PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 5, Pages 1275–1283
S 0002-9939(99)05398-8
Article electronically published on August 5, 1999

ANALYTIC CONTINUATION OF MULTIPLE ZETA FUNCTIONS

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(Communicated by Dennis A. Hejhal)

Abstract. In this paper we shall define the analytic continuation of the multiple (Euler-Riemann-Zagier) zeta functions of depth d:

$$\zeta(s_1,\dots,s_d):=\sum_{0<n_1<n_2<\dots<n_d}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_d^{s_d}},$$

where $\mathrm{Re}\,(s_d)>1$ and $\sum_{j=1}^d \mathrm{Re}\,(s_j)>d$. We shall also study their behavior near the poles and pose some open problems concerning their zeros and functional equations at the end.

1. Introduction

The Riemann zeta function $\zeta(s)$ is one of the most important objects in the study of number theory. It is a classical and well-known result that $\zeta(s)$, originally defined on the half plane $\mathrm{Re}\,(s)>1$, can be analytically continued to a meromorphic function on the entire complex plane with the only pole at $s=1$, which is a simple pole with residue 1.

One of the main reasons that people are interested in $\zeta(s)$ is that the special values of $\zeta(s)$ at integers have been proved/conjectured to have significant arithmetic meanings; for instance, Zagier's conjecture [10] concerning the relation between $\zeta(n)$ and the n-logarithms for $n\geq 2$ and Lichtenbaum's conjecture [9] connecting $\zeta(n)$ with motivic cohomology.

One way to generalize the Riemann zeta function is to define the "multiple (Euler-Riemann-Zagier) zeta function" of depth d as follows:

(1)

$$\zeta(s_1,\dots,s_d):=\sum_{0<n_1<n_2<\dots<n_d}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_d^{s_d}},\qquad \mathrm{Re}\,(s_d)>1,\sum_{j=1}^d \mathrm{Re}\,(s_j)>d.$$

Proposition 1. *The infinite sum on the right-hand of* (1) *converges absolutely when* $\mathrm{Re}\,(s_d)>1$ *and* $\sum_{j=1}^d \mathrm{Re}\,(s_j)>d$.

Proof. The case $d=1$ is trivial so let's assume $d=2$. Write $\sigma_i=\mathrm{Re}\,(s_i)$. If $\sigma_1>1$, then the result is standard. If $\sigma_1=1$, then our proposition follows from

Received by the editors June 21, 1998.
1991 *Mathematics Subject Classification.* Primary 11M99; Secondary 30D30, 30D10.
Key words and phrases. Analytic continuation, multiple zeta function, generalized function.

the well-known asymptotic formula

$$\sum_{n_1=1}^{n} 1 = \ln n + O(1).$$

If $\sigma_1 < 1$, then one has the following estimate:

$$\begin{aligned}\sum_{n_1=1}^{n} \frac{1}{n_1^{\sigma_1} n^{\sigma_2}} &= \sum_{k=1}^{\lceil \log_2 n \rceil} \sum_{n/2^k \le n_1 < n/2^{k-1}} \frac{1}{n_1^{\sigma_1} n^{\sigma_2}} \\ &< 2^{\max\{0,-\sigma_1\}} \sum_{k=1}^{\infty} \frac{n/2^k}{(n/2^k)^{\sigma_1} n^{\sigma_2}} \\ &< \frac{2^{\max\{0,-\sigma_1\}}}{n^{\sigma_1+\sigma_2-1}} \sum_{k=1}^{\infty} \frac{1}{2^{(1-\sigma_1)k}}.\end{aligned}$$

This finishes the proof of the proposition in the case $d = 2$ since $\sigma_1 + \sigma_2 - 1 > 1$. The general case follows from this immediately. □

One may wonder if the above domain of convergence is optimal. Nevertheless it is our primary goal in this note to define the analytic continuation of the multiple zeta function on all of $\mathbb{C}^n$.

