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SPLITTINGS OF BANACH SPACES INDUCED BY CLIFFORD ALGEBRAS

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Abstract. Let H be an infinite-dimensional Hilbert space of density character $\mathfrak{m}$. By representing H as a module over an appropriate Clifford algebra, it is proved that H possesses a family $\{A_\alpha\}_{\alpha\in\mathfrak{m}}$ of proper closed nonzero subspaces such that

$$d(S_{A_\alpha}, S_{A_\beta}) = d(S_{A_\alpha^\perp}, S_{A_\beta}) = d(S_{A_\alpha^\perp}, S_{A_\beta^\perp}) = \sqrt{2-\sqrt{2}} \qquad (\alpha \neq \beta).$$

Analogous results are proved for L_p spaces and for $c_0(X)$ and $\ell_p(X)$ $(1 \leq p \leq \infty)$ when X is an arbitrary nonzero Banach space.

1. Introduction and notation

Let us begin with some notation. Let C_1 and C_2 be nonempty subsets of a Banach space $(X, \|\cdot\|)$. The distance between C_1 and C_2, denoted $d(C_1, C_2)$, is defined as follows:

$$d(C_1, C_2) = \inf\{\|c_1 - c_2\| : c_1 \in C_1, c_2 \in C_2\}.$$

For a closed subspace A of X, its *unit sphere*, denoted S_A, is the set $\{a \in A : \|a\| = 1\}$.

A decomposition of X into the Banach space direct sum $X = A \oplus B$ of two nonzero closed subspaces A and B will be called a *splitting* of X, denoted (A, B). For a given family $\{(A_\gamma, B_\gamma) : \gamma \in \Gamma\}$ of splittings of X, a convenient measure of the extent to which the splittings in this family differ from one another is afforded by the quantity

$$\delta = \inf\{d(S_{A_\alpha}, S_{A_\beta}), d(S_{B_\alpha}, S_{B_\beta}), d(S_{A_\alpha}, S_{B_\gamma}) : \alpha, \beta, \gamma \in \Gamma \quad (\alpha \neq \beta)\}.$$

If $\delta > 0$, then the unit spheres of all the subspaces occurring in the splittings are separated from each other by a distance δ. Such a family of splittings will be said to be *well-separated*.

This paper proves the existence of infinite well-separated families of splittings for certain Banach spaces. First the case of an infinite-dimensional Hilbert space H is considered; here it is more natural to consider only orthogonal splittings (i.e. orthogonal decompositions) of H. It is proved that if H has density character $\mathfrak{m}$, then there exists a family of orthogonal splittings of H of cardinality $\mathfrak{m}$ for which $\delta = \sqrt{2-\sqrt{2}}$, which is best possible. The Hilbert space argument is then

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generalized to prove that if X is an arbitrary nonzero Banach space, then $c_0(X)$ and $\ell_p(X)$ $(1 \leq p \leq \infty)$ admit infinite well-separated families of contractively complemented splittings.

The main idea in the proof is to represent H (or $\ell_p(X)$) as a module over an appropriate infinite-dimensional Clifford algebra. The existence of the required splittings is then a consequence of algebraic identities in the Clifford algebra. The proof is self-contained and presupposes only an acquaintance with the terminology of elementary abstract algebra.

The Banach space notation and terminology employed throughout are standard. Let us only recall that the space $\ell_p(X)$ $(1 \leq p \leq \infty)$ is the space of sequences $\langle x_n \rangle$ $(x_n \in X)$ equipped with the norm

$$\|\langle x_n \rangle\| = \begin{cases} (\sum \|x_n\|^p)^{1/p} & \text{for } p < \infty; \\ \sup_n \|x_n\| & \text{for } p = \infty. \end{cases}$$

The space $c_0(X)$ is the subspace of $\ell_\infty(X)$ whose elements consist of sequences which tend to zero in norm.

The proofs are valid for both real and complex Banach spaces: the underlying field of scalars (either $\mathbb{R}$ or $\mathbb{C}$) will be denoted by $\mathbb{F}$.

