

$(Z_2)^k$ -ACTIONS WITH FIXED POINT SET
OF CONSTANT CODIMENSION $2^k + 1$

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(Communicated by Ralph Cohen)

ABSTRACT. The groups $J_{n,k}^{2^k+1}$ of cobordism classes in the unoriented cobordism group MO_n containing a representative M^n admitting a $(Z_2)^k$ -action with fixed point set of constant codimension $2^k + 1$ are determined.

1. INTRODUCTION

Let $J_{*,k}^r = \sum_{n \geq r} J_{n,k}^r$ denote the ideal in the unoriented cobordism ring MO_* of classes containing a representative that admits a $(Z_2)^k$ -action with fixed point set of constant codimension r . In [3] and [4] $J_{*,k}^r$ for $1 \leq r \leq 2^k$ were determined. From [3, 2.2] one might suspect that in $J_{*,k}^r$ for $r > 2^k$ it is hard to find indecomposable classes in dimensions close to r , so the determination of $J_{*,k}^r$ for $r > 2^k$ is difficult. In this paper we will study $J_{*,k}^{2^k+1}$.

If the cobordism class $[M^n]$ of the smooth closed manifold M^n can be expressed as a sum of products of lower dimensional cobordism classes, then $[M^n]$ is decomposable; otherwise it is indecomposable. Let $RP(n)$ be the real projective space of dimension n . Let $\chi : MO_* = \sum MO_n \rightarrow Z_2$ denote the mod 2 Euler characteristic and D_n the set of decomposable classes $[M^n]$ in MO_n . The present work proves the following:

Theorem 1. For $k \geq 4$

- (1) $J_{n,k}^{2^k+1} = MO_n \cap \ker \chi$ for $n \geq 2^k + 2$;
- (2) $J_{n,k}^{2^k+1} = D_n \cap \ker \chi$ for $n = 2^k + 1$.

Theorem 2. For $k = 3$, if there is an indecomposable class of dimension 11 in $J_{11,k}^9$, then

- (1) $J_{n,k}^9 = MO_n \cap \ker \chi$ for $n \geq 10$;
- (2) $J_{n,k}^9 = D_n \cap \ker \chi$ for $n = 9$.

Received by the editors June 3, 1997 and, in revised form, June 19, 1998.

1991 *Mathematics Subject Classification.* Primary 57S17; Secondary 57R85.

Key words and phrases. Indecomposable cobordism class, fixed point set, projective space bundle, $(Z_2)^k$ -action.

This work is supported by HNSF.

Theorem 3. For $k = 3$, if there is not any indecomposable class of dimension 11 in $J_{11,k}^9$, then

- (1) $J_{n,k}^9 = MO_n \cap \ker \chi$ for $n \geq 10, n \neq 11$;
- (2) $J_{n,k}^9 = D_n \cap \ker \chi$ for $n = 9, 11$.

Theorem 4. For $k = 2$

- (1) $J_{n,k}^5 = MO_n \cap \ker \chi$ for $n \geq 7$;
- (2) $J_{n,k}^5 = D_n \cap \ker \chi$ for $n = 5, 6$.

2. BACKGROUND

Let $\phi : (Z_2)^k \times M^n \rightarrow M^n$ be a smooth action of the group $(Z_2)^k$ on a closed n -dimensional manifold. Here $(Z_2)^k$ is considered as the group generated by k commuting involutions. The fixed point set of the action of $(Z_2)^k$ on M^n , i.e.

$$F = \{x \in M^n \mid \phi(g, x) = x, \forall g \in (Z_2)^k\},$$

is a disjoint union of closed submanifolds of M^n . If each component of F is $(n - r)$ -dimensional, then F has codimension r . $J_{n,k}^r$ consisting of cobordism classes in MO_n containing a representative M^n admitting a $(Z_2)^k$ -action with fixed point set F of constant codimension r is a subgroup in MO_n and $J_{*,k}^r = \sum J_{n,k}^r$ forms an ideal in MO_* . It is immediate that $J_{*,k}^r \subset J_{*,k+1}^r$ and $J_{*,k}^r J_{*,k}^{r'} \subset J_{*,k}^{r+r'}$.

