| < \frac{\delta}{\lambda}$. This would imply that $(y_1, y_k) \cap U(u, \delta)$ is nonempty. But such an implication would contradict the fact that y_k sees y_1 via S . Hence, no such u can exist and $\Upsilon(y_1)$ holds. Almost the same argument holds, mutatis mutandi, for every $y_j \in Y$.

(b) $\Upsilon(z)$ is **valid** $\forall z \in D$. Let x_k be a generic point of D and denote $R_k = R(x_k \rightarrow x_k + v_o)$. As in part (a) assume the existence of $u \in R_k \cap S^C$. There would exist $\delta > 0$ such that $U(u, \delta) \subset S^C$. Let n be the first index such that y_n can see x_k . We know by construction that y_m would be able to see x_k for every index $m \geq n$. Furthermore, we can pick an integer m large enough so that v_m is close enough to

v_o and $\|x_k - y_m\|$ large enough to allow that the segment $[x_k, y_m]$ punctures the neighborhood $U(u, \delta)$. But this situation would contradict the fact that x_k and y_m are mutually visible. Hence no such point u can exist, and the claim (b) is proved.

(c) $\Upsilon(z)$ is valid $\forall z \in S$. Now let z be a generic point of S . There exists a sequence $\{z_n\}$ of points of D that converges to z . Hence, every point $p \in R(z \rightarrow z + v_o)$ is the limit of a translated sequence $\{p_n\}$ such that each $p_n \in R(z_n \rightarrow z_n + v_o)$. Since statement (b) implies that such points are in S which is closed, $p \in S$. \square

The geometrical idea of this result is the existence of a fixed direction such that from every point of S “hangs” a halfline (or a cone) entirely included in S and pointing in this direction. Thus the term “comet”.

4. CONSEQUENCES OF THE MAIN THEOREM

Theorem 6 is a result about unbounded sets, hence it is reasonable to try to express it using tools devised precisely to deal with such sets. We will use two of these specific tools:

- The *family of cones* associated to a certain set S . This aspect of Convex Geometry has been largely studied by the Belgian school of Jongmans and Bair (for further information see Jongmans [6]). From the large family of cones studied we will use only three types: the *infinitude cone* $\mathbf{I}(S)$, the *recession cone* $\mathbf{R}(S)$ and the *inscribed cone* at a point s , $I(S, s)$. $\mathbf{I}(S)$ is the cone with vertex at the origin θ formed by the translates of all the halflines included in S . $\mathbf{R}(S)$ is the union of all “rays of recession” of S , i.e. all the rays $R(\theta \rightarrow v)$ issuing from θ and such that $S + R(\theta \rightarrow v) \subset S$. The inscribed cone is defined by $I(S, s) = \{s\} \cup \bigcup_{v \in \Omega_d} \{R(s \rightarrow s + v) | R(s \rightarrow s + v) \subset S\}$ and $I_o(S, s)$ denotes its translate to the origin θ . Then, it is clear that $\mathbf{I}(S) = \bigcup_{s \in S} I_o(S, s)$ and $\mathbf{R}(S) = \bigcap_{s \in S} I_o(S, s)$. Any cone of these three types is considered *trivial* when it is reduced to its vertex. We denote by $J(S, s)$ the cone formed by the opposite directions of $I(S, s)$.
- The *enlarged affine space*, recently introduced by G. Hansen (see [4]), is an integrated approach that allows a unified treatment of bounded and unbounded convex sets. It is essentially a compactification of the affine space E^d , different from the Alexandrov compactification and from the projective space. The enlarged d -dimensional affine space *enl* E^d is formed by the regular space E^d and the set of “improper points” or “directions” defined as equivalence classes of parallel halflines in E^d . This space is provided with a Hausdorff topology that extends the natural topology of E^d and yields *enl* E^d compact. If S is a subset of E^d , the *enlarged of* S is the set *enl* S obtained by adjoining to S the family of directions corresponding to its infinitude cone.

