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THE SPECTRAL PROPERTIES OF CERTAIN LINEAR OPERATORS AND THEIR EXTENSIONS

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ABSTRACT. Let H be a Hilbert space with inner-product (x, y), and let R be a bounded positive operator on H which determines an inner-product, $\langle x, y\rangle = (Rx, y)$, $x, y \in H$. Denote by H^- the completion of H with respect to the norm $\|x\| = \langle x, x\rangle^{1/2}$. In this paper, operators having certain relationships with R are studied. In particular, if $T = SR^{1/2}$ where $S \in B(H)$, then T has an extension $T^- \in B(H^-)$, and T and T^- have essentially the same spectral and Fredholm properties.

INTRODUCTION

Throughout this paper, H is a Hilbert space with inner-product (x, y) and norm $\|x\|_H = (x, x)^{1/2}$. Assume that $\langle x, y\rangle$ is a bounded inner-product on H, so there exists $c > 0$ such that for all $x, y \in H$: $|\langle x, y\rangle| \leq c\|x\|_H\|y\|_H$. Let $\|x\| = \langle x, x\rangle^{1/2}$, and let H^- be the completion of H with respect to the norm $\|x\|$. Since the inner-product $\langle x, y\rangle$ is bounded, it is well-known that there exists a positive operator $R \in B(H)$ such that

$$\langle x, y\rangle = (Rx, y) \quad \text{for all } x, y \in H.$$

For future reference we note that

$$(*) \qquad \|R^{1/2}x\|_H = \|x\| \quad \text{for all } x \in H.$$

Early work in this setting centered on operators of the form $T = SR$ where $S = S^* \in B(H)$; see Chapters 15–17 of [Z]. For such T, $\langle Tx, y\rangle = (RSRx, y) = (Rx, SRy) = \langle x, Ty\rangle$ for all $x, y \in H$. An operator T is *symmetrizable* with respect to an inner-product $\langle x, y\rangle$, if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Thus the operator $T = SR$ above is symmetrizable.

The concept of a symmetrizable operator makes sense whenever there is a bounded inner-product on a Banach space. P. Lax studied symmetrizable operators in this more general setting in [L]. He proved that when T is symmetrizable, then T has an extension $T^- \in B(H^-)$ and $\sigma(T) \supseteq \sigma(T^-)$. Istratescu's book [I, Chapter 11] is a good source of information about symmetrizable operators and related ideas.

Here we restrict attention to the case where the underlying space is the Hilbert space H. Our main results concern operators of the form $T = SR^{1/2}$ where S is

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an arbitrary operator in $B(H)$. It is shown that $T = SR^{1/2}$ has an extension to an operator $T^- \in B(H^-)$, and that T and T^- have essentially the same basic operator properties (for example, they have the same spectrum).

Results

We use the notation from the Introduction in what follows. In particular, R is the positive operator determined by the bounded inner-product $\langle x, y\rangle$. We use the fact that $\mathbf{R}(R^{1/2})$ is dense in H (here, and in what follows, $\mathbf{R}(S)$ denotes the range of the operator S).

Theorem. (1)–(4) *are equivalent for* $T \in B(H)$*:*

(1) RTR^{-1} *is bounded on* $\mathbf{R}(R)$*;*
(2) *there exists an operator* $S \in B(H)$ *such that* $\langle Tx, y\rangle = \langle x, Sy\rangle$ *for all* $x, y \in H$*;*
(3) *there exists an operator* $S \in B(H)$ *such that* $RT = SR$*;*
(4) $T^*(\mathbf{R}(R)) \subseteq \mathbf{R}(R)$.

(5)–(8) *are equivalent for* $T \in B(H)$*:*

(5) $R^{1/2}TR^{-1/2}$ *is bounded on* $\mathbf{R}(R^{1/2})$*;*
(6) T *has an extension to a bounded operator* T^- *on* H^-*;*
(7) *there exists an operator* $S \in B(H)$ *such that* $R^{1/2}T = SR^{1/2}$*;*
(8) $T^*(\mathbf{R}(R^{1/2})) \subseteq \mathbf{R}(R^{1/2})$.