It is well known that the unoriented cobordism ring MO_* is a Z_2 -polynomial algebra on a single generator x_i of each dimension not of the form $2^u - 1$ ([1]). Indecomposable classes can be used as generators of MO_* . Indecomposability is recognized as follows. Given M^n write the total Stiefel-Whitney class of the tangent bundle in factored form $(1 + t_1)(1 + t_2) \cdots (1 + t_n)$. Then $s_n(M^n) \in H^n(M^n; Z_2)$ is the polynomial in the Stiefel-Whitney classes corresponding to the symmetric function $t_1^n + t_2^n + \cdots + t_n^n$. It was shown that $[M^n]$ is indecomposable if and only if the characteristic number $s_n[M^n] \neq 0$ in Z_2 . For our purpose we will exhibit indecomposable classes as generators of MO_* . They will come from the following source.

Lemma 2.1 ([2, 3.4]). Let $RP(n_1, n_2, \dots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ over $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_l)$, where λ_i is the pullback of the canonical line bundle over the i th factor. Then for $l > 1$, $RP(n_1, n_2, \dots, n_l)$ is indecomposable in MO_* if and only if

$$S = \binom{n+l-2}{n_1} + \binom{n+l-2}{n_2} + \cdots + \binom{n+l-2}{n_l} \equiv 1 \pmod{2},$$

where $n = n_1 + n_2 + \cdots + n_l$ □

The manifold $RP(n_1, n_2, \dots, n_l)$ has dimension $n + l - 1$. If

$$n_{i+1} = n_{i+2} = \cdots = n_l = 0,$$

then $RP(n_1, n_2, \dots, n_l)$ will sometimes be written $RP(n_1, \dots, n_i; l)$.

To calculate binomial coefficients mod 2 the following lemma is extremely useful.

Lemma 2.2 ([3, 2.4]). If $m = \sum_{i=0}^l m_i 2^i$ and $n = \sum_{i=0}^l n_i 2^i$ with $0 \leq m_i, n_i \leq 1$, then

$$\binom{m}{n} \equiv 1 \pmod{2} \text{ if and only if } n_i \leq m_i \text{ for every } i. \quad \square$$

Lemma 2.3 ([3, 3.1]). *Let $\lambda_i \rightarrow X_i$ be line bundles, and let $(Z_2)^{k_i}$ ($k_i \geq 0$) act on λ_i as bundle maps with fixed point set F_i on X_i for all $1 \leq i \leq l$ with $\sum_{i=1}^l 2^{k_i} \leq 2^k$. Then $RP(\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l)$ over $X_1 \times X_2 \times \cdots \times X_l$ admits a $(Z_2)^k$ -action with fixed point set $F_1 \times F_2 \times \cdots \times F_l \times E$ with E a set of l points. \square*

3. EXISTENCE OF INDECOMPOSABLES

Lemma 3.1. *For $l + 1 \leq 2^k$, $[RP(2m + 1, n_2, n_3, \dots, n_l)] \in J_{*,k}^{m+l}$.*

Proof. Let $(Z_2)^1$ act on $RP(2m + 1)$ with fixed point set F_1 two copies of $RP(m)$ and $(Z_2)^0$ act as the identity on the rest of the base. By Lemma 2.3 the lemma is established. \square

Lemma 3.2. *For a positive integer l , $[RP(n_1, n_2, \dots, n_{2l})] \in J_{*,k}^l$.*

Proof. Let $\xi = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$, $\eta = \lambda_{l+1} \oplus \cdots \oplus \lambda_{2l}$. $RP(\xi \oplus \eta)$ has an involution T induced by multiplication by -1 in the fibers of ξ and 1 in the fibers of η . The fixed point set of T is $RP(\xi) \cup RP(\eta)$ which has codimension l . Therefore $[RP(n_1, n_2, \dots, n_{2l})] \in J_{*,1}^l \subset J_{*,k}^l$. \square

Lemma 3.3. *There exist indecomposable classes $x_n \in J_{n,k}^{2^k+1}$ for $k \geq 3$, $n = 2^k + 2$.*

Proof. Take $x_n = [RP(7; 2^k - 4)]$. Let

$$\begin{aligned} T_1[y_1, y_2, \dots, y_8] &= [-y_1, -y_2, -y_3, -y_4, y_5, \dots, y_8], \\ T_2[y_1, y_2, \dots, y_8] &= [-y_1, -y_2, y_3, y_4, -y_5, -y_6, y_7, y_8]. \end{aligned}$$

Then (T_1, T_2) defines a $(Z_2)^2$ -action on $RP(7)$ with fixed point set of four copies of $RP(1)$. Let $(Z_2)^0$ act as the identity on the rest of the base. By Lemma 2.1 and Lemma 2.3 x_n is as required. \square

Lemma 3.4. *There exist indecomposable classes $x_n \in J_{n,k}^{2^k+1}$ for n even and $n \geq 2^k + 4$.*

Proof. We take x_n as follows, and by Lemma 2.1 and Lemma 3.1 x_n is as required.