2. Splittings of Hilbert spaces

Let H be a separable infinite-dimensional Hilbert space. For $2 \leq n \leq \infty$, define δ_n as follows:

$$\delta_n = \sup\{\inf\{d(S_{A_j}, S_{A_k}),\, d(S_{A_j^\perp}, S_{A_k}),\, d(S_{A_j^\perp}, S_{A_k^\perp}) : 0 \leq j, k < n, j \neq k\}\},$$

where the supremum is taken over all n-tuples $\{A_j\}_{0 \leq j < n}$ of proper closed nonzero subspaces of H. Clearly, $\{\delta_n\}_{n \geq 1}$ is a decreasing sequence of nonnegative numbers with $0 \leq \delta_\infty \leq \lim \delta_n$.

We prove below (Theorem 3) that $\delta_n = \sqrt{2 - \sqrt{2}}$ for all $2 \leq n \leq \infty$.

Proposition 1. *Let A be a subspace of H and let P be the orthogonal projection onto A.*

(a) *If $x \in S_H$, then*

$$\min\{d(x, S_A), d(x, S_{A^\perp})\} \leq \sqrt{2 - \sqrt{2}} \tag{1}$$

with equality if and only if $\|Px\| = 1/\sqrt{2}$ (in which case $d(x, S_A) = d(x, S_{A^\perp}) = \sqrt{2 - \sqrt{2}}$).

(b) *Let B be the closed linear span of an orthonormal sequence $\{e_n\}$. Then*

$$d(S_A, S_B) = d(S_{A^\perp}, S_B) = \sqrt{2 - \sqrt{2}} \tag{2}$$

if and only if $\{\sqrt{2} P e_n\}$ is an orthonormal sequence in H.

Proof. (a) Let $a = Px$ and $a' = (I - P)x$. Then $1 = \|x\|^2 = \|a\|^2 + \|a'\|^2$, and so $\max(\|a\|, \|a'\|) \geq 1/\sqrt{2}$. Without loss of generality, we shall assume that $\|a\| \geq 1/\sqrt{2}$. Let $y \in S_A$. Then

$$\|y - x\|^2 = \|y - a\|^2 + \|a'\|^2.$$

The distance $\|y-a\|$ is minimized when $y = a/\|a\|$; in this case, $\|y-a\| = 1-\|a\|$, and so

$$\begin{aligned}\|y-x\|^2 &= (1-\|a\|)^2 + \|a'\|^2 \\ &= (1-\|a\|)^2 + (1-\|a\|^2) \\ &= 2-2\|a\| \\ &\leq 2-\sqrt{2},\end{aligned}$$

since we are assuming that $\|a\| \geq 1/\sqrt{2}$. It follows that $d(x, S_A) \leq \sqrt{2-\sqrt{2}}$, with equality if and only if $\|a\| = \|a'\| = 1/\sqrt{2}$.

(b) Suppose that (2) holds. Then from (a) we deduce that $d(x, S_A) = d(x, S_{A^\perp}) = \sqrt{2-\sqrt{2}}$ for all $x \in S_B$, whence $\|Px\| = 1/\sqrt{2}$ (by (a) again) for all $x \in S_B$. It follows that $\sqrt{2}P$ is an isometry from B into A, and hence that $\{\sqrt{2}Pe_n\}$ is an orthonormal sequence in A. Conversely, if $\{\sqrt{2}Pe_n\}$ is an orthonormal sequence in A, then $\|Px\| = 1/\sqrt{2}$ for all $x \in S_B$, and so (by (a)) $d(x, S_A) = d(x, S_{A^\perp}) = \sqrt{2-\sqrt{2}}$ for all $x \in S_B$, which gives (2). □

Let $\{e_n\}$ be an orthonormal basis for H, let A be the closed linear span of the orthonormal sequence $\{e_{2n}\}_{n\geq 1}$, and let B be the closed linear span of the orthonormal sequence $\{(1/\sqrt{2})(e_{2n} + e_{2n-1})\}$. It follows from Proposition 1 that

$$d(S_A, S_B) = d(S_{A^\perp}, S_B) = d(S_A, S_{B^\perp}) = d(S_{A^\perp}, S_{B^\perp}) = \sqrt{2-\sqrt{2}}, \tag{3}$$

and also that this pair of splittings is the best possible in the sense that the constant $\sqrt{2-\sqrt{2}}$ cannot be improved (i.e., increased). This proves that $\delta_2 = \sqrt{2-\sqrt{2}}$.

