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TOEPLITZ ALGEBRAS ON DISCRETE ABELIAN QUASILY ORDERED GROUPS

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ABSTRACT. In this note, Toeplitz operators on discrete abelian quasily ordered groups are studied, and a theorem of R. Douglas is generalized.

1. INTRODUCTION

Throughout the note, we assume that G is a discrete abelian group. For any subset G_+ of G, we say that (G, G_+) is a quasi-partial ordered group if $0 \in G_+, G_+ + G_+ \subseteq G_+$ and $G = G_+ - G_+$; further, (G, G_+) is referred to as a quasily ordered group if $G = G_+ \cup (-G_+)$. Note when $G_+^0 = G_+ \cap (-G_+) = \{0\}$, then a quasi-partial ordered group (resp. quasily ordered group) (G, G_+) is known as a partially ordered (resp. ordered) group.

Let $\{ e_g \mid g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$e_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for } g, h \in G.$$

For any $E \subseteq G$, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{ e_g \mid g \in E \}$; its projection is denoted by p^E. For any $g \in G$, we define a unitary u_g on $\ell^2(G)$ by $u_g(e_h) = e_{g+h}$ for $h \in G$. For any subset E of G, the C^*-algebra generated by $\{ p^E u_g p^E \mid g \in G \}$ is denoted by $\mathcal{T}^E(G)$ and is called the *Toeplitz algebra* with respect to E.

Toeplitz algebras on the quarter-planes have been studied by many mathematicians; see [1] and [2] for example. Associated with such Toeplitz algebras are the usual quasily ordered groups $(\mathbb{Z}^2, \mathbb{Z}^2_\alpha)$ for $\alpha \in \mathbb{R}^1$, where $\mathbb{Z}^2_\alpha = \{ (m,n) \in \mathbb{Z}^2 \mid \alpha m + n \geq 0 \}$. R. Douglas proved in [2], among other things, that when α is a rational number $\mathcal{T}^{\mathbb{Z}^2_\alpha}(\mathbb{Z}^2) \cong \mathcal{T}^{\mathbb{Z}_+}(\mathbb{Z}) \otimes C(T)$, where T is the unit circle in $\mathbb{C}^1$ and $\mathcal{T}^{\mathbb{Z}_+}(\mathbb{Z})$ is the classical Toeplitz algebra (see Corollary 4 below). In this note, under a setting of quasi-partial ordered groups, we generalize such a result.

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2. The main results

Let G be a discrete abelian group and (G, G_+) a quasily ordered group. By an isometric representation (V, H) of G_+, we mean H is a Hilbert space and $V : G_+ \to \mathbb{B}(H)$ is a map satisfying

(1) $V(0) = 1, V^*(x)V(x) = 1$ for $x \in G_+$;
(2) $V(x + y) = V(x)V(y)$ for $x, y \in G_+$;
(3) $V(x)V(x)^* = 1$ for $x \in G_+^0$.

Lemma 1 (cf. [9], Theorem 3.5). *Let G be a discrete abelian group and (G, G_+) a quasily ordered group. Then for any isometric representation $V : G_+ \to \mathbb{B}(H)$, there is a C^*-algebra morphism $\pi_V : \mathcal{T}^{G_+}(G) \to \mathbb{B}(H)$ such that $\pi_V(p^{G_+}u_g p^{G_+}) = V(g)$ for all $g \in G_+$.*

Remark. (1) Let (G, G_+) and (E, E_+) be two quasily ordered groups. If there is an isomorphism of groups $\rho : G \to E$ such that $\rho(G_+) = E_+$ and $\rho(G_+^0) = E_+^0$, then by the above Lemma, we know that $\mathcal{T}^{G_+}(G) \cong \mathcal{T}^{E_+}(E)$.

(2) Let $\widehat{G}$ be the dual group of G. Since G is discrete and abelian, $\widehat{G}$ is compact and it is connected if and only if G is torsion-free. By [6], Proposition 7.1.6, we know that the reduced group C^*-algebra $C_r^*(G) = \mathcal{T}^G(G) \cong_\rho C(\widehat{G})$, with the property $\rho(u_g) = \varepsilon_g$ for any $g \in G$, where $\varepsilon_g(\gamma) = \gamma(g)$ for $\gamma \in \widehat{G}$. By the Stone-Weierstrass Theorem, $C(\widehat{G})$ is generated by $\{\varepsilon_g \,|\, g \in G\}$. Applying the above Lemma to (G, G), we get the following Corollary.

Corollary 2 (cf. [3], Lemma 1.2). *Let G be a discrete abelian group. Then for any unitary representation (V, H) of G, there is a C^*-algebra morphism $\pi_V : C(\widehat{G}) \to \mathbb{B}(H)$ such that $\pi_V(\varepsilon_g) = V(g)$ for all $g \in G$.*

Now let (G, G_+) be a quasily ordered group. By definition, we know that $G{=}G_+ \cup (-G_+)$ and $0 \in G_+^0 = G_+ \cap (-G_+)$. Let $G_+^* = G_+ \setminus G_+^0$; then it is easy to show that

$$(*) \qquad G_+^* + G_+ = G_+^*, \qquad G = G_+^* \cup G_+^0 \cup (-G_+^*).$$

So if we set $G_1 = G_+^* \cup \{0\}$, then for any $x \in G_+^0$ and $y \in G_+^*$, we have $x = (x + y) - y \in G_1 - G_1$; therefore (G, G_1) is actually a partially ordered group. Further, by $(*)$ we know that $(G/_{G_+^0}, G_1/_{G_+^0})$ is an ordered group.