Corollary 3. *If S is a comet in E^d , then $\mathbf{R}(S)$ is not trivial.*

Proof. The direction found in Theorem 6 belongs to $\mathbf{R}(S)$. \square

Corollary 4. *Let S be a comet in E^d . Then *enl* S is a starshaped subset of *enl* E^d . Moreover, *mir enl* S is the set of improper points of *enl* $\mathbf{R}(S)$.*

Proof. The direction yielded by Theorem 6 is an improper element of $mir\ enl\ S$. Furthermore, the definition of the recession cone implies that for any unbounded set M with $\mathbf{R}(M)$ not trivial, the set of improper points of this cone is included in $mir\ enl\ M$, and since S is a comet, $mir\ enl\ S$ cannot include proper points. \square

Corollary 5. *If S is a comet in E^d , then $mir\ enl\ S$ is a convex set in the enlarged affine space.*

Proof. This statement follows immediately from Corollary 3 and the general theory of cones (see [6]) that asserts the convexity of $\mathbf{R}(S)$. \square

Corollary 6. *If S is a comet in E^d and K a convex component of S , then $\mathbf{R}(K)$ is not trivial.*

Proof. Once more, this is an easy consequence of Theorem 6. \square

The next two results connect this study with the idea of external visibility, i.e. visibility inside the complement of a set.

Corollary 7. *Let S be a comet in E^d . Then $enl\ S^C$ is a starshaped set in the enlarged affine space.*

Proof. From Theorem 6 there exists at least one improper point (direction of recession) $v \in enl\ \mathbf{R}(S)$. Then it is easy to verify that $-v \in mir\ enl\ S^C$. \square

Corollary 8. *Let S be a comet in E^d and A a connected component of S^C . Then A is unbounded.*

Proof. This statement follows immediately from the previous result. \square

To understand the visibility behaviour inside nonconvex sets (for example, star-shaped sets) several concepts are used (nova, visibility cell, inner stem, etc.) that define particular subsets of the set considered. We obtain some properties of these subsets when the set considered is a comet. Given a set $S \subset E^d$ and a point $x \in S$ we define the *nova* of x in S as the set $nova(x, S) = \{y \in st(x, S) \mid \exists U_x \text{ such that } U_x \cap S \subset st(y, S)\}$. The *visibility cell* of x in S is the set $vis(x, S) = \{y \in S \mid st(x, S) \subset st(y, S)\}$. If $x \in S$, a ray $R(x \rightarrow y)$ is *inward through* y if there exists $t \in R(xy \rightarrow)$ such that $(y, t) \subset int\ S$. Otherwise we say that $R(x \rightarrow y)$ is *outward through* y . If $x \in \partial S$, we define the *inner stem* of x in S as the set $ins(x, S)$ formed by x and all the points $y \in st(x, S)$ such that $R(y \rightarrow x)$ is outward through x . The *strong inner stem* of a point $x \in \partial S$ is the set $sins(x, S) = J(cl\ S^C, x) \cap st(x, S)$ (see [8]). It is easy to verify that $sins(x, S) \subset ins(x, S)$. If $S \subset E^d$, we define the *algebraic hull* of S as the set ${}^a S = \{y \in E^d \mid \exists x \in S \text{ such that } [x, y] \subset S\}$. A set S such that ${}^a S = S$ and ${}^a(S^C) = cl\ S^C$ has the *shining boundary property* if S^C is free from bounded connected components and for each point $x \in \partial S$ there exists a ray issuing from x and disjoint from the interior of S (see again [8]).

Theorem 7. *Let S be a comet in E^d and $x \in \partial S$. Then $\mathbf{I}(ins(x, S))$ is not trivial.*

Proof. Let v be the direction of recession of S provided by Theorem 6, and assume that $x + v \notin ins(x, S)$. There would exist $t \in R(x \rightarrow x - v)$ such that $(x, t) \subset int\ S$. Taking $t' \in (x, t)$ and $U \subset int\ S$ an open neighborhood of t' , we get from Theorem 6 that the “cylinder” $C = \bigcup_{z \in U} R(z \rightarrow z + v)$ is included in S , whence $x \in int\ S$, a contradiction. Hence, $x + v \in ins(x, S)$ and $(R(x \rightarrow x + v))^\sim \{x\} \subset \mathbf{I}(ins(x, S))$. \square