In fact, (3) determines A and B uniquely up to an isomorphism of H, as the following result shows.

Corollary 2. *Suppose that H is a separable infinite-dimensional Hilbert space. Let A and B be closed subspaces of H which satisfy equality (3). Then there exist orthonormal bases $\{e_n\}$ and $\{f_n\}$ of A and $A^\perp$ such that $\{(1/\sqrt{2})(e_n + f_n)\}$ and $\{(1/\sqrt{2})(e_n - f_n)\}$ are orthonormal bases of B and $B^\perp$.*

Proof. Let P be the orthogonal projection of H onto A and let $Q = I - P$. Let $\{g_n\}$ be an orthonormal basis for B. Then, from Proposition 1 and (3), it follows that $\{e_n\}$ and $\{f_n\}$ are orthonormal sequences in A and $A^\perp$, where $e_n = (\sqrt{2})P(g_n)$ and $f_n = (\sqrt{2})Q(g_n)$. Let us show that $\{e_n\}$ is in fact an orthonormal *basis* of A. So suppose that $e \in A$ and that $\langle e, e_n\rangle = 0$ for all n. Then $\langle e, P(y)\rangle = 0$ for all $y \in B$. From Proposition 1 and (3) we have $e = y + z$ where $y \in B$, $z \in B^\perp$, and $\|y\| = \|z\| = (1/\sqrt{2})\|e\|$. Hence $e = P(e) = P(y) + P(z)$, and so

$$\frac{\|e\|^2}{4} = \|P(z)\|^2 = \|e\|^2 + \|P(y)\|^2 = \|e\|^2 + \frac{\|e\|^2}{4}.$$

Thus $e = 0$, which implies that $\{e_n\}$ is an orthonormal basis of A. Similarly $\{f_n\}$ is an orthonormal basis of $A^\perp$. Note that $g_n = (1/\sqrt{2})(e_n + f_n)$ and recall that $\{g_n\}$ is an orthonormal basis of B. Clearly, $\{(1/\sqrt{2})(e_n - f_n)\}$ is an orthonormal basis of $B^\perp$. □

Theorem 3. *Let H be a separable infinite-dimensional Hilbert space. There exists a sequence $\{A_n\}$ of subspaces of H such that*

$$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{A_m^\perp}) = d(S_{A_n^\perp}, S_{A_m^\perp}) = \sqrt{2-\sqrt{2}} \qquad (n \neq m).$$

In particular, $\delta_n = \sqrt{2-\sqrt{2}}$ for all $2 \leq n \leq \infty$.

Proof. We begin our construction by defining a group G which is generated by an element -1 and by the sequence of symbols $\{e_n\}_{n\geq 1}$ satisfying the generating relations:

$$e_n^2 = -1 \qquad \text{and} \qquad (-1)^2 = 1.$$

The element -1 belongs to the center of G and we define for any $g \in G$:

$$-g = (-1)(g) = (g)(-1).$$

Multiplication among the e_n's is defined to be anti-commutative:

$$e_m\, e_n = -e_n\, e_m \qquad (n \neq m).$$

The elements of G consist of finite strings of the generators. For any finite subset $I \subset \mathbb{N}$, let $e_I = e_{n_1}\, e_{n_2} \cdots e_{n_k}$, where $I = \{n_1, n_2, \ldots, n_k\}$ and $1 \leq n_1 < n_2 < \cdots < n_k$, and let $e_I = 1$ for $I = \emptyset$. Finally, setting $W = \{e_I : I \text{ is a finite subset of } \mathbb{N}\}$, we can list the group elements thus:

$$G = \{w, -w : w \in W\}.$$

Consider the collection C of all finite sums of the form $\sum_{w\in W} \lambda_w\, w$, where $\lambda_w \in \mathbb{F}$. Next define scalar multiplication on $G \setminus W = \{-w : w \in W\}$ as one would expect:

$$\lambda\,(-w) = (-\lambda)\, w \qquad (w \in W, \lambda \in \mathbb{F}).$$

Then C becomes an algebra over $\mathbb{F}$ with multiplication defined in the obvious way using the group multiplication of G and the distributive law. In fact, if we denote by V the inner product space over $\mathbb{F}$ which has orthonormal basis $\{e_n\}$, then C is the *universal Clifford algebra* associated to V. We refer the reader to [3] for a discussion of Clifford algebras and for a proof of the associativity of the algebra multiplication.

Let the symbols x and y be the generators of a free left-module over C, which we shall denote by M. It follows that every $m \in M$ is uniquely expressible in the form $c_1\, x + c_2\, y$, where c_1, $c_2 \in C$. Note that the elements of M are of the form

$$m = \Big(\sum_{w\in W} \lambda_w\, w\Big)\, x + \Big(\sum_{w\in W} \mu_w\, w\Big)\, y = \sum_{w\in W} (\lambda_w)\, w\, x + \sum_{w\in W} (\mu_w)\, w\, y,$$

where the sums are finite and the λ's and μ's belong to $\mathbb{F}$. Next we turn the module M into an inner product space over $\mathbb{F}$ by taking the set $\Xi = \{wx, wy : w \in W\}$ to be an orthonormal spanning set. Let H be the completion of this inner product space, so that H is a separable infinite-dimensional Hilbert space with orthonormal basis Ξ. Note that the inner product $\langle\ ,\ \rangle$ on H is defined by the relations

$$\langle wx, w'y\rangle = 0 \qquad (w, w' \in W)$$

and

$$\langle wx, w'x\rangle = \langle wy, w'y\rangle = \begin{cases} 1, & \text{for } w = w', \\ 0, & \text{for } w \neq w'. \end{cases}$$

For each $w' \in W$ there exists a permutation $\pi_{w'}$ of W such that the mapping $w \mapsto w'w$ is given by

$$w'w = \pm\pi_{w'}(w),$$

where the choice of signs depends, of course, on the element w. For convenience, let us call such a mapping a *sign-changing permutation.* It follows that the linear operator corresponding to the action of $w' \in W$ on H given by

$$w'\Big\{\sum_{w}(\lambda_w\, w\, x + \gamma_w\, w\, y)\Big\} = \sum_{w}\Big\{\lambda_w(w'wx) + \gamma_w(w'wy)\Big\} \tag{4}$$

is in fact an isomorphism of H, as it is the linear extension of a sign-changing permutation of the orthonormal basis Ξ. We may now regard H as a left C-module by extending the action of (4) to the whole of C by linearity.

Let A_n be the norm-closed left C-submodule generated by the element $x + e_n y$. One easily sees that

$$\Xi_n = \left\{\frac{wx + (we_n)y}{\sqrt{2}}\ :\ w \in W\right\}$$

is an orthonormal basis for A_n, and that

$$\Xi_n' = \left\{\frac{wx - (we_n)y}{\sqrt{2}}\ :\ w \in W\right\}$$

is an orthonormal basis for $A_n^{\perp}$. Having defined the subspaces that make up our sequence of splittings, it remains to show that the distances $d(S_{A_n}, S_{A_m})$, $d(S_{A_n}, S_{A_m^{\perp}})$ and $d(S_{A_n^{\perp}}, S_{A_m^{\perp}})$ are all equal to $\sqrt{2-\sqrt{2}}$ for $n \neq m$.