Next we generalize R. Douglas's result to a more general setting, but first we need a definition.

Definition. Let G be a discrete abelian group and (G, G_+) a quasily ordered group. A morphism of groups $\phi : G \to G_+^0$ is said to be a *deformation retraction* if $\phi(h) = h$ for all $h \in G_+^0$.

Theorem 3. *Let G be a discrete abelian group and (G, G_+) a quasily ordered group. Denote $G/_{G_+^0} = [G]$ and $G_1/_{G_+^0} = [G_1]$. If there is a deformation retraction $\phi : G \to G_+^0$, then $\mathcal{T}^{G_+}(G) \cong \mathcal{T}^{[G_1]}([G]) \otimes C(\widehat{G_+^0})$.*

Proof. For any $g \in G$, let $T_g^{G_+} = p^{G_+}u_g p^{G_+}, T_{[g]}^{[G_1]} = p^{[G_1]}u_{[g]}p^{[G_1]}$. For $g \in G_+$, define

$$V(g) = T_{[g-\phi(g)]}^{[G_1]} \otimes \varepsilon_{\phi(g)} = T_{[g]}^{[G_1]} \otimes \varepsilon_{\phi(g)};$$

then V is an isometric representation of G_+. By Lemma 1, we know that there is a C^*-algebra morphism $\pi : \mathcal{T}^{G_+}(G) \to \mathcal{T}^{[G_1]}([G]) \otimes C(\widehat{G_+^0})$ such that

$$\pi(T_g^{G_+}) = V(g) = T_{[g]}^{[G_1]} \otimes \varepsilon_{\phi(g)} \text{ for any } g \in G_+,$$

which implies

$$\pi(T_g^{G_+}) = T_{[g]}^{[G_1]} \otimes \varepsilon_{\phi(g)} \text{ for any } g \in G.$$

On the other hand, by Lemma 1 and Corollary 2, we know that there are C^*-algebra morphisms $\rho : \mathcal{T}^{[G_1]}([G]) \to \mathcal{T}^{G_+}(G)$ and $\lambda : C(\widehat{G_+^0}) \to \mathcal{T}^{G_+}(G)$, such that

$$\rho\big(T_{[g]}^{[G_1]}\big) = T_{g-\phi(g)}^{G_+} \text{ for any } g \in G, \quad \lambda(\varepsilon_h) = T_h^{G_+} \text{ for any } h \in G_+^0.$$

It is easy to show that for any $g \in G_1$ and $h \in G_+^0$, $\rho\big(T_{[g]}^{[G_1]}\big)\,\lambda(\varepsilon_h) = \lambda(\varepsilon_h)\,\rho\big(T_{[g]}^{[G_1]}\big)$, and since $C(\widehat{G_+^0})$ is nuclear, by [5], Corollary T.6.9, we know that there is a C^*-algebra morphism $\rho \otimes \lambda : \mathcal{T}^{[G_1]}([G]) \otimes C(\widehat{G_+^0}) \to \mathcal{T}^{G_+}(G)$ such that

$$\big(\rho \otimes \lambda\big)\big(T_{[g]}^{[G_1]} \otimes \varepsilon_h\big) = T_{g-\phi(g)}^{G_+} T_h^{G_+} \text{ for any } g \in G \text{ and } h \in G_+^0.$$

For any $g \in G_+$ and $h \in G_+^0$,

$$\big((\rho \otimes \lambda) \circ \pi\big)(T_g^{G_+}) = (\rho \otimes \lambda)(T_{[g]}^{[G_1]} \otimes \varepsilon_{\phi(g)}) = T_{g-\phi(g)}^{G_+} T_{\phi(g)}^{G_+} = T_g^{G_+},$$

and

$$\begin{aligned}&\big(\pi \circ (\rho \otimes \lambda)\big)(T_{[g]}^{[G_1]} \otimes \varepsilon_h) = \pi(T_{g-\phi(g)}^{G_+} T_h^{G_+})\\ &= \pi(T_{g-\phi(g)}^{G_+})\,\pi(T_h^{G_+}) = (T_{[g]}^{[G_1]} \otimes 1)(1 \otimes \varepsilon_h) = T_{[g]}^{[G_1]} \otimes \varepsilon_h.\end{aligned}$$

Therefore, $\mathcal{T}^{G_+}(G) \cong \mathcal{T}^{[G_1]}(\,[G]\,) \otimes C(\widehat{G_+^0})$. □