Theorem 8. *Let S be a comet in E^d such that ${}^a S = S$ and ${}^a(S^C) = cl S^C$. Then S enjoys the shining boundary property.*

Proof. Notice that S^C has no bounded connected components, according to Corollary 8. Besides, in the proof of Theorem 7 we have seen that the ray $R(x \rightarrow x - v)$ has empty intersection with $int S$. \square

Corollary 9. *Let S be a comet in E^d and $x \in \partial S$. Then $\mathbf{I}(sins(x, S))$ is not trivial.*

Proof. This statement follows immediately from Theorem 7 and the remark made in the proof of Theorem 8. \square

Theorem 9. *Let S be a comet in E^d and $x \in S$. Then $\mathbf{R}(vis(x, S))$ is not trivial.*

Proof. Let v be the direction of recession provided by Theorem 6. According to [10], the visibility cell of x is the intersection of all convex components of S that include the point x . But Corollary 6 says that each of these components has a nontrivial cone of recession, since all of them include the ray $R(x \rightarrow x + v)$. Clearly, the same is true for $vis(x, S)$. \square

It would have been nice to prove for the nova of a boundary point of S a result similar to those just proved for convex components, inner stems, strong inner stems and visibility cells. Unfortunately, such a result is false as an appropriate counterexample (Conjecture 4) shows below.

5. A CHARACTERIZATION OF PLANAR COMETS

More than thirty years ago it was proved (see Theorem 3, above) that the mirador of a starshaped set is the intersection of any covering family of convex components. It is important to remark that such a characterization imposed absolutely no topological or dimensional restriction on the set or on the space. This statement provides an immediate characterization of starshaped sets via their convex components.

Theorem 10. *The set S is starshaped if and only if every covering family of convex components has nonempty intersection.*

We intend to obtain in this section, and in the planar case, a similar result that characterizes finitely starshaped sets and, a fortiori, comets. In order to get this generalization we need a technical lemma that can be labelled “*the surveillance cone lemma*” in the sense that a certain cone $C \subset S$ is selected with the property that any point of S can be seen from some point of C .

Lemma 2. *Let S be a planar comet and $\mathbf{R}(S)$ its recession cone. Then there exists $x_o \in S$ such that any point of S is visible from some point of $\mathbf{R}(S) + \{x_o\}$.*

Proof. We consider two alternatives:

[1] $\exists x_o \in S$ such that $st(x_o, S) = \mathbf{R}(S) + \{x_o\}$.

Let w be a generic point of S . Since S is finitely starshaped, there must exist $y \in S$ that sees both x_o and w . From the assumption [1], $y \in \mathbf{R}(S) + \{x_o\}$ and the thesis holds.

[2] $\forall x_o \in S \exists w \in st(x_o, S)$ such that $w \notin \mathbf{R}(S) + \{x_o\}$.

Let x_o be a point of S and let w be a point that verifies statement [2]. Denote by t a point that sees both x_o and w , and split statement [2] into two subalternatives:

[2.a] $w \in \mathbf{R}(S) + \{t\}$.

In this case $\left(\bigcup_{x \in [t, x_o]} \mathbf{R}(S) + \{x\}\right) \subset S$. This implies that $\mathbf{R}(S) + \{x_o\} \subset st(w, S)$, and the thesis holds.

[2.b] $w \notin \mathbf{R}(S) + \{t\}$.

