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THE CONVERSE OF THE INVERSE–CONJUGACY THEOREM FOR UNITARY OPERATORS AND ERGODIC DYNAMICAL SYSTEMS

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Abstract. We show that the converse to the main theorem of *Ergodic transformations conjugate to their inverses by involutions*, by Goodson et al. (Ergodic Theory and Dynamical Systems **16** (1996), 97–124), holds in the unitary category. Specifically it is shown that if U is a unitary operator defined on an L^2 space which preserves real valued functions, and if $U^{-1}S = SU$ implies $S^2 = I$ whenever S is another such operator, then U has simple spectrum. The corresponding result for measure preserving transformations is shown to be false. The counter-example we have involves Gaussian automorphisms. We show that a Gaussian automorphism is always conjugate to its inverse, so that the Inverse-Conjugacy Theorem is applicable to such maps having simple spectrum. Furthermore, there are Gaussian automorphisms having non-simple spectrum for which every conjugation of T with T^{-1} is an involution.

§0. Introduction

Let $U : H \to H$ be a unitary operator defined on a separable Hilbert space H. U is said to have *simple spectrum* if there exists an $h \in H$ for which $Z(h) = H$, where $Z(h)$ is the closed linear span of the set $\{U^n h : n \in \mathbb{Z}\}$. We are mainly interested in the case where the Hilbert space is a function space. Let $(X, \mathcal{F}, \mu)$ denote a standard Borel probability space, and let $T : X \to X$ be an invertible measure preserving transformation (*automorphism*). To say that the automorphism T has simple spectrum means that the unitary operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$, $U_T f(x) = f(Tx)$, induced by T, has simple spectrum. It is known that if T has simple spectrum, then T is ergodic, and that unitary operators preserving real valued functions are conjugate to their inverses.

The following theorem and corollary were given in [2] (see also [3] and [4] for related results). We use I to denote both the identity operator, and the identity automorphism.

Inverse Conjugacy Theorem. *Let $U : L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{F}, \mu)$ be a unitary operator having simple spectrum and preserving real valued functions. If S is another unitary operator preserving real valued functions and $U^{-1}S = SU$, then $S^2 = I$.*

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Corollary. *Let $T : X \to X$ be an automorphism having simple spectrum. If an automorphism S satisfies $T^{-1}S = ST$, then $S^2 = I$.*

Our aim is to show that the converse of the Inverse Conjugacy Theorem is also true, but that the converse of the corollary is false. This tells us that in some sense the unitary centralizer of an operator is big enough to distinguish operators which have simple spectrum, while the measure theoretic centralizer is not. This is analogous to the well known result that the unitary centralizer of U is abelian if and only if U has simple spectrum, while the measure–theoretic centralizer of an automorphism T may be abelian even when T has non–simple spectrum (see [3]). Specifically we prove:

Theorem 1. *Let $U : L^2(X, \mathcal{F}, \mu) \to L^2(X, \mathcal{F}, \mu)$ be a unitary operator preserving real valued function for which*

$$US = SU^{-1} \quad \Rightarrow \quad S^2 = I$$

whenever S is a unitary operator preserving real valued functions. Then U has simple spectrum.

Our counter–example to the converse of the corollary is obtained by investigating the spectral properties of Gaussian automorphisms. It is shown that a Gaussian automorphism is conjugate to its inverse (see also [5] for a different proof of this) and we deduce conditions under which any conjugation between a Gaussian automorphism and its inverse is an involution. Then we show that some Gaussian automorphisms having infinite maximal spectral multiplicity have this property, giving us the desired counter–examples.

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§1. Preliminaries

Let $U : H \to H$ be a unitary operator defined on a separable Hilbert space H, and having simple spectrum. Suppose $h \in H$ satisfies $Z(h) = H$; then there exists a finite Borel measure σ_h (called the *spectral type of* h), defined on the unit circle S^1 in the complex plane for which

$$(U^n h, h) = \int_{S^1} z^n \, d\sigma_h, \quad n \in \mathbb{Z},$$

and such that U is unitarily equivalent to $V : L^2(S^1, \sigma_h) \to L^2(S^1, \sigma_h)$ defined by $Vf(z) = zf(z)$.