As a side remark, the special values of multiple zeta functions at positive integers have come to the foreground in recent years ([2, 3, 4], etc.), both in connection with theoretical physics (Feynman diagrams) and the theory of mixed Tate motives. Historically, Euler [5] already investigated the double zeta values in the eighteenth century. Using our notation he considered $\zeta(n, m) + \zeta(n + m)$ for positive integers n and m. Though the multiple zeta values are extremely important we do not want to elaborate on this topic in this short note.

Apostol and Vu [1] investigated Euler's double zeta function and found its analytic continuation by a method different from ours.

We also want to remark that our definition of multiple zeta functions is not the same as that of Kurokawa (cf. [7]). In particular we do not know any kind of Euler product for our multiple zeta functions.

This note grew out of a reading of Gelfand and Shilov's book [6] and Zagier's paper [11]. The author would like to thank Prof. Goncharov for providing the great book [6] as a reference in his spring 1998 course at Brown University. He also thanks the referee for informing him of the joint work of Apostol and Vu and Prof. Apostol for pointing out a few inaccuracies in the first version of this paper.

2. Preliminary on generalized functions

For any positive integer n, we define the space K of test functions as formed by all the complex-valued smooth functions on $\mathbb{R}^n$ which decrease to zero faster than any negative power of $|x|$ as $|x| \to \infty$. Here and throughout we use the abbreviations $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. We say that a generalized function $g(x; s)$ is entire in s if the inner product $\langle g(x; s), \varphi(x)\rangle$ is an entire function of s for any test function $\varphi(x)$.

Let us define the generalized function

$$x_+^s = \begin{cases} x^s & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

whose value on a test function $\varphi(x)$ is given by

$$\langle x_+^s, \varphi(x)\rangle := \int_0^\infty x^s \varphi(x) dx.$$

The following lemmas will play crucial roles in this note.

Lemma 2 ([6, Ch. I, §3.2, Formula (3)]). *If* $\mathrm{Re}\,(s) > -n-1,\ s \neq -1, -2, \ldots, -n,$ *then*

$$\int_0^\infty x^{s-1}\varphi(x)\,dx = \int_0^1 x^{s-1}\left[\varphi(x) - \varphi(0) - x\varphi'(0) - \cdots - \frac{x^n}{n!}\varphi^{(n)}(0)\right] dx + \int_1^\infty x^{s-1}\varphi(x)\,dx + \sum_{j=0}^{n} \frac{\varphi^{(j)}(0)}{j!(s+j)}.$$

As an easy corollary of Lemma 2 one has

Lemma 3 ([6, Ch. I, §3.5, p. 57]). *The generalized function* $x_+^{s-1}/\Gamma(s)$ *has an analytic continuation to an entire function in* s *such that*

$$\left.\frac{x_+^{s-1}}{\Gamma(s)}\right|_{s=-n} = \delta^{(n)}(x), \qquad n \geq 0,$$

where $\langle \delta^{(n)}(x), \varphi(x)\rangle := (-1)^n \varphi^{(n)}(0)$.

Lemma 4. *The generalized function*

$$f(x; s, u) = \frac{(1-x)_+^{s-1} x_+^{u-1}}{\Gamma(s)\Gamma(u)}$$

can be extended to an entire function in complex variables s *and* u.

Proof. We need to show that for any smooth function $\varphi(x)$ on $[0,1]$ the function of two complex variables s and u defined by

$$\int_0^1 \frac{(1-x)^{s-1} x^{u-1}}{\Gamma(s)\Gamma(u)} \varphi(x)\,dx, \qquad \mathrm{Re}\,(u) > 0,\ \mathrm{Re}\,(s) > 0,$$

can be analytically continued to an entire function on all of $\mathbb{C}^2$. This follows easily from Lemma 2 since we can break the interval $[0,1]$ into $[0,1/2] \cup [1/2,1]$. □

The Riemann ζ-function can be obtained via the Mellin-transformation by the integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-2} \cdot x dx}{e^x - 1} = \frac{1}{s-1} \int_0^\infty \frac{x^{s-2}}{\Gamma(s-1)} \cdot \frac{x}{e^x - 1}\, dx.$$

In light of Lemma 3, one sees that $(s-1)\zeta(s)$ can be thought of as the value of the generalized function $x_+^{s-2}/\Gamma(s-1)$ (which is entire) on the test function $x/(e^x-1)$. In this way one immediately recovers the analytic continuation of $\zeta(s)$. In the next section we can analytically continue the multiple zeta functions of any depth by using the same idea.