(9)–(12) *are equivalent for* $T \in B(H)$*:*

(9) $TR^{-1/2}$ *is bounded on* $\mathbf{R}(R^{1/2})$*;*
(10) T *has an extension to a bounded operator linear operator* $T^\sim : H^- \to H$*;*
(11) *there exists an operator* $S \in B(H)$ *such that* $T = SR^{1/2}$*;*
(12) $T^*(H) \subseteq \mathbf{R}(R^{1/2})$.

Proof. Clearly, (3)⇒(1). Suppose that (1) holds. Let S denote the bounded extension of RTR^{-1} to all of H. It follows that $RT = SR$. Therefore (3) holds.

Assume that (2) holds. Then $(RTx, y) = (Rx, Sy) = (S^*Rx, y)$ for all $x, y \in H$. Therefore, $RT = S^*R$, so (3) holds. Conversely, if $RT = S^*R$, then reversing the argument above, we have that (2) is true.

That (3)⇒(4) is clear. Now assume that $T^*(\mathbf{R}(R)) \subseteq \mathbf{R}(R)$. Then $\mathbf{R}(T^*R) \subseteq \mathbf{R}(R)$, so by the Douglas Range Inclusion Theorem [D], it follows that $T^*R = RS$ for some operator $S \in B(H)$. Taking adjoints, we have $RT = S^*R$, and thus (3) holds.

Assume that (5) holds. Then there exists $M > 0$ such that

$$\|R^{1/2}TR^{-1/2}(R^{1/2}x)\|_H \leq M\|R^{1/2}x\|_H$$

for all $x \in H$. Thus by (*), $\|Tx\| \leq M\|x\|$ for all $x \in H$, and this implies (6). Also, this argument is reversible, so (6)⇒(5).

Again, assume that (5) holds. Let S be the bounded extension of $R^{1/2}TR^{-1/2}$ on H. It follows immediately that $SR^{1/2} = R^{1/2}T$. Thus (7) holds. Clearly (7)⇒(5).

That (7)⇒(8) is clear. Now assume that $T^*(\mathbf{R}(R^{1/2})) \subseteq \mathbf{R}(R^{1/2})$. Then $\mathbf{R}(T^*R^{1/2}) \subseteq \mathbf{R}(R^{1/2})$. Again, applying the Range Inclusion Theorem, we have $T^*R^{1/2} = R^{1/2}S$ for some operator $S \in B(H)$. Taking adjoints in this equality we see that (7) is true.

Assume that (9) holds. Then there exists $M > 0$ such that $\|TR^{-1/2}(R^{1/2}x)\|_H \leq M\|R^{1/2}x\|_H$ for all $x \in H$. Thus by (*), $\|Tx\|_H \leq M\|x\|$ for all $x \in H$, and this implies (10). Also, this argument is reversible, so (10)⇒(9).

Again, assume that (9) holds. Let S be the bounded extension of $TR^{-1/2}$ on H. Then $T = ST^{1/2}$. Clearly (11)$\Rightarrow$(9).

Finally, making use of the Range Inclusion Theorem as before, it is straightforward to check that (11)$\Leftrightarrow$(12).

Corollary. *Assume $T = SR^{1/2}$, where $S \in B(H)$.*

(a) *The operator T has a bounded extension $T^- \in B(H^-)$ with the property that $T^-(H^-) \subseteq H$.*

(b) *Let $E\colon H \to H^-$ be the continuous embedding map, $Ex = x$ for all x. Let $T^\sim : H^- \to H$ be as in part (10) of the Theorem. Then*

$$T^\sim \in B(H^-, H); \quad T = T^\sim E; \quad T^- = ET^\sim.$$

Proof. Since $R^{1/2}T = (R^{1/2}S)R^{1/2}$, the operator T satisfies (7) in the Theorem. Then by (7)$\Rightarrow$(6), T has a bounded extension T^- on H^-. Also by hypothesis, T satisfies (11), so by (10), T has a bounded extension $T^\sim \in B(H^-, H)$. Then clearly $T^- = ET^\sim$. It follows that $T^-(H^-) \subseteq H$. It is also clear that $T = T^\sim E$. This verifies both parts (a) and (b) of the Corollary.