In addition, there is a natural unitary equivalence

$$\Theta : (U, Z(h)) \to (V, L^2(S^1, \sigma_h)),$$

determined by $\Theta(U^n h) = p_n(z)$, where $p_n(z) = z^n$, $n \in \mathbb{Z}$.

We use the symbol $\cong$ (as in $(U, Z(h)) \cong (V, L^2(S^1, \sigma_h))$) to indicate unitary equivalence.

In the case where $U : H \to H$ does not necessarily have simple spectrum, there is a way of representing H as a direct sum of cyclic subspaces: Let A_n denote the subspace of those elements of H whose spectral types have uniform multiplicity n, $n \in \mathbb{Z}^+$, so that A_n may be trivial. Also denote by A_∞ those elements with uniformly infinite multiplicity. Each A_n, $1 \le n \le \infty$, is a U–invariant subspace, and if non–trivial may be decomposed into the direct sum of n cyclic subspaces generated by elements $x_1, x_2, \dots, x_n$, $A_n = \bigoplus_{i=1}^{n} Z(x_i)$, such that each x_i has the same spectral type σ_n say. The type of the measure $\bigoplus_{n=1}^{\infty} \sigma_n \oplus \sigma_\infty$ is called the *maximal spectral type* of U (see Halmos [6]).

The spectral properties of an automorphism $T : X \to X$ are those of the induced unitary operator $U_T : L^2(X,\mu) \to L^2(X,\mu)$, $U_T f(x) = f(Tx)$.

§2. Spectral properties of unitary operators equivalent to their inverses

In this section we prove Theorem 1. Let $U : H \to H$ be a unitary operator defined on a separable Hilbert space H. Denote by σ_h the maximal spectral type of U. σ_h is a Borel measure defined on the unit circle S^1 in the complex plane, which we assume is symmetric, i.e., $\sigma_h(A) = \sigma_h(\overline{A})$ for each Borel set $A \subseteq S^1$ (where $\overline{A}$ indicates that we have taken the complex conjugate of each element of A). This condition is sufficient to ensure that U is unitarily equivalent to its inverse. A unitary operator $U : L^2(X,\mu) \to L^2(X,\mu)$ preserving real valued functions always has this property (see [2]).

Lemma 1. *Let $U : H \to H$ be a unitary operator and suppose that $H = \bigoplus_{n=1}^{\infty} A_n \oplus A_\infty$ is the decomposition of H into the U–invariant subspaces A_n of elements whose spectral types have uniform multiplicity n. If each corresponding spectral measure σ_n, $1 \le n \le \infty$, is symmetric, then*

(i) U is unitarily equivalent to its inverse, and there exists a unitary operator S with $U^{-1}S = SU$ and $S^2 = I$,

(ii) if U has non-simple spectrum, then the unitary operator S may be chosen so that $U^{-1}S = SU$, $S^4 = I$ and $S^2 \neq I$.

Proof. (i) For each n, $1 \le n \le \infty$, if A_n is non–trivial we can write $A_n = \bigoplus_{i=1}^{n} Z(h_i)$ where each h_i has the same spectral type σ_n, which is symmetric.

Now $U|Z(h_i)$ is unitarily equivalent to the operator $V_n : L^2(S^1,\sigma_n) \to L^2(S^1,\sigma_n)$ defined by $V_n f(z) = zf(z)$. It will suffice to prove the lemma for the operator V_n. Note that $V_n^{-1} f(z) = \overline{z} f(z)$.

Define $S_n : L^2(S^1,\sigma_n) \to L^2(S^1,\sigma_n)$ by $S_n f(z) = f(\overline{z})$. Then S_n is a unitary operator since the measure σ_n is symmetric. Also

$$S_n V_n f(z) = S_n(zf(z)) = \overline{z} f(\overline{z}) = V_n^{-1}(f(\overline{z})) = V_n^{-1} S_n f(z),$$

i.e., $S_n V_n = V_n S_n^{-1}$, and clearly $S_n^2 = I$.