Corollary 4 ([10]). *Let $p, q \in \mathbb{Z}$ such that there exist $r, s \in \mathbb{Z}$ with the property $rp + sq = 1$. Denote $G = \mathbb{Z}^2$ and $G_+ = \mathbb{Z}^2_{p,q} = \{\, (m,n) \in \mathbb{Z}^2 \,\big|\, pm + qn \geq 0 \,\}$. Then $\mathcal{T}^{\mathbb{Z}^2_{p,q}}(\mathbb{Z}^2) \cong \mathcal{T}^{\mathbb{Z}_+}(\mathbb{Z}) \otimes C(T)$.*

Proof. Define a deformation retraction $\phi : G \to G_+^0$ by

$$\phi(m,n) = \big(-q(-sm + rn), p(-sm + rn)\big).$$

Since $rp + sq = 1$, we know that $\{\, -sm + rn \,\big|\, (m,n) \in \mathbb{Z}^2 \,\} = \mathbb{Z}$. Further, if $(k,l) \in G_+^0$, i.e. $pk + ql = 0$, then $k = qa$ and $l = -pa$ for some $a \in \mathbb{Z}$. Choose $m, n \in \mathbb{Z}$ such that $sm - rn = a$, then $\phi(m,n) = (k,l)$. Therefore,

$$G_+^0 = \{\, \big(-q(-sm + rn), p(-sm + rn)\big) \,\big|\, (m,n) \in \mathbb{Z}^2 \,\}.$$

So if we set $\rho : G_+^0 \to \mathbb{Z}$ by $\rho\,\big(-q(-sm + rn), p(-sm + rn)\big) = -sm + rn$, then ρ is an isomorphism of groups. By Corollary 2, we know $C(\widehat{G_+^0}) \cong C(T)$. Obviously, $G_1 = \{\, (m,n) \,\big|\, pm + qn > 0 \,\} \cup \{(0,0)\}$. Let $\theta : [G] \to \mathbb{Z}$, $\theta\,[(m,n)] = pm + qn$ for $(m,n) \in \mathbb{Z}^2$; then θ is an order isomorphism of groups. By Lemma 1, we know that $\mathcal{T}^{[G_1]}([G]) \cong \mathcal{T}^{\mathbb{Z}_+}(\mathbb{Z})$, so the conclusion holds. □

Remark. Let $\widehat{G}$ be the dual group of G; then $\{\varepsilon_x \mid x \in G\}$ is an orthonormal basis for $L^2(\widehat{G})$. When (G, G_+) is a quasi-partial ordered group, we may also regard $\mathcal{T}^{G_+}(G)$ as a subalgebra of $B(L^2(\widehat{G}))$ as follows:

Let $H^{G_+}(\widehat{G})$ be the closed subspace of $L^2(\widehat{G})$ generated by $\{\varepsilon_x \mid x \in G_+\}$; its projection is denoted by $p^{G_+}(\widehat{G})$. For any $\varphi \in C(\widehat{G})$, define $T_\varphi^{G_+}$ on $H^{G_+}(\widehat{G})$ by $T_\varphi^{G_+}(f) = p^{G_+}(\varphi f)$ for $f \in H^{G_+}(\widehat{G})$. The C^*-algebra generated by $\{T_\varphi^{G_+} \mid \varphi \in C(\widehat{G})\}$ is denoted by $\mathcal{T}_r^{G_+}(\widehat{G})$ and is also called the *Toeplitz algebra* with respect to G_+. Clearly, $\mathcal{T}_r^{G_+}(\widehat{G})$ is generated by $\{T_{\varepsilon_x}^{G_+} | x \in G\}$ and it can be regarded as a C^*-subalgebra of $\mathbb{B}(L^2(\widehat{G}))$ through zero-extension. So if we define a unitary $u : \ell^2(G) \to L^2(\widehat{G})$ by $ue_g = \varepsilon_g$ for $g \in G$, then $u^* \circ T_{\varepsilon_g}^{G_+} \circ u = p^{G_+} u_g p^{G_+}$; therefore the two Toeplitz algebras are unitarily equivalent.

Now let (G, E) be an ordered group, in this case G is torsion-free, so $\widehat{G}$ is connected. For any $\varphi \in C(\widehat{G})$, if it does not vanish anywhere, then by [7] there exist some x in G and $\psi \in C(\widehat{G})$ such that $\varphi = \varepsilon_x e^\psi$. As observed in [4], such an x is unique. We denote this group element $\mathrm{ind}_t(\varphi)$ and call it the *topological index* of φ. This index depends only on the connected component of $C(\widehat{G})^{-1}$ in which φ lies. The following proposition is stated in [10], and as we know when $(G, G_+) = (\mathbb{Z}^2, \mathbb{Z}^2_{p,q})$, it reduces to [2], Theorem 5.

Proposition ([10]). *Let (G, G_+) be a quasily ordered group (G need not admit a deformation retraction onto G_+). If G is torsion-free, then for any $\phi \in C(\widehat{G})$, $T_\phi^{G_+}$ is invertible if and only if ϕ does not vanish anywhere and* $\mathrm{ind}_t(\phi) \in G_+^0$.

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