Let P_n denote the orthogonal projection from H onto A_n. To prove that $d(S_{A_n}, S_{A_m}) = \sqrt{2-\sqrt{2}}$, we shall invoke Proposition 1. It suffices to prove that P_n maps the basis Ξ_m of A_m onto an orthogonal sequence of vectors in A_n, all having norm $1\sqrt{2}$. To this end, let us express the vector $(x + e_m y)/\sqrt{2}$ in the form

$$\frac{x + e_m y}{\sqrt{2}} = \left(\frac{1 - e_m e_n}{2}\right)\left(\frac{x + e_n\, y}{\sqrt{2}}\right) + \left(\frac{1 + e_m e_n}{2}\right)\left(\frac{x - e_n\, y}{\sqrt{2}}\right).$$

Then, for $w \in W$, we have

$$\frac{wx + (we_m)y}{\sqrt{2}} = \left(\frac{w(1 - e_m e_n)}{2}\right)\left(\frac{x + e_n\, y}{\sqrt{2}}\right) + \left(\frac{w(1 + e_m e_n)}{2}\right)\left(\frac{x - e_n\, y}{\sqrt{2}}\right).$$

Note that in this form the vector $(wx + (we_m)y)/\sqrt{2}$ is written as the sum of vectors from A_n and $A_n^{\perp}$, and so it is easy to compute its projection onto A_n:

$$\begin{aligned} P_n\left(\frac{wx + (we_m)\, y}{\sqrt{2}}\right) &= \frac{w(1 - e_m e_n)}{2} \cdot \frac{(x + e_n\, y)}{\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}}\Big\{wx - (we_m e_n)\, x + (we_n)\, y - (we_m e_n e_n)\, y\Big\} \\ &= \frac{1}{2\sqrt{2}}\Big\{wx - (we_m e_n)\, x + (we_n)\, y + (we_m)\, y\Big\}. \end{aligned} \tag{5}$$

First observe that from (5)

$$\left\| P_n\left(\frac{wx + (we_m)\,y}{\sqrt{2}}\right)\right\| = \frac{1}{2\sqrt{2}}(1^2 + 1^2 + 1^2 + 1^2)^{1/2} = \frac{1}{\sqrt{2}},$$

which proves that P_n maps each member of Ξ_m onto a vector of norm $1/\sqrt{2}$.

Now let us show that P_n maps Ξ_m onto an orthogonal sequence; that is, that

$$\left\langle P_n\left(\frac{wx + (we_m)\,y}{\sqrt{2}}\right),\ P_n\left(\frac{w'x + (w'e_m)\,y}{\sqrt{2}}\right)\right\rangle = 0$$

for all $w \neq w'$.

Replacing w by w' in (5), we get

$$(6)\qquad P_n\left(\frac{w'x + (w'e_m)\,y}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\Big\{w'x - (w'e_me_n)\,x + (w'e_n)\,y + (w'e_m)\,y\Big\}.$$

The four vectors occurring on the right-hand side of (5) and the four on the right-hand side of (6) belong to the orthonormal basis Ξ of H. If these sets of basis vectors are disjoint, then clearly the inner product is zero. These sets of basis vectors will be disjoint unless

$$(7)\qquad w' = \pm\, we_me_n \quad \text{or} \quad w = \pm\, w'e_me_n.$$

It remains to show that the inner product is zero if one of the two conditions of (7) holds. Without loss of generality, let us assume that $w' = we_me_n$. In this case, we have

$$\begin{aligned} w'x - &(w'e_me_n)\,x + (w'e_n)\,y + (w'e_m)\,y \\ &= (we_me_n)\,x - (we_me_ne_me_n)\,x + (we_me_ne_n)\,y + (we_me_ne_m)\,y \\ &= (we_me_n)\,x - (we_m(-e_me_n)e_n)\,x + (we_m(-1))\,y + (we_m(-e_me_n))\,y \\ &= (we_me_n)\,x + (w(-1)(-1))x - (we_m)\,y - (w(-1)e_n)\,y \\ &= (we_me_n)\,x + wx - (we_m)\,y + (we_n)\,y, \end{aligned}$$

and it follows from (6) that

$$P_n\left(\frac{w'x + (w'e_m)\,y}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\{wx + (we_me_n)\,x - (we_m)\,y + (we_n)\,y\}.$$