(ii) If U has non-simple spectrum, then some A_m, $2 \le m \le \infty$, will be non–trivial, and hence there are $j, k \in \mathbb{Z}^+$, $j \neq k$, such that

$$U|Z(h_j) \cong U|Z(h_k) \quad \text{and} \quad Z(h_j) \perp Z(h_k),$$

where $Z(h_j)$ and $Z(h_k)$ are cyclic subspaces contained in A_m.

We define $S : H \to H$ in the following way. For each $n, 1 \le n \le \infty$, $n \neq m$, define S on the n cyclic subspace making up A_n as in (i) above. Then we see that $S^2|A_n = I$ for $n \neq m$. Define S in the same way on the other cyclic subspaces of A_m (except for $Z(h_j)$ and $Z(h_k)$).

Since $\sigma_{h_j} \equiv \sigma_{h_k} = \sigma_m$, we can think of $L^2(S^1,\sigma_{h_j})$ and $L^2(S^1,\sigma_{h_k})$ as two copies of the same space $L^2(S^1,\sigma_m)$, with V_m acting on each of them. Define $S'_m : L^2(S^1,\sigma_m) \oplus L^2(S^1,\sigma_m) \to L^2(S^1,\sigma_m) \oplus L^2(S^1,\sigma_m)$ by

$$S'_m \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} 0 & S_m \\ -S_m & 0 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} f_2(\overline{z}) \\ -f_1(\overline{z}) \end{pmatrix}.$$

It is now easy to check that $S'_m \circ (V_m \oplus V_m) = (V_m^{-1} \oplus V_m^{-1}) \circ S'_m$, and $(S'_m)^2 = -I$. If we now lift S'_m back to $Z(h_j) \oplus Z(h_k)$, we see that we have a unitary operator S defined on this subspace, which when combined with the way S is defined on the other subspaces, gives the required result. $\square$

Remarks. Now suppose that $H = L^2(X, \mu)$ and that $U : H \to H$ preserves real valued functions. Let $f \in H$, where σ_f denotes its *spectral measure*, i.e., the unique Borel measure on S^1 satisfying

$$\widehat{\sigma}_f(n) = \int_{S^1} z^n \, d\sigma_f(z) = \int_X U^n f \cdot \overline{f} \, d\mu. \tag{1}$$

Since U preserves real valued functions, it also preserves complex conjugation, i.e., $U\overline{f} = \overline{Uf}$ for all $f \in H$. It follows that

$$\widehat{\sigma}_{\overline{f}}(n) = \int_X U^n \overline{f} \cdot f \, d\mu = \int_X \overline{f} \cdot U^{-n} f \, d\mu = \widehat{\sigma}_f(-n).$$

This means that $\sigma_{\overline{f}}$ is a measure on the circle which is the image of σ_f via the map $z \to \overline{z}$. If f realizes the maximal spectral type of U, then $\sigma_{\overline{f}} \ll \sigma_f$ and consequently σ_f has to have the type of a symmetric measure. So, if we select f in such a way that σ_f is a symmetric measure (such an f does exist), then in particular we have $\widehat{\sigma}_f(n) = \overline{\int_{S^1} z^n \, d\sigma_f(z)} \in \mathbf{R}$, $\forall n \in \mathbb{Z}$. Finally, we point out that it was shown in [2] that the maximal spectral type σ_h can be chosen in such a way that h is a real valued function.

We now apply Lemma 1 to the case we are dealing with. First, notice that the maximal spectral type of U is symmetric. It is easy to see that each measure in the spectral decomposition is symmetric, so the Lemma is applicable. To prove Theorem 1, it suffices to show that S can be chosen so as to preserve real valued functions. The proof which now follows, uses ideas and notation introduced in [2].

Let us write $H_{\mathbf{R}} = L^2_{\mathbf{R}}(X, \mu)$, the real valued functions in H; then for each $f \in H_{\mathbf{R}}$, $Uf = U\overline{f} = \overline{Uf}$, since $U(H_{\mathbf{R}}) = H_{\mathbf{R}}$. Suppose that $h \in H_{\mathbf{R}}$ realizes the maximal spectral type of U; then we can write

$$\mathcal{H} = L^2(S^1, \sigma_h),$$

where σ_h is a symmetric measure. Now let

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(\sigma_h) = \{g \in \mathcal{H} : g(\overline{z}) = \overline{g(z)}\}.$$

For $V : \mathcal{H} \to \mathcal{H}$, $Vf(z) = zf(z)$, the following was proved in [2].