Dimensions $2^k + 4 \leq n \leq 2^k + 2^{k-1}$ ($k \geq 3$).

Take $x_n = [RP(2n - 2^{k+1} - 1; 2^{k+1} - n + 2)]$.

Dimensions $2^k + 2^{k-1} + 2 \leq n \leq 2^{k+1} + 2^k$ ($k \geq 2$).

Take $x_n = [RP(2^k + 1, n - 2^k - 2^{k-1} - 1; 2^{k-1} + 1)]$.

Dimensions $2^{k+1} + 2^k + 2 \leq n$ ($k \geq 2$).

To construct representatives we consider the following cases.

Case 1. $n = 2^{r_m} + \cdots + 2^{r_1}$, $r_m > \cdots > r_1 \geq k$.

Take $x_n = [RP(2^k + 1, n - 2^k - 2^{k-1} - 1; 2^{k-1} + 1)]$.

Case 2. $n = 2^{r_m} + \cdots + 2^{r_1} + 2^{v_q} + 2^{v_{q-1}} + \cdots + 2^{v_1}$,

$r_m > \cdots > r_1 > v_q \geq k > k - 1 \geq v_{q-1} > \cdots > v_1 \geq 1$, $v_{i+1} > v_i + 1$ for some i , and $q \geq 2$.

Let $p = 2^{v_i} + \cdots + 2^{v_1} (\geq 2)$; take $x_n = [RP(2p + 1, n - 2^k - p - 1; 2^k - p + 1)]$.

Case 3. $n = 2^{r_m} + \cdots + 2^{r_1} + 2^k + 2^{k-1} + \cdots + 2^2 + 2$, $r_m > \cdots > r_1 > k$.

Take $x_n = [RP(2^k + 1, n - 2^k - 2^{k-1} - 1; 2^{k-1} + 1)]$.

Case 4. $n = 2^{r_m} + \dots + 2^{r_1} + 2^{v_q} + 2^{v_{q-1}} + \dots + 2^{v_1}$, $r_m > \dots > r_1 > k = v_q, v_{i+1} = v_i + 1$ for every i , $v_1 \geq 2, q \geq 2$.

Take $x_n = [RP(2^{v_1+1} - 1, n - 2^k - 2^{v_1}; 2^k - 2^{v_1} + 2)]$. □

Lemma 3.5. *There exist indecomposable classes $x_n \in J_{n,k}^{2^k+1}$ for $k \geq 4, n = 2^k + 3$.*

Proof. Let $x_n = [RP(8; 2^k - 4)]$. Notice $k \geq 4$, and by Lemma 2.1 x_n is indecomposable. Suppose

$$\begin{aligned} T_1[y_1, y_2, \dots, y_9] &= [-y_1, -y_2, -y_3, y_4, \dots, y_9], \\ T_2[y_1, y_2, \dots, y_9] &= [y_1, \dots, y_6, -y_7, -y_8, -y_9]. \end{aligned}$$

Then (T_1, T_2) defines an action of $(Z_2)^2$ on $RP(8)$ with fixed point set of three copies of $RP(2)$. Let $(Z_2)^0$ act as the identity on the rest of the base. By Lemma 2.3 $x_n \in J_{n,k}^{2^k+1}$. □

Lemma 3.6. *There exist indecomposable classes $x_n \in J_{n,k}^{2^k+1}$ for n odd, $n \geq 2^k + 5, n \neq 2^u - 1$.*

Proof. Dimensions $2^k + 5 \leq n \leq 2^k + 2^{k-1} - 1$ ($k \geq 4$).

Let $n = 2^k + 2^{r_m} + \dots + 2^{r_1}, k - 2 \geq r_m > \dots > r_1 = 0$.

Take $x_n = [RP(2^{r_m+1} + 2^{r_m} + 2, n - 2^k - 2^{r_m} - 1; 2^k - 2^{r_m+1})]$.

Similar to the action of $(Z_2)^2$ on $RP(8)$, we have an action of $(Z_2)^2$ on $RP(2^{r_m+1} + 2^{r_m} + 2)$ with fixed point set of three copies of $RP(2^{r_m})$. Let $(Z_2)^0$ trivially act on the rest of the base. By Lemma 2.3 and Lemma 2.1 $x_n \in J_{n,k}^{2^k+1}$ and x_n is indecomposable.

