PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 5, Pages 1389–1390
S 0002-9939(99)05129-1
Article electronically published on August 5, 1999

# A REMARK ON ANALYTIC CONTINUATION

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(Communicated by Frederick W. Gehring)

ABSTRACT. A simple example is given to show that the space of germs obtained by analytic continuation of a given germ need not be a covering space in the topological sense.

There is a significant difference between the definition of a covering surface used by complex analysts and that used by topologists. In complex analysis one constructs the space $G$ of all germs that can be obtained by analytic continuation of a given analytic function $f$, and then $G$ is a covering space of the natural domain of the extended function in the sense that every germ $g$ in $G$ has a neighbourhood on which the natural map (taking a germ at $z_0$ to the point $z_0$) is a homeomorphism. In topology, however, a covering surface $\tilde{X}$ of a surface $X$ is given by a covering map $p : \tilde{X} \to X$ with the property that each $x$ in $X$ has a neighbourhood $N$ such that the restriction of $p$ to each component $N_j$ of $p^{-1}(N)$ is a homeomorphism of $N_j$ onto $N$. The topological notion of a covering surface is equivalent to what complex analysts often call a regular covering surface (see [1], p.29), although for topologists, a regular covering space has an entirely different meaning (see [2], p. 163). The difference between the two definitions of a covering surface can be seen by comparing the proofs of the two versions of the Monodromy Theorem in the standard references [1] and [2] (and on p.149 and Lemma 3.3, p.152, in [2]).

One way to show that the topologist's version of a covering space is not always applicable to analytic continuation is to construct a function $f$ whose analytic continuation has infinitely many branches, say $f_n$, at some point $z_0$, and which is such that the radius of convergence $r_n$ of the Taylor series for the branch $f_n$ at $z_0$ tends to zero as $n \to \infty$ (at least on a subsequence). I am not aware of such an example in the literature, and here we give an explicit and very simple example.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, and let $a_1, a_2, \dots$ be distinct points in $\mathbb{D}$ such that $\sum(1 - |a_n|)$ converges, and such that the $a_n$ accumulate at every point of the unit circle $\partial\mathbb{D}$. With these assumptions, the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \left( \frac{z - a_n}{1 - \bar{a}_n z} \right)$$

is an analytic map of $\mathbb{D}$ into itself with zeros at and only at the $a_n$ (each zero being a simple zero), and with natural boundary $\partial\mathbb{D}$. Further, as $B$ is a contraction of

Received by the editors April 21, 1998 and, in revised form, June 25, 1998.
1991 *Mathematics Subject Classification.* Primary 30B40.

the hyperbolic metric in $\mathbb{D}$ (the Schwarz-Pick Lemma), we see that

$$0 < |B'(a_n)| \leq \frac{1-|B(a_n)|^2}{1-|a_n|^2} = \frac{1}{1-|a_n|^2} \leq \frac{1}{1-|a_n|}. \tag{1}$$

For each $n$, $a_n$ is a simple zero of $B$ so we can define a branch $f_n$ of $B^{-1}$ at 0 that satisfies $f_n(0) = a_n$, and it is clear that the $f_n$ are analytic continuations of each other. Now let $r_n$ and $\Delta_n$ be the radius of convergence, and the disc of convergence, respectively, of the Taylor series for $f_n$ about 0, so that $\Delta_n$ is given by $|z| < r_n$. Then $f_n$ maps $\Delta_n$ into $\mathbb{D}$ (as otherwise, we could analytically continue $B$ beyond $\mathbb{D}$), and hence $f_n$ is a homeomorphism of $\Delta_n$ onto $f_n(\Delta_n)$ with inverse $B$. In particular, for each $n$, if $m \neq n$, then

$$a_m \notin f_n(\Delta_n). \tag{2}$$

Next, as $f_n$ is univalent on $\Delta_n$, Koebe's 1/4-Theorem is applicable; thus $f_n(\Delta_n)$ contains the open disc with centre $a_n$ and radius $r_n|f_n'(0)|/4$. From this, together with (1), (2) and the fact that $|f_n'(0)B'(a_n)| = 1$, we have for $m \neq n$,

$$4|a_n - a_m| \geq r_n|f_n'(0)| \geq r_n(1-|a_n|). \tag{3}$$

It is clear that we can construct the $a_n$ satisfying the earlier assumptions, and also such that (for example) for all $k$, $|a_{2k} - a_{2k+1}| \leq (1-|a_{2k}|)^2$, and this with $n = 2k$ and $m = 2k+1$ in (3) shows that $r_{2n} \to 0$. This concludes the example.

## References

[1] Ahlfors, L.V. and Sario, L., *Riemann surfaces*, Princeton, 1960. MR **22:**5729

[2] Massey, W.S., *Algebraic Topology: An Introduction*, GTM 56, Springer-Verlag, 1967. MR **35:**2271

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