3. Analytic continuation of multiple zeta functions

Let us start with the following example which has perplexed people for some time. By the power series definition clearly one has

$$\zeta(0, s_2) = \zeta(s_2 - 1) - \zeta(s_2). \tag{2}$$

On the other hand it is not difficult to see

$$\zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2) = \zeta(s_1)\zeta(s_2).$$

By the analytic continuation of $\zeta(s)$, plugging in $s_2 = 0$ in the above and using (2) one should have

$$\zeta(s_1, 0) = \zeta(s_1)\zeta(0) - \zeta(s_1 - 1). \tag{3}$$

Since $\zeta(-1) = -B_2/2 = -1/12$ and $\zeta(0) = B_1 = -1/2$ it follows from (2) that

$$\zeta(0, s_2)|_{s_2=0} = 5/12 \tag{4}$$

while (3) yields that

$$\zeta(s_1, 0)|_{s_1=0} = 1/3. \tag{5}$$

This kind of "absurdity" is very misleading (see the relevant remark in [11, p. 509]). However, we will see that the above phenomenon occurs exactly because we are in a multiple variable situation so that the limits of a function at some point along different routes in the variable space may not be the same. In fact, $\zeta(s_1, s_2)$ has poles along $s_1 + s_2 = 0$ by the next theorem.

Theorem 5. *The multiple zeta function $\zeta(s_1, \dots, s_d)$ of depth d can be analytically continued to a meromorphic function on all of $\mathbb{C}^n$, with possible poles at $s_d = 1$ and $s_j + \cdots + s_d = d - j + 2 - l$ for positive integers l and $1 \le j < d$. Moreover all the poles are simple.*

Proof. Clearly one may assume $d \ge 2$. For simplicity we put $s_d(j) = s_j + \cdots + s_d$. In particular, $s_d(1) = s_1 + \cdots + s_d$ and $s_d(d) = s_d$. By the identity

$$\begin{aligned} &\sum_{0<n_1<\cdots<n_d} e^{-n_1t_1-n_2t_2-\cdots-n_dt_d} \\ &= \sum_{n_j>0} e^{-n_1t_1-(n_1+n_2)t_2-\cdots-(n_1+\cdots+n_d)t_d} \\ &= \prod_{j=1}^{d} \sum_{n_j>0} e^{-n_j(t_j+\cdots+t_d)} = \prod_{j=1}^{d} (e^{t_j+\cdots+t_d} - 1)^{-1} \end{aligned}$$

one easily gets

$$\Gamma(s_1)\cdots\Gamma(s_d)\zeta(s) = \int_0^\infty \cdots \int_0^\infty \frac{t_1^{s_1-1}\cdots t_d^{s_d-1} dt_1 \cdots dt_d}{\prod_{j=1}^d (e^{t_j+\cdots+t_d} - 1)}. \tag{6}$$

Now let us put $x_{d+1} = 0$ and make the change of variables

$$x_1 \cdots x_j = t_j + \cdots + t_d;\ 1 \le j \le d \quad \Longleftrightarrow \quad t_j = x_1 \cdots x_j(1 - x_{j+1});\ 1 \le j \le d.$$