For a bounded linear operator S, we use the notation:

$$\begin{aligned}
\sigma(S) &= \text{the usual operator spectrum of } S;\\
\sigma_F(S) &\equiv \text{the Fredholm spectrum of } S\\
&\equiv \{\lambda \in \mathbf{C}\colon\ (\lambda - S) \text{ is not a Fredholm operator}\};\\
\sigma_W(S) &\equiv \text{the Weyl spectrum of } S\\
&\equiv \{\lambda \in \mathbf{C}\colon\ (\lambda - S) \text{ is not a Fredholm operator of index zero}\}.
\end{aligned}$$

In what follows, R, T, T^-, and $T^\sim$ are as in the Corollary.

Consequences. In I–III below, assume $T = SR^{1/2}$, where $S \in B(H)$.

I. By part (a) of the Corollary, $T^-(H^-) \subseteq H$. Applying [B1, Theorem 4(2)], we have:
(i) $\sigma(T) = \sigma(T^-)$;
(ii) $\sigma_F(T) = \sigma_F(T^-)$;
(iii) $\sigma_W(T) = \sigma_W(T^-)$.
Also: (iv) when $\lambda \neq 0$, $\mathbf{N}(\lambda - T) = \mathbf{N}(\lambda - T^-)$; here $\mathbf{N}(W)$ denotes the null space of the operator W.

II. By part (b) of the Corollary, $T = T^\sim E$ and $T^- = ET^\sim$. Therefore, T and T^- have all the common operator properties described in [B3]. In particular, when $\lambda \neq 0$:
(i) $\lambda - T$ has a pseudoinverse $\Leftrightarrow$ $\lambda - T^-$ has a pseudoinverse [B3, Theorem 4];
(ii) $\lambda - T$ has closed range $\Leftrightarrow$ $\lambda - T^-$ has closed range [B3, Theorem 5];
(iii) λ is a pole of finite rank of the resolvent of T $\Leftrightarrow$ λ is a pole of finite rank of the resolvent of T^- [B3, Theorem 9].

III. Let $\mathcal{K}$ denote the set of all linear operators $J \in B(H)$ such that J is compact, and J has an extension J^- on H^- which is also compact. By I (iii), $\sigma_W(T) = \sigma_W(T^-)$. It follows from [B2, Theorem 8] that
(i) $\sigma(T + J) = \sigma(T^- + J^-)$ for all $J \in K$.

IV. Now assume that $T = SR$ where $S = S^* \in B(H)$. Operators of this form are common in applications. As noted in the Introduction, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Since $T = (SR^{1/2})R^{1/2}$, consequences I–III hold for T and T^-, and in this case, T^- is selfadjoint.

Example 1. We give an example of a common situation in analysis where the results of this paper apply. Let μ be a measure defined on some σ-algebra of subsets of a set Ω. Let w be a weight function, $w \in L^\infty(\mu)$, with $w(x) > 0$ μ-a.e. Set $H = L^2(\mu)$. Consider the bounded inner-product on H defined by

$$\langle f, g\rangle \equiv \int_\Omega f g^- w \, d\mu \qquad (f, g \in H).$$

Let R be the multiplication operator defined by: $R(f) = wf$, $f \in H$. Then $\langle f, g\rangle = (Rf, g)$ for all $f, g \in H$.

Now let $K(x,t)$ be a kernel that determines a bounded integral operator S on $L^2(\mu)$:

$$S(f)(x) = \int_\Omega K(x,t) f(t) \, d\mu(t), \qquad f \in L^2(\mu).$$

Then consequences I–III apply to the operator $T = SR$,

$$T(f)(x) = \int_\Omega K(x,t) w(t) f(t) \, d\mu(t), \qquad f \in L^2(\mu).$$

When in addition $S = S^*$, then IV also applies to the operator T.