Dimensions $2^k + 2^{k-1} + 1 \leq n < 2^{k+1} - 1$ ($k \geq 3$).

Take $x_n = [RP(2^k - 3, n - 2^k - 2^{k-1} + 1; 2^{k-1} + 3)]$. By Lemma 2.1 and Lemma 3.1 $x_n \in J_{n,k}^{2^k+1}$ and x_n indecomposable.

Dimensions $2^{k+1} + 1 \leq n$ ($k \geq 2$).

Case 1. $n = 2^{r_m} + \dots + 2^{r_1} + 2^{v_q} + \dots + 2^{v_2} + 2^{v_1}$, $r_m > \dots > r_1 \geq k + 2, k \geq v_q > \dots > v_1 = 0$.

Take $x_n = [RP(n - 2^{k+1} - 1; 2^{k+1} + 2)]$. By Lemma 2.1 and Lemma 3.2 $x_n \in J_{n,k}^{2^k+1}$ and x_n indecomposable.

Case 2. $n = 2^{r_m} + \dots + 2^{r_1} + 2^{v_q} + \dots + 2^{v_1}$, $r_m > \dots > r_1 > k + 1 = v_q > \dots > v_1 = 0, v_{i+1} > v_i + 1$ for some i .

Take $x_n = [RP(2^{k+1} - 3, n - 2^{k+1} + 1, 0)]$. By Lemma 2.1 and Lemma 3.1 x_n is as required.

Case 3. $n = 2^{r_m} + \dots + 2^{r_1} + 2^{k+1} - 1$, $r_m > \dots > r_1 = k + 1. r_{i+1} > r_i + 1$ for some i .

Let $n_1 = n - 2^{r_i+1} - 2^{k+1} - 1$ and $x_n = [RP(2^{r_i+1}, n_1/2, n_1/2; 2^{k+1} + 2)]$.

By Lemma 2.1 and Lemma 3.2 $x_n \in J_{n,k}^{2^k+1}$ and x_n indecomposable. Therefore the lemma is established. □

Lemma 3.7. *$J_{6,2}^5$ does not contain 6-dimensional indecomposable classes.*

Proof. Let $[M^6] \in J_{6,2}^5$ and $(Z_2)^2$ act on M^6 , the fixed point set $F = \bigcup_{i=1}^m S_i^1$, where S_i^1 is a sphere of dimension 1 for every i . We will prove $s_6[M^6] = 0$ by making use of the formula in [5, p.727]; hence $[M^6]$ is decomposable ([1, p.76]).

We use the notations of [5]. The irreducible real representations of the group $(Z_2)^2 = \{T_1, T_2 | T_i^2 = 1, T_1 T_2 = T_2 T_1\}$ are all one-dimensional, and are given by homomorphisms $\rho_j : (Z_2)^2 \rightarrow Z_2 = \{+1, -1\}$, where T_i acts on R as multiplication by $\rho_j(T_i)$. $\rho_4 = 1$ is the trivial representation and ρ_j ($j = 1, 2, 3$) is not trivial.

Let α_1, α_2 be formal variables. According to [5] we have $\alpha_{\rho_1} = \alpha_1, \alpha_{\rho_2} = \alpha_2, \alpha_{\rho_3} = \alpha_1 + \alpha_2, \alpha_{\rho_4} = 0$. The normal bundle of S_i^1 in M^6 decomposes into subbundles under the action of $(Z_2)^2$,

$$\nu_{\rho_1}^{(i)} \oplus \nu_{\rho_2}^{(i)} \oplus \nu_{\rho_3}^{(i)} \rightarrow S_i^1$$

where $\dim \nu_{\rho_j}^{(i)} = q_{\rho_j}^{(i)}, q_{\rho_1}^{(i)} + q_{\rho_2}^{(i)} + q_{\rho_3}^{(i)} = 5$.

Set

$$A_j = \{(q_{\rho_1}^{(i)}, q_{\rho_2}^{(i)}, q_{\rho_3}^{(i)}) | q_{\rho_j}^{(i)} \text{ odd, } q_{\rho_l}^{(i)} (l \neq j) \text{ even, } 1 \leq i \leq m\} \text{ for } j = 1, 2, 3,$$

$$A_4 = \{(q_{\rho_1}^{(i)}, q_{\rho_2}^{(i)}, q_{\rho_3}^{(i)}) | q_{\rho_l}^{(i)} \text{ odd, } 1 \leq l \leq 3, 1 \leq i \leq m\}.$$

We denote by $f(t)$ a symmetric Z_2 -polynomial in the variables $\{t_1, t_2, \dots, t_6\}$.