Under this change of variables one has

$$\text{(7)}\quad \Gamma(s_1)\cdots\Gamma(s_d)\zeta(s_1,\dots,s_d) = \int_0^1\cdots\int_0^1\int_0^\infty \frac{\prod_{j=1}^d x_j^{s_d(j)-d+j-2}\prod_{j=2}^d(1-x_j)^{s_{j-1}-1}\cdot x_1^d\cdots x_d\cdot dx_1\cdots dx_d}{\prod_{j=1}^d(e^{x_1\cdots x_j}-1)},$$

because the Jacobian

$$\frac{\partial(t_1,\dots,t_d)}{\partial(x_1,\dots,x_d)} = x_1^{d-1}\cdots x_{d-1}.$$

We now define the following test function in K:

$$\varphi(x) := \frac{x_1^d x_2^{d-1}\cdots x_d}{\prod_{j=1}^d(e^{x_1\cdots x_j}-1)}.$$

Letting $u_j = s_d(j) - d + j - 1$ we can apply the generalized function

$$\text{(8)}\qquad \frac{x_{1+}^{u_1-1}}{\Gamma(u_1)}\cdot\prod_{j=2}^d\frac{(1-x_j)_+^{s_{j-1}-1}\cdot x_{j+}^{u_j-1}}{\Gamma(s_{j-1})\Gamma(u_j)}$$

on $\varphi(x)$ and get exactly $(s_d-1)\zeta(s_1,\dots,s_d)/\prod_{j=1}^{d-1}\Gamma(u_j)$ since $\Gamma(s_d) = (s_d-1)\Gamma(u_d)$.

By Lemma 3 and Lemma 4 all of the following generalized functions

$$f_1(x;s) = \frac{x_{1+}^{u_1-1}}{\Gamma(u_1)};\qquad f_j(x;s) = \frac{(1-x_j)_+^{s_{j-1}-1}x_{j+}^{u_j-1}}{\Gamma(s_{j-1})\Gamma(u_j)},\ 2\le j\le d,$$

can be extended to entire functions in s. Further, f_j depends only on x_j as a function of x. Hence one can set the entire function

$$\xi(s) := \Big\langle\prod_{j=1}^d f_j(x;s),\varphi(x)\Big\rangle$$

on $\mathbb{C}^n$ and define the analytic continuation of $\zeta(s)$ on $\mathbb{C}^n$ by

$$\text{(9)}\qquad \zeta(s) = \frac{\prod_{j=1}^{d-1}\Gamma(u_j)}{s_d-1}\xi(s).$$

This completes the proof of the theorem because gamma functions have poles at and only at non-positive integers. □

Let us now return to the example in the beginning of this section. In what follows we will use $R_i(s_1,s_2)$ to represent entire functions of (s_1,s_2) near $(0,0)$ (say with radius 1/2) such that $\lim_{(s_1,s_2)\to(0,0)}R_i(s_1,s_2)=0$. From the identity

$$\frac{x^2y}{(e^x-1)(e^{xy}-1)} = 1 - \frac{1}{2}(1+y)x + \frac{1}{12}(y^2+3y+1)x^2 + O(x^3)$$

and equation (7) one has

$$\begin{aligned}\zeta(s_1,s_2) &= \int_0^1 \frac{(1-y)^{s_1-1}y^{s_2-2}}{\Gamma(s_1)\Gamma(s_2)} \int_0^\infty \frac{x^{s_1+s_2-3}\cdot x^2y}{(e^x-1)(e^{xy}-1)}\,dxdy\\ &= \int_0^1 \frac{(1-y)^{s_1-1}y^{s_2-2}}{\Gamma(s_1)\Gamma(s_2)}\cdot\frac{y^2+3y+1}{12(s_1+s_2)}\,dy + R_1(s_1,s_2)\\ &= \frac{B(s_1-1,s_2)+3B(s_1-1,s_2-1)+B(s_1-1,s_2+1)}{12(s_1+s_2)\Gamma(s_1)\Gamma(s_2)} + R_1(s_1,s_2)\\ &= \frac{4s_1+5s_2}{12(s_1+s_2)} + R_2(s_1,s_2).\end{aligned}$$