Example 2. There exist symmetrizable operators T for which $\sigma(T)$ and $\sigma(T^-)$ can be very different. Now we modify an example due to J. Nieto in [N] to verify this in our particular setting. Let H be the weighted l^2-space of sequences $\{a_k\}_{k\geq 1}$ such that $\sum_1^\infty 4^k |a_k|^2 < \infty$. The inner-product on H is:

$$(\{a_k\}, \{b_k\}) = \sum_1^\infty 4^k a_k b_k^-.$$

Consider the inner-product on H defined by:

$$\langle \{a_k\}, \{b_k\}\rangle = \sum_1^\infty a_k b_k^-.$$

It is easy to check that this inner-product is bounded on H, and that the positive operator R such that $(Ra, b) = \langle a, b\rangle$ is the multiplication operator $R(\{a_k\}) = \{4^{-k} a_k\}$.

Let S and B be the shift and backward shift on H, so $S(a_1, a_2, a_3, \dots) = (0, a_1, a_2, \dots); B(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$. Let $T = S + B$, and note that T is selfadjoint relative to the inner-product $\langle a, b\rangle$. Let H^- be the completion of H relative to the norm determined by the inner-product $\langle a, b\rangle$, so H^- is the usual sequence space l^2. Let S^-, B^-, and T^- denote the extensions of S, B, and T to l^2. The extension T^- has real spectrum (in fact, $\sigma(T^-) = [-2, 2]$). Now we compute the spectrum of T in $B(H)$. Let $W\colon\ H \to l^2$ be defined by $W(\{a_k\}) = \{2^k a_k\}$. Note that W is a linear isometry that maps H onto l^2. A straightforward computation verifies that:

$$WSW^{-1} = 2S^-; \quad WBW^{-1} = \frac{1}{2}B^-; \quad \text{so, } WTW^{-1} = 2S^- + \frac{1}{2}B^-.$$

These operators act on l^2. The spectrums of these operators have been computed; see [N, Prop. 2]. Using this result, we have that $\sigma(WTW^{-1}) = \Gamma \equiv$ {all the numbers in the complex plane which are inside or on the ellipse $\frac{4}{25}x^2+\frac{4}{9}y^2 = 1$}. Thus,

$$\sigma(T) = \Gamma \supseteq [-2, 2] = \sigma(T^-).$$

We note that in contrast, the weighted shift and weighted backward shift, $SR^{1/2}$ and $BR^{1/2}$, have the same spectral properties on H and H^-.

Example 3. Let H, H^-, and R be as in Example 2. We construct an example of an operator $W \in B(H)$ such that $T = WR^{1/2}$ does not have an adjoint in $B(H)$ with respect to the inner-product $\langle a, b\rangle$. Not only does T have a bounded extension T^- on H^- [Corollary], but also T and T^- satisfy the consequences I, II, and III.

Let e_k denote the vector in H with kth coordinate 1 and with all other coordinates 0. Note that $\|e_k\|_H = 2^k$ for all k, and that the sequence $\{2^{-k}e_k\}_{k\geq 1}$ is an orthonormal basis for H. Define W on this sequence by $W(e_k) = 0$ if $k \neq m^2$ for $m \geq 1$, and $W(2^{-m^2}e_{m^2}) = 2^{-m}e_m$, otherwise. Clearly $W \in B(H)$. Now we show that $RWR^{1/2}R^{-1}$ is not bounded on $\mathbf{R}(R)$, so part (1) of the Theorem cannot hold for T. Since $R^{-1/2}(e_k) = 2^k e_k$,

$$RWR^{-1/2}(2^{-m^2}e_{m^2}) = RW(e_{m^2}) = R(2^{(m^2-m)}e_m) = 4^{-m}2^{(m^2-m)}e_m.$$

Finally, $\|4^{-m}2^{(m^2-m)}e_m\|_H = 4^{-m}2^{m^2} \to \infty$ as $m \to \infty$. This proves that $T = WR^{1/2}$ does not have an adjoint in $B(H)$ with respect to the inner-product $\langle a, b\rangle$.

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