Computing the inner product, we get

$$\begin{aligned} &\left\langle P_n\left(\frac{wx + (we_m)\,y}{\sqrt{2}}\right), P_n\left(\frac{w'x + (w'e_m)\,y}{\sqrt{2}}\right)\right\rangle \\ &= \frac{1}{8}\Big\langle wx - (we_me_n)\,x + (we_m)\,y + (we_n)\,y, \\ &\qquad\qquad wx + (we_me_n)\,x - (we_m)\,y + (we_n)\,y\Big\rangle \\ &= \frac{1}{8}\Big\{\langle wx, wx\rangle - \langle (we_me_n)\,x, (we_me_n)\,x\rangle \\ &\qquad\qquad - \langle (we_m)\,y, (we_m)\,y\rangle + \langle (we_n)\,y, (we_n)\,y\rangle\Big\} \\ &= \frac{1}{8}\Big\{\|wx\|^2 - \|(we_me_n)\,x\|^2 - \|(we_m)\,y\|^2 + \|(we_n)\,y\|^2\Big\} \\ &= \frac{1}{8}\{1 - 1 - 1 + 1\} = 0. \end{aligned}$$

Thus, P_n maps Ξ_m onto an orthogonal sequence of vectors in A_n, which completes the proof that $d(S_{A_n}, S_{A_m}) = \sqrt{2-\sqrt{2}}$. Similar computations prove that $d(S_{A_n}, S_{A_m^\perp})$ and $d(S_{A_n^\perp}, S_{A_m^\perp})$ are also equal to $\sqrt{2-\sqrt{2}}$. □

The proof of Theorem 3 readily extends to nonseparable Hilbert spaces to yield the result stated in the abstract (or a paraphrase thereof).

Theorem 4. *Let Γ be an infinite set. Then there exists a family $\{A_\gamma\}_{\gamma\in\Gamma}$ of subspaces of $\ell_2(\Gamma)$ such that*

$$d(S_{A_\gamma}, S_{A_\delta}) = d(S_{A_\gamma}, S_{A_\delta^\perp}) = d(S_{A_\gamma^\perp}, S_{A_\delta^\perp}) = \sqrt{2-\sqrt{2}}$$

for all $\gamma, \delta \in \Gamma$ with $\gamma \neq \delta$.

Proof. The proof is analogous to the proof of Theorem 3, with the modification that the group G is now generated by the (possibly uncountable) set $\{e_\gamma\}_{\gamma\in\Gamma}$. We construct the Clifford algebra C that is associated to the inner product space which has $\{e_\gamma\}_{\gamma\in\Gamma}$ as an orthonormal basis. Next we define the C-module M and the Hilbert space H, as in the proof of Theorem 3, and one sees that H has an orthonormal spanning set of the same cardinality as Γ. Then we define a collection of subspaces A_γ ($\gamma \in \Gamma$), as in the proof of Theorem 3. The distance calculations are identical. □

Remark 5. Obviously, the cardinality of Γ is the largest possible cardinality of any well-separated family of splittings of $\ell_2(\Gamma)$.

3. Splittings of Banach spaces

Recall that there exists an infinite-dimensional Banach space X [1] which is *hereditarily indecomposable*; that is, X has the property that no closed subspace of X can be expressed as a direct sum of two further infinite-dimensional closed subspaces. Let $X = A_1 \oplus B_1$ and $X = A_2 \oplus B_2$ be a pair of splittings of X such that both A_1 and A_2 are infinite-dimensional. Since X is indecomposable, both B_1 and B_2 are finite-dimensional, whence $A_1 \cap A_2$ is nonzero, and in particular $d(S_{A_1}, S_{A_2}) = 0$. This shows that Theorem 3 cannot be extended to the category of Banach spaces simply by replacing orthogonal projections by bounded projections.