Hence (4) should be interpreted as

$$\lim_{s_2\to 0}\lim_{s_1\to 0}\zeta(s_1,s_2) = 5/12$$

while (5) as

$$\lim_{s_1\to 0}\lim_{s_2\to 0}\zeta(s_1,s_2) = 1/3.$$

4. Residues of multiple zeta functions at poles

From the last section we know that $s_d = 1$ is a simple pole of $\zeta(s_1,\ldots,s_d)$, so we would like to calculate its corresponding residue. This is given by

Theorem 6. *The residue of the multiple zeta function $\zeta(s_1,\ldots,s_d)$ at $s_d=1$ is 1 or $\zeta(s_1,\ldots,s_{d-1})$ according to whether $d=1$ or $d>1$.*

Proof. The case $d=1$ being classical one may assume $d>1$. Let $\varphi(x)$ and $\xi(s)$ be as in the proof of Theorem 5. By our definition (9) of analytic continuation we have

$$\mathrm{res}_{s_d=1}\,\zeta(s_1,\ldots,s_d) = \Gamma(s_{d-1}(1)-d)\cdots\Gamma(s_{d-1}(d-1)-1)\lim_{s_d\to 1}\xi(s_1,\ldots,s_d).$$

It follows from (9) and

$$\big\langle f_d(x;s_1,\ldots,s_{d-1},1),\varphi(x)\big\rangle = \frac{1}{\Gamma(s_{d-1})\cdot x_1\cdots x_{d-1}}$$

that

$$\mathrm{res}_{s_d=1}\,\zeta(s_1,\ldots,s_d) = \zeta(s_1,\ldots,s_{d-1}),$$

as desired. □

Finally, we want to compute the residues of the multiple zeta function at all the other poles. Using the generating function of Bernoulli numbers one can prove

Theorem 7. *For depth $d\ge 2$ and any integers $1\le i\le d-1$ and $l\ge 1$, the residue of the multiple zeta function $\zeta(s_1,\ldots,s_d)$ on the hyperplane*

$$s_d(i) = d-i+2-l \tag{10}$$

in $\mathbb{C}^n$ is equal to (using the convention $\zeta(s_0)=1$)

$$\zeta(s_1,\ldots,s_{i-1})\sum_{\substack{a_d(i+1)=l-1\\ a_{i+1},\ldots,a_d\ge 0}}\left(\prod_{j=i+1}^{d}\frac{B_{a_j}\Gamma(a_d(j)+u_j)}{a_j!\Gamma(a_d(j+1)+u_j+1)}\right). \tag{11}$$

Here we have set $a_d(j)=a_j+\cdots+a_d$, $a_d(d+1)=0$, and $u_j=s_d(j)-d+j-1$.

Proof. Let us first assume $i \geq 2$ and rewrite the multiple zeta function $\zeta(s)$ as

$$\int_0^1 \cdots \int_0^1 \int_0^\infty \prod_{j=1}^{i-1} \frac{x_j^{s_{i-1}(j)+u_i-1}}{e^{x_1\cdots x_j}-1} \prod_{j=2}^{i-1} (1-x_j)^{s_{j-1}-1} \frac{f(x_1,\ldots,x_{i-1};s)}{\prod_{j=1}^d \Gamma(s_j)}\, dx_1\cdots dx_{i-1} \tag{12}$$

where

$$f(x_1,\ldots,x_{i-1};s) = \int_0^1 \cdots \int_0^1 \prod_{j=i}^d x_j^{u_j-1}(1-x_j)^{s_{j-1}-1} \frac{x_1\cdots x_j}{e^{x_1\cdots x_j}-1}\, dx_i\cdots dx_d.$$

Now we want to calculate the residue of f when s lies in the hyperplane defined by (10). We have

$$\prod_{j=i}^d \frac{x_1\cdots x_j}{e^{x_1\cdots x_j}-1} = \sum_{a_i,\ldots,a_d\geq 0} \left\{ (x_1\cdots x_{i-1})^{a_d(i)} \prod_{j=i}^d \frac{B_{a_j}}{a_j!} x_i^{a_d(j)} \right\}.$$