Lemma 2. *(i)* $\tilde{\mathcal{H}}$ *is a subspace over* $\mathbf{R}$*, which is closed and* V*–invariant.*

(ii) $\tilde{\mathcal{H}}$ *is the closure of* $\{\sum_{-n}^{n} a_k z^k : a_k \in \mathbf{R}, n \geq 0\}$.

(iii) If $f \in H_{\mathbf{R}}$*, then* $\Theta(H_{\mathbf{R}} \cap Z(f)) = \tilde{\mathcal{H}}(\sigma_f)$*, where* Θ *is the natural unitary equivalence between* $Z(f)$ *and* $L^2(S^1, \sigma_f)$.

Proof of Theorem 1. Let $L^2(X, \mu) = \bigoplus_{n=1}^{\infty} A_n \cup A_\infty$, $A_n = \bigoplus_{i=1}^{n} Z(h_i^{(n)})$, $1 \leq i \leq n$, $1 \leq n \leq \infty$, be a spectral decomposition for U, where each $h_i^{(n)}$, $1 \leq i \leq n$, $1 \leq n \leq \infty$, may be chosen to be real, so that the corresponding measures σ_n are symmetric.

For each $i = 1, 2, \ldots$, there is a natural unitary equivalence

$$\Theta_i^{(n)} : (U, Z(h_i^{(n)})) \cong (V_n, L^2(S^1, \sigma_n)),$$

where $V_n f(z) = zf(z)$. Write $\Theta^{(n)} = \bigoplus_{i=1}^{n} \Theta_i^{(n)}$, and $\Theta = \bigoplus_{n=1}^{\infty} \Theta^{(n)} \oplus \Theta^{(\infty)}$, the equivalence between $U : H \to H$ and $V : \bigoplus_{n,i} L^2(S^1, \sigma_n) \to \bigoplus_{n,i} L^2(S^1, \sigma_n)$, where $V = \bigoplus_{n,i} V_n = (\bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{n} V_n) \oplus (\bigoplus_{m=1}^{\infty} V_\infty)$.

Suppose that U has non-simple spectrum, and define an equivalence $\tilde{S}$ in the following way using the idea of Lemma 1(ii):

We may assume that there exists m such that A_m is non–trivial. Using the natural unitary equivalence there are $k, j \in \mathbb{Z}^+$, $j \neq k$, and $\sigma_{h_j^{(m)}} \equiv \sigma_{h_k^{(m)}} = \sigma_m$ (say), so that

$$L^2(S^1, \sigma_{h_j^{(m)}}) \equiv L^2(S^1, \sigma_{h_k^{(m)}}) = L^2(S^1, \sigma_m).$$

Define $S'_m : L^2(S^1, \sigma_m) \oplus L^2(S^1, \sigma_m) \to L^2(S^1, \sigma_m) \oplus L^2(S^1, \sigma_m)$ by

$$S'_m \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} 0 & S_m \\ -S_m & 0 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} f_2(\overline{z}) \\ -f_1(\overline{z}) \end{pmatrix}.$$

Now define $S : \bigoplus_{n,i} L^2(S^1, \sigma_i) \to \bigoplus_{n,i} L^2(S^1, \sigma_i)$ by:

$$S | L^2(S^1, \sigma_i) = S_i, \quad i \neq m.$$

When $i = m$, S is defined in the same way except that we define S to be S'_m for S restricted to $L^2(S^1, \sigma_{h_j^{(m)}}) \oplus L^2(S^1, \sigma_{h_k^{(m)}})$ (where S_i is defined by $S_i : L^2(S^1, \sigma_i) \to L^2(S^1, \sigma_i)$, $S_i f(z) = f(\overline{z})$). It is clear that $S \circ V = V^{-1} \circ S$ and that $S^2 \neq I$.