We may assume that $(\mathbf{R}(S) + \{x_o\}) \cap (\mathbf{R}(S) + \{w\}) = \emptyset$ (*), since otherwise the thesis holds immediately. As we are in the plane, statement (*) implies that $\mathbf{R}(S)$ reduces to a halfline having direction v , whence $\mathbf{R}(S) + \{x_o\} = R(x_o \rightarrow x_o + v)$ and $\mathbf{R}(S) + \{w\} = R(w \rightarrow w + v)$. From (*) it follows that $w \notin R(x_o \rightarrow x_o + v)$ and $x_o \notin R(w \rightarrow w + v)$. Denote $L = L(x_o \rightarrow w)$. It is clear that the direction of L is not v . The whole plane split into four convex regions, $(^1) L^+$, the closed halfplane bounded by L that includes neither $R(w \rightarrow w + v)$ nor $R(x_o \rightarrow x_o + v)$; $(^2) F = \bigcup_{x \in [w, x_o]} R(x \rightarrow x + v)$; $(^3) Z_w$, the connected component of $(L^+ \cup F)^C$ that includes w on its boundary; $(^4) Z_{x_o}$, the connected component of $(L^+ \cup F)^C$ that includes x_o on its boundary. We have to check the validity of the thesis according with the position of the point t .

- If $t \in L^+$, then $\bigcup_{x \in [x_o, t] \cup [t, w]} R(x \rightarrow x + v) \subset S$ and the thesis holds.
- If $t \in F \cup Z_{x_o}$, define a such that $R(w \rightarrow t) \cap R(x_o \rightarrow x_o + v) = \{a\}$. This point verifies the thesis.
- If $t \in Z_w$, consider the open halfplane H_w bounded by $L(w \rightarrow w + v)$ that includes x_o . In the case that $\exists p \in st(w, S) \cap H_w$ the validity of the thesis follows from an argument similar to that used in the first case, since

$$\bigcup_{x \in [x_o, t] \cup [t, w] \cup [w, p]} R(x \rightarrow x + v) \subset S.$$

Hence we may assume that $st(w, S) \subset (H_w)^C$ (\blackboxtimes). Furthermore, there cannot exist two different points w_1 and w_2 satisfying property (\blackboxtimes) and such that $H_{w_1} \cap H_{w_2} = \emptyset$, since this would contradict the finite starshapedness of S . In this case we may use the unique point w_o in the role of x_o of the thesis. Let z be a generic point of S and pick $q \in S$ such that q sees both w_o and z . If $z \in H_{w_o}$, then z and q are strictly separated by $L(w_o \rightarrow w_o + v)$ and a point $b \in [q, z] \cap R(w_o \rightarrow w_o + v)$ would satisfy the thesis. Otherwise, if $z \notin H_{w_o}$, it holds that $\bigcup_{x \in [z, q] \cup [q, w_o]} R(x \rightarrow x + v) \subset S$, which implies the thesis.

Since all the possible alternatives have been considered, the lemma is proved. \square

Theorem 11. *Let S be a closed planar set. The following statements are equivalent:*

- (1) S is finitely starshaped.
- (2) S admits a covering family of convex components which enjoys fip.

Proof. (1) \implies (2). If S is starshaped, Theorem 3 asserts that any covering family of convex components has nonempty intersection, and statement (2) is trivial. Hence, we may assume that S is a comet. From the previous result there exist a point x_o and a cone $C = \mathbf{R}(S) + \{x_o\}$ such that any point of S is visible from some point of that cone. It is possible to associate to a generic point $x \in S$ a point $\hat{x} \in C$ such that $[x, \hat{x}] \subset S$, and we can obtain a convex component K_x that includes this interval. Define $\mathcal{K} = \{K_x \mid x \in S\}$ that is clearly a covering family of convex

components. For each $p \in C$ denote $C_p = \mathbf{R}(S) + \{p\}$. It is easy to verify that given $p', p'' \in C$ there exists a unique point $p_o \in C$ such that $C_{p_o} = C_{p'} \cap C_{p''}$. This *infimum* operation on C is clearly associative. Let $\mathcal{F} = \{K_{x_1}; K_{x_2}; \dots; K_{x_n}\}$ be a finite subfamily of \mathcal{K} . By construction we obtain $\{p_1; p_2; \dots; p_n\} \subset C$ such that each p_i can see the corresponding x_i . Let p_o be the infimum of the finite set $\{p_1; p_2; \dots; p_n\}$. Then it holds that $C_{p_o} \subset \bigcap_{i=1}^n C_{p_i} \subset \bigcap_{i=1}^n K_{x_i}$.