Taking $f(t) = 1$, by the corollary ([5, p.727]) in the quotient field of $Z_2[\alpha_1, \alpha_2]$ we have a characteristic number

$$0 = f(t)[M^6] = \sum_{i=1}^m \frac{f(z, \alpha + y)}{\prod(\alpha + y)} [S_i^1] = B_1 + B_2 + B_3 + B_4$$

$$\text{where } B_j = \sum_{(q_{\rho_1}^{(i)}, q_{\rho_2}^{(i)}, q_{\rho_3}^{(i)}) \in A_j} \frac{f(z, \alpha + y)}{\prod(\alpha + y)} [S_i^1].$$

Further taking $f(t) = \sum_{i=1}^6 t_i^2, \sum_{i=1}^6 t_i^4$, we obtain

$$0 = f(t)[M^6] = \alpha_1^2 B_1 + \alpha_2^2 B_2 + (\alpha_1 + \alpha_2)^2 B_3 + 0,$$

$$0 = f(t)[M^6] = \alpha_1^4 B_1 + \alpha_2^4 B_2 + (\alpha_1 + \alpha_2)^4 B_3 + 0,$$

and so

$$B_1 + B_2 + B_3 + B_4 = 0,$$

$$\alpha_1^2(B_1 + B_3) + \alpha_2^2(B_2 + B_3) = 0,$$

$$\alpha_1^4(B_1 + B_3) + \alpha_2^4(B_2 + B_3) = 0.$$

From the above equations, we have $B_1 = B_2 = B_3 = B_4$, so

$$\begin{aligned} s_6[M^6] &= (t_1^6 + t_2^6 + \dots + t_6^6)[M^6] \\ &= \alpha_1^6 B_1 + \alpha_2^6 B_2 + (\alpha_1 + \alpha_2)^6 B_3 + (\alpha_1^6 + \alpha_2^6 + (\alpha_1 + \alpha_2)^6) B_4 \\ &= (2\alpha_1^6 + 2\alpha_2^6 + 2(\alpha_1 + \alpha_2)^6) B_1 \equiv 0 \pmod{2}. \end{aligned}$$

Therefore $[M^6]$ is decomposable. □

4. PROOFS OF THE THEOREMS

Proof of Theorem 1. (a) Let $x_2 = [RP(2)]$, and hence $\chi(x_2^i) = 1$. By [3, 2.1,5.1] $x_2 \in J_{*,k}^2$ and $x_2^i \notin J_{*,k}^{2^k+1}$.

(b) For $3 \leq n \leq 2^k + 1, n \neq 2^u - 1$, we can choose indecomposable classes x_n such that $\chi(x_n) = 0$ (otherwise, replacing x_n by $x_n + x_2^{n/2}$). From [3, 2.2,5.1], $x_n \in J_{*,k}^r$ for $2 \leq r \leq \min(n, 2^k - 1)$ and $x_{2^k+1} \notin J_{*,k}^{2^k+1}$.

(c) For $n \geq 2^k + 2, n \neq 2^u - 1$, let x_n be as in the Lemmas 3.3–3.6; hence $x_n \in J_{*,k}^{2^k+1}$.

We have chosen the system of generators $\{x_n\}$ of MO_* . Since $J_{*,k}^{2^k+1}$ is an ideal in MO_* , to complete the proof it is necessary only to show that it contains the decomposable classes $x_{i_1}x_{i_2} \cdots x_{i_m} (\neq x_2^i), m \geq 2, \sum_{j=1}^m i_j \geq 2^k + 1, 2 \leq i_1 \leq \cdots \leq i_m \leq 2^k + 1$.

If $i_m \geq 2^k - 1$, then by (a) and (b)

$$x_{i_1}x_{i_2} \cdots x_{i_{m-1}} \in J_{*,k}^2, x_{i_m} \in J_{*,k}^{2^k-1}, \text{ so } x_{i_1}x_{i_2} \cdots x_{i_m} \in J_{*,k}^2 J_{*,k}^{2^k-1} \subset J_{*,k}^{2^k+1}.$$