To complete the proof it suffices to show that $\tilde{\mathcal{H}}$ is S–invariant, where

$$\tilde{\mathcal{H}} = \bigoplus_{n,i} \tilde{\mathcal{H}}_n \; ; \quad \tilde{\mathcal{H}}_n = \{g \in L^2(S^1, \sigma_n) : g(\overline{z}) = \overline{g(z)}\}.$$

This is clear, for if $f \in \tilde{\mathcal{H}}_n$, then $(S_n f)(\overline{z}) = f(z) = \overline{f(\overline{z})} = \overline{(S_n f)(z)}$, for $n \neq m$. A similar argument works when $n = m$.

Now, if we map S back to H, i.e., put $\tilde{S} = \Theta^{-1} \circ S \circ \Theta$, then $H_{\mathbf{R}}$ is $\tilde{S}$–invariant and $\tilde{S}^2 \neq I$. The theorem now follows. □

§3. Gaussian automorphisms

Let $T : X \to X$ be an ergodic automorphism on a standard Borel probability space $(X, \mathcal{F}, \mu)$. We examine the case of Gaussian automorphisms, showing that such an automorphism is always conjugate to its inverse (see also [5]). It follows that the corollary to the Inverse Conjugacy Theorem is applicable when such an automorphism has simple spectrum. We shall see that this analysis also applies to certain Gaussian automorphisms having infinite spectral multiplicity. We will need some basic information from the theory of Gaussian automorphisms, which can be found in Cornfeld-Fomin-Sinai [1, Chapters 8 and 14]. First, we give a result from [2] which is needed.

Definition. Let $\mathcal{A} \subseteq L^2_{\mathbf{R}}(X, \mu)$ be a non-empty family. We say that $\mathcal{A}$ *generates* the σ-algebra $\mathcal{F}$ if the smallest σ-algebra containing all sets

$$\{a^{-1}(C) : C \subseteq \mathbf{R} \;\; \text{Borel}, \;\; a \in \mathcal{A}\}$$

is equal to $\mathcal{F}$.

Proposition 1 ([2]). *Assume that there exists a T-invariant subspace $\mathcal{A}\subset L^2_{\mathbf{R}}(X,\mu)$ generating $\mathcal{F}$ on which T has simple spectrum. If the maximal spectral type of T on $\mathcal{A}^\perp$ is orthogonal to the one on $\mathcal{A}$, then each S conjugating T to T^{-1} is an involution.*

Definition. An automorphism $T:(X,\mathcal{F},\mu)\to(X,\mathcal{F},\mu)$ is said to be *Gaussian* if there exists a real function $h_0\in L^2(X,\mathcal{F},\mu)$ with zero mean and which is a *Gaussian element*, i.e., for every family of integers $k_1,k_2,\dots,k_n$ and Borel sets $A_1,\dots,A_n\subset\mathbf{R}$,

$$\mu(\{x\in X: h_{k_j}(x)\in A_j, j=1,\dots n\}) = c\cdot\int\dots\int_{A_1\times\dots\times A_n} e^{-\frac{1}{2}(Dt,t)}\,dt,$$

where c is a constant, $t=(t_1,\dots,t_n)\in\mathbf{R}^n$, $h_s=h_0T^s$, $s\in\mathbb{Z}$, and D is the matrix inverse to $B=[b_{ij}]_{1\le i,j\le n}$, $b_{ij}=\int_X h_0T^{k_j-k_i}h_0\,d\mu$, such that the Gaussian element generates $\mathcal{F}$, meaning that the smallest σ-algebra containing all the sets of the form $h_n^{-1}(A)$, $n\in\mathbb{Z}$, $A\subset\mathbf{R}$ is Borel, is equal to $\mathcal{F}$.

The measure σ on S^1 determined by

$$\widehat{\sigma}(n)=\int_X h_nh_0\,d\mu,\quad n\in\mathbb{Z},$$

is called the *spectral measure* of the Gaussian automorphism T.

Notice that since h_0 is real, for each $f\in L^2(S^1,\sigma)$

$$\int_{S^1} f(z)\,d\sigma(z)=\int_{S^1} f(\overline{z})\,d\sigma(z),$$

or, in other words, σ is a symmetric measure. We will always assume that σ is non-atomic (which implies that T is weakly mixing).

