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CURVATURE RESTRICTIONS ON CONVEX, TIMELIKE SURFACES IN MINKOWSKI 3-SPACE

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Abstract. Suppose that K and H are Minkowski Gauss curvature and Minkowski mean curvature respectively on a timelike surface S that is C^2 immersed in Minkowski 3-space E_1^3. Suppose also that $0 \not\equiv K < 0$ and that S is complete as a surface in the underlying Euclidean 3-space E^3. It is shown that neither K nor H can be bounded away from zero on such a surface S.

1. Introduction

The global behavior of timelike surfaces in Minkowski 3-space E_1^3 is not yet fully understood. In particular, the effects of many standard restrictions on mean curvature H or Gauss curvature K are still not known. In this paper, we study the implications of such restrictions on timelike surfaces in E_1^3 that are convex as surfaces in Euclidean 3-space E^3.

A surface S is convex if it is the boundary ∂C of a closed, convex set C with non-empty interior in E^3. (The Euclidean geometry of such surfaces has been extensively studied.) If $0 \not\equiv K < 0$ on a timelike surface S in E_1^3 that is complete as a surface in E^3, then S is convex as a surface in E^3. We exploit this fact to show that if $0 \not\equiv K < 0$ on a timelike surface S in E_1^3 that is complete as a surface in E^3, then neither H nor K can be bounded away from zero on S. As a corollary, we show that (up to a rigid motion of E_1^3) a timelike surface S in E_1^3 must be a circular cylinder or a hyperbolic cylinder in case S is complete as a surface in E^3 with $H \equiv \textit{constant} \neq 0$ and $K \leq 0$ on S.

Our result, as it applies to K, can be compared with the Bonnet-Hopf-Rinow theorem, which states that any complete surfaces S in E^3 on which Euclidean Gauss curvature $K_\epsilon \geq \textit{constant} > 0$ must be compact. The conclusions match, since there are no compact timelike (or spacelike) surfaces in E_1^3.

As applied to H, our result can be compared with theorems stating that a complete convex hypersurface in Euclidean n-space E^n splits into a product with a non-trivial linear factor if Euclidean mean curvature $H_\epsilon \equiv 1$ (see [7] and [3]) or if $1 \leq H_\epsilon \leq c$ for a certain constant $c < (n-1)/(n-2)$ (see [8]), or if $1 \leq H_\epsilon \leq (n-1)/(n-2)$ with the upper limit $(n-1)/(n-2)$ sharp (see [9] and [10]). For related results involving spacelike or timelike convex hypersurfaces in Minkowski n-space E_1^n, see [11].

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Entire, spacelike surfaces in E^3_1 that are convex as surfaces in E^3 need not be cylindrical when $H \equiv \textit{constant} \neq 0$. (See [17].) This is not the only instance in which Euclidean results carry over to spacelike or timelike surfaces in E^3_1, but not to both. Bernstein's theorem holds in the spacelike case (see [2] and [4]) but fails in the timelike case (see [5] or [13]).

It should also be noted that timelike surfaces in E^3_1 on which $H \equiv 0$ and $K \leq 0$ need not be cylindrical if they are complete as surfaces in E^3. (See [14] and [19].) One knows, however, that any such surface is conformally equivalent to E^2_1. (See [13].)

2. Preliminaries

Use the Minkowski scalar product $\langle , \rangle$ determined by the metric $ds^2 = dx^2 + dy^2 - dz^2$ on x, y, z-space R^3 to define Minkowski 3-space E^3_1. Similarly, use the Euclidean inner product $\langle , \rangle_\epsilon$ determined by the metric $ds^2 = dx^2 + dy^2 + dz^2$ on x, y, z-space R^3 to define Euclidean 3-space E^3. A surface C^2 immersed in E^3_1 is automatically C^2 immersed in E^3 as well (and vice versa). We work with the geometry induced on the surface by both ambient spaces. (For background, see 7.1-7.2 in [19].)