However, we are able to modify our construction in order to obtain a result for $\ell_p(X)$, when X is an *arbitrary* nonzero Banach space. Regarding X and M as vector spaces over $\mathbb{F}$, we may form the vector space tensor product $X \otimes M$. A typical element of $X \otimes M$ may be expressed uniquely as a finite sum of the form:

$$a = \sum_{w\in W} \{a_w \otimes (wx) + a'_w \otimes (wy)\} \qquad (a_w, a'_w \in X). \tag{8}$$

We now equip $X \otimes M$ with the norm

$$|||a|||_p = \begin{cases} \left(\sum \|a_w\|^p + \|a'_w\|^p\right)^{1/p} & \text{for } p < \infty, \\ \sup\{\|a_w\|, \|a'_w\| : w \in W\} & \text{for } p = \infty. \end{cases}$$

Taking the completion of $|||\cdot|||_p$ we obtain, for $p < \infty$, a Banach space X_p that is isometrically isomorphic to $\ell_p(X)$, and, for $p = \infty$, a space X_∞^0 that is isometrically isomorphic to $c_0(X)$. The elements of X_p have the unique representation given by (8) when the sums are allowed to be infinite.

Similarly, one defines the space X_∞, which is isometrically isomorphic to $\ell_\infty(X)$, in the obvious fashion.

The action of the Clifford algebra C on M extends to $X \otimes M$ by means of the following definition:

$$c\left(\sum\{a_w \otimes (wx) + a'_w \otimes (wy)\}\right) = \sum\{a_w \otimes (cwx) + a'_w \otimes (cwy)\}.$$

Extending this action to X_p (by continuity) turns these Banach spaces into C-modules for which each $w \in W$ acts as an *isometric isomorphism* of X_p.

We can now define, for each $n \geq 1$, a pair of complementary subspaces in X_p. Let A_n consist of all vectors $a \in X_p$ of the form

$$a = \sum_{w\in W} a_w \otimes (wx + we_n y),$$

and let B_n consist of all vectors b of the form

$$b = \sum_{w\in W} b_w \otimes (wx - we_n y).$$

Proposition 6. *Let $1 \leq p \leq \infty$. Then A_n and B_n are contractively complemented in X_p and*

$$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_n}, S_{B_m}) \geq \frac{1}{2} \qquad (m \neq n).$$

The same result holds for X^0_∞.

Proof. We give the proof only for $p < \infty$. Fix $m, n \in \mathbb{N}$. Consider the projection P_n on X_p given by

$$P_n(a) = \sum_{w\in W} ((1/2)a_w + (1/2)a'_w) \otimes (wx + we_n y),$$

when a is represented (uniquely) as:

$$a = \sum_{w\in W} \{a_w \otimes (wx) + a'_w \otimes (we_n y)\} \qquad (a_w, a'_w \in X).$$

Clearly A_n is the range of P_n, and P_n is contractive by convexity of the norm in X. The complementary projection $Q_n = I - P_n$ is also contractive and has range B_n. Suppose that $m \neq n$. Observe that W can be expressed as the disjoint union $W = W_1 \cup W_2$, in which each $w \in W_1$ corresponds to a unique element $w' = \pm we_m e_n$ in W_2. Thus each $a \in A_m$ can be written uniquely in the form

$$a = \sum_{w\in W_1} \{a_w \otimes (w(x + e_m y)) + b_w \otimes ((we_m e_n)(x + e_m y))\}.$$

From the identity

$$wx + (we_m)y = \left(\frac{w(1 - e_m e_n)}{2}\right)(x + e_n\, y) + \left(\frac{w(1 + e_m e_n)}{2}\right)(x - e_n\, y),$$

we obtain

$$\begin{aligned} P_n(a) &= \sum_{w\in W_1} a_w \otimes \left(\frac{w(1-e_m e_n)}{2}\right)(x+e_n\, y) \\ &\qquad + \sum_{w\in W_1} b_w \otimes \left(\frac{w e_m e_n(1-e_m e_n)}{2}\right)(x+e_n\, y) \\ &= \sum_{w\in W_1} \left\{ \left(\frac{a_w+b_w}{2}\right)\otimes w(x+e_n y) + \left(\frac{b_w-a_w}{2}\right)\otimes w e_m e_n(x+e_n y)\right\}, \end{aligned}$$