Denote by $H^{(r)}$ the *real Gaussian subspace* of $L^2(X,\mu)$, that is, the (real) closure of all finite sums

$$\sum_{j=-k}^{k} r_jh_j,\quad r_j\in\mathbf{R},\ k\in\mathbf{N},$$

and by $H^{(c)}=\{g_1+ig_2: g_1,g_2\in H^{(r)}\}$ the Gaussian space. Both of these subspaces are T-invariant.

Let $\tilde{\mathcal{H}}\subset L^2(S^1,\sigma)$ denote the real subspace of functions $f\in L^2(S^1,\sigma)$ satisfying $f(\overline{z})=\overline{f(z)}$ a.s. σ (cf. the definition of $\tilde{\mathcal{H}}$ in Section 2). We again see that if $Vf(z)=zf(z)$, then $\tilde{\mathcal{H}}$ is V-invariant and that for the natural equivalence

$$\Theta:(H^{(c)},T)\to(L^2(S^1,\sigma),V)$$

given by $\Theta(h_n)=z^n$, $n\in\mathbb{Z}$, we furthermore have $\Theta(H^{(r)})=\tilde{\mathcal{H}}$, and hence $\tilde{\mathcal{H}}$ is the real closure of the polynomials $\sum_{j=-k}^{k} r_jz^j$, $r_j\in\mathbf{R}$, $k\in\mathbf{N}$.

Proposition 2. *Suppose that T_1 and T_2 are two Gaussian automorphisms with real Gaussian spaces $H_1^{(r)}$, $H_2^{(r)}$ respectively. Assume moreover that $W:H_1^{(r)}\to H_2^{(r)}$ is unitary and $WU_{T_1}=U_{T_2}W$. Then there is a unique automorphism $S:X\to X$ such that $W=U_S$ on $H_1^{(r)}$.*

Proof. Let $h_0^{(1)}$ and $h_0^{(2)}$ denote the Gaussian elements of the respective automorphisms. If we denote by (g_n) the process given by $g_n=Wh_n^{(1)}$, $n\in\mathbb{Z}$, then we have that (g_n) is Gaussian and its spectral measure is equal to the spectral measure of

$(h_n^{(1)})$. Therefore these two processes have the same distributions. Consequently, the corresponding actions must be conjugate, and since (g_n) generates the full σ-algebra, we conclude that there exists a unique extension of W to a measure preserving conjugation S of T_1 and T_2. □

As a corollary we deduce that a Gaussian automorphism is always conjugate to its inverse. Lemańczyk and de Sam Lazaro in [5] give a direct proof that this conjugation may be realized by an involution. Below, we show that quite generally, every conjugation is an involution.

Corollary 1. *For each Gaussian automorphism $T : (X, \mathcal{F}, \mu) \to (X, \mathcal{F}, \mu)$, T and T^{-1} are conjugate.*

Proof. Let W be a unitary equivalence between U_T and $U_{T^{-1}}$ which preserves the real Gaussian space $H^{(r)}$ (these always exist). From Proposition 2 we see that $W = U_S$ on $H^{(r)}$ for some automorphism $S : X \to X$. The uniqueness implies that W and U_S coincide on $L^2(X, \mu)$, and the result follows. □

Assume that the spectral measure σ satisfies

$$\sigma \perp \underbrace{\sigma * \ldots * \sigma}_{n}, \quad n \geq 2 \tag{2}$$

(this is the case if T has simple spectrum).

Corollary 2. *If σ satisfies condition* (2) *above, then each conjugating map is an involution.*

Proof. The maximal spectral type of T on $(H^{(c)})^\perp$ is equal to

$$\frac{\sigma * \sigma}{2!} + \frac{\sigma * \sigma * \sigma}{3!} + \ldots ,$$

and so is orthogonal to the one on $H^{(c)}$. The result now follows immediately from Proposition 1. □

The following proposition now gives a counter-example to the converse of the corollary to the Inverse Conjugacy Theorem.

Proposition 3. *There is a Gaussian automorphism T having infinite spectral multiplicity, and for which every conjugation of T with T^{-1} is an involution.*

Proof. By Corollary 2, every Gaussian automorphism T is conjugate to its inverse. If T is constructed so that the measure σ is singular, and is such that $\sigma * \sigma$ is Lebesgue, then T will have infinite spectral multiplicity. Since condition (2) is satisfied, every conjugation of T with T^{-1} is an involution. □

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