If $i_m \leq 2^k - 2$, then notice that

$$2^k + 1 - i_1 \leq i_2 + \cdots + i_m, 2^k + 1 - i_1 \leq 2^k - 1, \text{ by (b) and [3, 5.1],}$$

$$x_{i_1} \in J_{*,k}^{i_1}, x_{i_2} \cdots x_{i_m} \in J_{*,k}^{2^k+1-i_1}, \text{ and } x_{i_1}x_{i_2} \cdots x_{i_m} \in J_{*,k}^{i_1} J_{*,k}^{2^k+1-i_1} \subset J_{*,k}^{2^k+1}.$$

The proof is completed. □

Proof of Theorem 2. For $2 \leq n \leq 2^k + 1, n \neq 2^u - 1$, take x_n as in the proof of Theorem 1. For $n \geq 2^k + 2, n \neq 2^u - 1$, we can choose indecomposable classes $x_n \in J_{n,k}^{2^k+1}$ by Lemmas 3.3–3.4, Lemma 3.6 and the assumption. Then repeating the statement of the proof of Theorem 1, we obtain the result. □

Proof of Theorem 3. (a) Let x_2 be the cobordism class of $RP(2)$; then $\chi(x_2^i) = 1, x_2 \in J_{*,k}^2, x_2^i \notin J_{*,k}^{2^k+1}$.

(b) For $3 \leq n \leq 2^k + 1, n = 2^k + 3, n \neq 2^u - 1$, we can choose indecomposable classes x_n such that $\chi(x_n) = 0$. From [3, 2.2,5.1], $x_n \in J_{*,k}^r$ for $2 \leq r \leq \min(n, 2^k - 1)$ and $x_{2^k+1} \notin J_{*,k}^{2^k+1}, x_{2^k+3} \notin J_{*,k}^{2^k+1}$.

(c) For $n \geq 2^k + 2, n \neq 2^k + 3, n \neq 2^u - 1$, by Lemmas 3.3–3.4 and Lemma 3.6 there exist indecomposable classes $x_n \in J_{*,k}^{2^k+1}$.

The above $\{x_n\}$ form the generators of MO_* . To complete the proof we need only to show that it contains the decomposable classes $x_{i_1}x_{i_2} \cdots x_{i_m} (\neq x_2^i), m \geq 2, \sum_{j=1}^m i_j \geq 2^k + 1, 2 \leq i_1 \leq \cdots \leq i_m \leq 2^k + 3(i_j \neq 2^k + 2)$.

Then repeating the statement of the proof of Theorem 1, we complete the proof. □

Proof of Theorem 4. Let x_2 be the cobordism class of $RP(2)$, then $\chi(x_2^i) = 1, x_2 \in J_{*,2}^2, x_2^i \notin J_{*,2}^5$. For $4 \leq n \leq 6$, take indecomposable classes x_n such that $\chi(x_n) = 0$. For $n \geq 8$, by Lemma 3.4 and Lemma 3.6 there exist indecomposable classes $x_n \in J_{*,2}^5$.

Similarly we need only to show that $J_{*,2}^5$ contains the decomposable classes $x_{i_1}x_{i_2} \cdots x_{i_m} (\neq x_2^i), m \geq 2, \sum_{j=1}^m i_j \geq 5, 2 \leq i_1 \leq \cdots \leq i_m \leq 6$.

From $i_1 \geq 2, i_m \geq 4$, we obtain $x_{i_1} \in J_{*,2}^2, x_{i_m} \in J_{*,2}^3$, and $x_{i_1}x_{i_2} \cdots x_{i_m} \in J_{*,2}^2 J_{*,2}^3 \subset J_{*,2}^5$. The proof is completed. \square

Corollary 4.1. *If $k \geq 4, n \geq 2^k + 1$, then a cobordism class $x_n \in J_{n,k}^{2^k+1}$ if and only if characteristic numbers $s_{2^k+1}[x_n] = w_n[x_n] = 0$.* \square

Corollary 4.2. *If there is an indecomposable class of dimension 11 in $J_{11,3}^9, n \geq 9$, then a cobordism class $x_n \in J_{n,3}^9$ if and only if characteristic numbers $s_9[x_n] = w_n[x_n] = 0$.* \square

Corollary 4.3. *If there is not any indecomposable class of dimension 11 in $J_{11,3}^9, n \geq 9$, then a cobordism class $x_n \in J_{n,3}^9$ if and only if characteristic numbers $s_9[x_n] = s_{11}[x_n] = w_n[x_n] = 0$.* \square

Corollary 4.4. *If $n \geq 5$, then a cobordism class $x_n \in J_{n,2}^5$ if and only if characteristic numbers $s_5[x_n] = s_6[x_n] = w_n[x_n] = 0$.* \square

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