whence

$$\begin{aligned} |||P_n(a)|||^p &= \sum_{w\in W_1} \{\|(1/2)(a_w+b_w)\|^p + \|(1/2)(a_w-b_w)\|^p\} \\ &\geq 2^{-p} \sum_{w\in W_1} \{\|a_w\|^p + \|b_w\|^p\} \qquad (9) \\ &= 2^{-p}|||a|||^p, \end{aligned}$$

and so $|||P_n(a)||| \geq (1/2)|||a|||$. A similar calculation shows that $|||Q_n(a)||| \geq (1/2)|||a|||$. Finally,

$$d(a, S_{A_n}) \geq |||Q_n(a)||| \geq (1/2)|||a|||,$$

which implies that $d(S_{A_m}, S_{A_n}) \geq 1/2$. By symmetry, it follows that

$$d(S_{B_m}, S_{A_n}) = d(S_{B_m}, S_{B_n}) = d(S_{A_m}, S_{A_n}). \qquad \square$$

Since X_p is isometrically isomorphic to $\ell_p(X)$, we obtain the following theorem.

Theorem 7. *Let $1 \leq p < \infty$ and let X be an arbitrary nonzero Banach space. For each $n \geq 1$, there exist closed subspaces A_n and B_n of $\ell_p(X)$ such that the following hold:*

(a) *$\ell_p(X) = A_n \oplus B_n$ and the corresponding projections are contractions;*
(b) *$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_n}, S_{B_m}) \geq 1/2$ for $m \neq n$.*

The same result holds for $c_0(X)$ and $\ell_\infty(X)$.

Next we consider splittings of Lebesgue L_p spaces for which we require the following lemma, whose proof is an easy deduction from Clarkson's inequalities (see e.g. [4]), which we leave to the reader.

Lemma 8. *Let (X, Σ, μ) be a measure space and let $1 < p < \infty$. For all $f, g \in L_p(\mu)$, we have*

$$\left(\left\|\frac{f+g}{2}\right\|^p + \left\|\frac{f-g}{2}\right\|^p\right) \geq 2^{-p/p^*}(\|f\|^p + \|g\|^p),$$

where $p^* = \min(p, p/(p-1))$.

Theorem 9. *Let (X, Σ, μ) be a separable σ-finite measure space which has either no atoms or infinitely many atoms and let $1 < p < \infty$. For each $n \geq 1$, there exist closed subspaces A_n and B_n of $L_p(\mu)$ such that the following hold:*

(a) *$L_p(\mu) = A_n \oplus B_n$ and the corresponding projections are contractions;*
(b) *$d(S_{A_n}, S_{A_m}) = d(S_{A_n}, S_{B_m}) = d(S_{B_n}, S_{B_m}) \geq 2^{-1/p^*}$ for $m \neq n$, where $p^* = \min(p, p/(p-1))$.*

The same result holds for $L_1(\mu)$ and $L_\infty(\mu)$ with constant $1/2$.

Proof. The assumption on (S, Σ, μ) implies that $L_p(\mu)$ is isometrically isomorphic to $\ell_p(L_p(\mu))$. (In fact $L_p(\mu)$ will be isometrically isomorphic to either ℓ_p, $L_p(0,1)$ or $\ell_p \oplus_p L_p(0,1)$.) Using Lemma 8 we can replace the factor of 2^{-p} in equation (9) of Proposition 6 by the larger constant of $2^{-p/p^*}$. With this change the argument of Proposition 6 yields the desired result. □

Remark 10. The constant $2^{-1/p^*}$ is probably not best possible. In fact, we have already seen that for $p = 2$ the best constant is $\sqrt{2-\sqrt{2}} = 0.7653\ldots$ (as opposed to $1/\sqrt{2} = 0.7071\ldots$). Finally, let us observe one further consequence.

Corollary 11. *Let K be an infinite compact metric space. Then $C(K)$ admits an infinite well-separated family of splittings.*

Proof. This follows from Theorem 7 since $C(K)$ is linearly *isomorphic* to $c_0(C(K))$ whenever K is an infinite compact metric space (see e.g. [4]). □

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