Hence by the identity $u_j + s_{j-1} = u_{j-1}+1$ and the beta integrals one has

$$\begin{aligned} f(x_1,&\ldots,x_{i-1};s)\\ &= \sum_{a_i,\ldots,a_d\geq 0} \left\{ (x_1\cdots x_{i-1})^{a_d(i)} \prod_{j=i}^d \frac{B_{a_j}}{a_j!}\cdot \frac{\Gamma(a_d(j)+u_j)\Gamma(s_{j-1})}{\Gamma(a_d(j)+u_j+s_{j-1})} \right\}\\ &= \sum_{a_j,\ldots,a_d\geq 0} \left\{ (x_1\cdots x_{i-1})^{a_d(i)} \cdot \frac{B_{a_i}}{a_i!} \frac{\Gamma(a_d(i)+u_i)}{\Gamma(a_d(i+1)+u_i+1)} \right.\\ &\quad \left. \cdot \frac{1}{\Gamma(a_d(i)+u_i+s_{i-1})} \cdot \prod_{j=i+1}^d \frac{B_{a_j}}{a_j!} \frac{\Gamma(a_d(j)+u_j)}{\Gamma(a_d(j+1)+u_j+1)} \cdot \prod_{j=i-1}^d \Gamma(s_j) \right\}. \end{aligned}$$

Note that only the terms with $a_d(i) = l-1$ contribute to the residue we want. Moreover for every such term with $a_i > 0$

$$\frac{\Gamma(a_d(i)+u_i)}{\Gamma(a_d(i+1)+u_i+1)} = \prod_{n=1}^{a_i} \big(a_d(i)+u_i-n\big)$$

has no poles at all. Now the case $i \geq 2$ of the theorem follows immediately.

The proof of the theorem in the case $i=1$ is essentially the same as above. One only needs to set $i=2$ in equation (12) and use

$$\lim_{u\to 0} \int_0^\infty \frac{ux^u}{e^x-1}\, dx = \lim_{u\to 0} u\zeta(u+1)\Gamma(u+1) = 1.$$

This completes the proof of the theorem. □

Remark 8. In the special case $d=2$ we find the residue of $\zeta(s_1,s_2)$ at $s_1+s_2 = 2-k$ for $k \geq 0$ is

$$\frac{B_k\Gamma(s_2+k-1)}{k!\Gamma(s_2)} = \frac{B_k\Gamma(1-s_1)}{k!\Gamma(2-k-s_1)}$$

which agrees with the results in [1, p. 89] when one notices that their function $H(s,z)$ is $\zeta(z,s)+\zeta(s+z)$ in our notation.

5. Some open problems

Let $B_j(x)$ be the j-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{j=0}^{\infty}B_j(x)\frac{t^j}{j!}.$$

For $j>0$ one has the following identities:

$$1^j+2^j+\cdots+(n-1)^j=\frac{1}{j+1}(B_{j+1}(n)-B_{j+1})=\frac{1}{j+1}\sum_{i=0}^{j}\binom{j+1}{i}B_i n^{j+1-i}.$$

It follows from the properties of the Bernoulli numbers that the double zeta function $\zeta(s_1,s_2)$ has the trivial zeros

$$(s_1,s_2)=(0,-k),\quad (-1,1-k),\quad (-2j,1-k),\quad (-2j-1,2-k)$$

where j and k run through the set of positive integers. To derive the above trivial zeros from the power series definition one should verify that the double zeta function has zero residue at those points by formula (11) though they lie inside the hyperplanes (10) for suitable l.

For the triple zeta function $\zeta(s_1,s_2,s_3)$ it is likely that the following zeros exhaust the list of the trivial zeros:

$$\begin{aligned}(s_1,s_2,s_3)=&(0,0,-k),\quad (1-j_1,1-j_2,1-k)\text{ with } j_1\ge 2 \text{ or } j_2\ge 2,\\ &(-1,1,-k),\quad (-2j,1,-k),\quad (-2j-1,1,1-k),\quad (-2j-1,2,-k)\end{aligned}$$

where j_1, j_2, j and k run through the set of positive integers.