(2) \implies (1). Assume that \mathcal{F} is the covering family of convex components of S . Given a finite subset $\{t_1; t_2; \dots; t_k\} \subset S$ we may get for each index i a convex component $K_i \in \mathcal{F}$ such that $t_i \in K_i$. It is clear that $\forall i K_i \subset st(t_i, S)$. Hence, any point $x \in \bigcap_{i=1}^k K_i$ can see every t_i . □

The conjunction of Theorem 10 and Theorem 11 yields the following result.

Theorem 12. *Let S be a closed planar set. Then the following statements are equivalent:*

- (1) S is a comet.
- (2) S admits a covering family of convex components which enjoy *fip* but have empty intersection.

6. FALSE CONJECTURES

In this section we present some plausible but false conjectures regarding comets. We prove its falsehood by exhibiting appropriate counterexamples. Throughout this section we will consider a comet $S \subset E^d$.

Conjecture 1. S must contain a line in some direction of recession.

Let $S = \{(x, y) \in E^2 \mid y \leq x^2\}$. The set $\mathbf{R}(S)$ is formed only by the ray issuing from θ and pointing downwards, and clearly S does not include any line in this direction.

Conjecture 2. $\dim(\text{mir enl } S) \leq d - 2$.

Let $S = \{(x, y) \in E^2 \mid y \leq x^2, x \geq 0\} \cup \{(x, y) \mid y \leq -x, -1 \leq x \leq 0\}$. Notice that $\text{mir enl } S$ is the convex arc of directions (or improper points) between $(0, -1)$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then $\dim(\text{mir enl } S) = 1$.

A point $x \in S$ is a *point of local nonconvexity* of S if given any neighborhood U_x of x , it holds that $U_x \cap S$ is not a convex set. Let $\text{lnc } S$ denote the set of all the points of local nonconvexity of S .

Conjecture 3. If $\text{lnc } S$ is compact, then S is starshaped.

If $S = \bigcup_{i \in \mathbf{N}} \left\{ (x, y) \in E^2 \mid y \leq \left(\frac{-2^i - 4^i}{2} \right) x + \frac{4^i - 2^i}{2}, 1 - \frac{1}{2^{i-1}} \leq x \leq 1 - \frac{2}{1+2^i} \right\} \cup \{(x, y) \in E^2 \mid x \geq 0, y \leq 0\} \cup \{(x, y) \in E^2 \mid x = 1, y \leq 1\}$, this is a comet, and its set of points of local nonconvexity is the following one, which is compact,

$$\begin{aligned} \text{lnc } S = & \bigcup_{i \in \mathbf{N}} \left\{ (x, y) \in E^2 \mid x = 1 - \frac{2}{1+2^i}, y = 0 \right\} \\ & \cup \left\{ (x, y) \in E^2 \mid x = 1 - \frac{1}{2^{i-1}}, y = 0 \right\} \cup \{(x, y) \in E^2 \mid x = 1, 0 \leq y \leq 1\}. \end{aligned}$$

Conjecture 4. If $x \in S$, then $\mathbf{I}(\text{nova}(x, S))$ is not trivial.

The set is the one defined in the previous counterexample, and the point is $(1, 0)$.

Conjecture 5. The intersection of any finite family of convex components is non-empty.

Let

$$S = \left\{ (x, y) \in E^2 \mid y \leq \frac{1}{1-x^2}, -1 < x < 1 \right\} \cup \{ (x, y) \in E^2 \mid x = 1 \} \\ \cup \{ (x, y) \in E^2 \mid x = -1 \}.$$

Let us define the following subsets of S :

$K_1 = \{ (x, y) \in E^2 \mid x = 1 \}$ and $K_2 = \{ (x, y) \in E^2 \mid x = -1 \}$. These sets are convex components of S and clearly $K_1 \cap K_2 = \emptyset$. The refinement of the statement of this false conjecture led to the characterization of planar comets given in Section 5. Still open is the possibility of generalizing this result to dimension greater than two.

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