Although it might be possible to apply the preceding argument to pin down all the trivial zeros of the multiple zeta functions of depth $d\ge 4$, numerous cases will present themselves if one considers the possibilities of $s_2=1$, $s_2=2,\dots$, etc. For simplicity we are content with the following list:

$$(s_1,\dots,s_{d-1},s_d)=(0,\dots,0,-k),\quad (1-j_1,\dots,1-j_{d-1},1-k)$$

where $j_1,\dots,j_{d-1}$ and k run through the set of positive integers such that $j_\alpha\ge 2$ for some $1\le\alpha\le d-1$. Once again one needs to show that the residues of the zeta function of depth d at the above points are zero by formula (11). Clearly the above list of trivial zeros is not complete. However, we do not venture to make any conjecture in the larger depth case because of lack of comprehensive numerical support.

By analogy with the Riemann zeta function, we would also like to know the solutions to the following problems:

Problem 1. Determine the complete set of trivial (resp. nontrivial) zeros of the multiple zeta functions.

Problem 2. Determine the functional equations (if any) of the multiple zeta functions which generalize the classical functional equation of the Riemann zeta function.

At present we do not even know how to get a handle on these two problems. For the second problem we do not mean to find identities involving multiple zeta functions of different depths such as various forms of shuffle relations. Such identities at positive integer points are very interesting for their own sake and are currently studied by a number of mathematicians and physicists.

References

[1] T. M. Apostol, and T. H. Vu, *Dirichlet series related to the Riemann zeta function*, J. of Number Theory **19** (1984), pp. 85-102. MR **85j:**11106

[2] D. Borwein, J.M. Borwein, and R. Girgensohn, *Explicit evaluation of Euler sums*, Proc. Edinburgh Math. Soc. **38** (1995), pp. 277–294. MR **96f:**11106

[3] D.J. Broadhurst, *Conjectured enumeration of irreducible multiple zeta values, from knots and Feynman diagrams*, preprint hep-th/9612012, available at http://xxx.lanl.gov/list/hep-th/9612.

[4] A. A. Beilinson and P. Deligne, *Interpretation motivique de la conjecture de Zagier*, in: Symp. in Pure Math. vol. 55, part 2, 1994, pp. 23–41. MR **95a:**19008

[5] L. Euler, *Meditationes circa singvlare seriervm genus*, Novi Comm. Acad. Sci. Petropol **20** (1775), pp. 140–186.

[6] I. M. Gelfand, and G. E. Shilov, *Generalized Functions* vol. I, Academic Press, New York, London, 1964. MR **55:**8786a

[7] N. Kurokawa, *Multiple zeta functions: an example*, in: Zeta Functions in Geometry, (Tokyo, 1990), Adv. Stud. Pure. Math. vol. 21, Kinokuniya, Tokyo, 1992, pp. 219–226. MR **94f:**11084

[8] T. Q. T. Le and J. Murakami, *Kontsevich integral for the Homely polynomial and relations between values of multiple zeta functions*, Topology and Appl. **62** (1995), pp. 193–206. MR **96c:**57017

[9] S. Lichtenbaum, *Values of zeta functions at non-negative integers*, in: Number theory, Noordwijkerhout 1983, Lecture Notes in Math. vol. 1068, Springer-Verlag, 1984, pp. 127–138. CMP 16:17

[10] D. Zagier, *Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields*, in: Arithmetic algebraic geometry (Texel, 1989), Prog. in Math. vol. 89, B. van der Geer, F. Oort and J. Steenbrink (eds.), Birkhäuser Boston, MA, pp. 391–430. MR **92f:**11161

[11] D. Zagier, *Values of zeta functions and their applications*, in: Proc. of the First European Congress of Math., Prog. in Math. vol. 120, part II (Paris, 1992), Birkhäuser, Basel, pp. 497–512. MR **96k:**11110

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