Suppose a surface S is C^2 immersed in E^3_1. Call S timelike (resp. spacelike) if all its tangent planes make a Euclidean angle of more than (resp. less than) $\pi/4$ with the x, y-plane. Assume S is timelike. Then, modulo a rigid motion of E^3_1, S can be locally represented as the graph of a C^2 function $x = f(y, z)$ over a neighborhood U in the y, z-plane. Thus $\varphi(y, z) = (f(y, z), y, z)$ is a C^2 imbedding $\varphi : U \to E^3_1$ with first fundamental form I, unit normal vector field ν, second fundamental form II, Gauss curvature K and mean curvature H given by

$$I = \langle d\varphi, d\varphi\rangle = ({f_y}^2 + 1)dy^2 + 2f_y f_z dydz + ({f_z}^2 - 1)dz^2,$$

$$\nu = (1, -f_y, f_z)/(1 + {f_y}^2 - {f_z}^2)^{1/2},$$

$$II = -\langle d\varphi, d\nu\rangle = (f_{yy}dy^2 + 2f_{yz}dydz + f_{zz}dz^2)/(1 + {f_y}^2 - {f_z}^2)^{1/2}, \tag{1}$$

$$K = detII/detI = (f_{yy}f_{zz} - {f_{yz}}^2)/(1 + {f_y}^2 - {f_z}^2)^2,$$

$$2H = tr_I II = (({f_y}^2 + 1)f_{zz} - 2f_y f_z f_{yz} + ({f_z}^2 - 1)f_{yy})/(1 + {f_y}^2 - {f_z}^2)^{3/2},$$

with $\langle \nu, \nu\rangle = 1$ and

$$detI = -(1 + {f_y}^2 - {f_z}^2) < 0,$$

since S is timelike. For the imbedding $\varphi : U \to E^3$, one gets the Euclidean first fundamental form I_ϵ, unit normal vector field ν_ϵ, second fundamental form II_ϵ, Gauss curvature K_ϵ and mean curvature H_ϵ given by

$$I_\epsilon = \langle d\varphi, d\varphi\rangle_\epsilon = ({f_y}^2 + 1)dy^2 + 2f_y f_z dydz + ({f_z}^2 + 1)dz^2,$$

$$\nu_\epsilon = (1, -f_y, -f_z)/(1 + {f_y}^2 + {f_z}^2)^{1/2},$$

$$II_\epsilon = -\langle d\varphi, d\nu_\epsilon\rangle_\epsilon = (f_{yy}dy^2 + 2f_{yz}dydz + f_{zz}dz^2)/(1 + {f_y}^2 + {f_z}^2)^{1/2}, \tag{2}$$

$$K_\epsilon = detII_\epsilon/detI_\epsilon = (f_{yy}f_{zz} - {f_{yz}}^2)/(1 + {f_y}^2 + {f_z}^2)^2,$$

$$2H_\epsilon = tr_{I_\epsilon} II_\epsilon = (({f_y}^2 + 1)f_{zz} - 2f_y f_z f_{yz} + ({f_z}^2 + 1)f_{yy})/(1 + {f_y}^2 - {f_z}^2)^{3/2}.$$

Note that (1) and (2) give

$$signK = -signK_\epsilon \tag{3}$$

on a timelike surface S that is C^2 immersed in E^3_1.

By a **cylinder**, we mean the set of all points on all lines perpendicular (in the Euclidean sense) to a plane Π in E^3 that intersect a fixed curve γ in Π. A surface C^2 immersed in E^3 with I_ϵ complete on which $K_\epsilon \equiv 0$ is a cylinder. (See [6].)

By a **convex surface**, we mean the boundary ∂C of a closed, convex set C in E^3 with non-empty interior. A surface that is C^2 immersed in E^3 with I_ϵ complete and $0 \not\equiv K_\epsilon \geq 0$ is actually imbedded and convex. (See [18] and [16].) Suppose now that S is a timelike surface C^2 immersed in E^3_1. It follows from (3) that S is convex and non-cylindrical as a surface in E^3 provided that $0 \not\equiv K \leq 0$ on S and S is $\mathbf{E^3}$**-complete** (meaning that I_ϵ is complete on S).

A subset T of the 2-sphere S^2 in E^3 is **convex on** $\mathbf{S^2}$ if and only if the shorter great circular arc joining any two non-antipodal points in T lies in T, and at least one great circular arc joining any two antipodal points in T lies in T. For any subset T of S^2 (resp. on R^2), $Int(T)$ denotes the interior of T on S^2 (resp. on R^2). For any subset T of E^3, $cl(T)$ denotes the closure of T in E^3. For an oriented surface S that is C^2 immersed in E^3 (or in E^3_1), the **Euclidean Gauss map** $G_\epsilon : S \to S^2$ sends each point p on S to the point on S^2 described by the Euclidean unit normal vector ν_ϵ to S at p.

The following lemma summarizes known facts.

Lemma 1. *Suppose a non-compact, oriented surface S is C^2 imbedded in E^3 as the boundary ∂C of a convex subset C of E^3 with non-empty interior.*

(i) *If C contains no lines, then S is homeomorphic to R^2. (See* [1]*.)*

(ii) *If S is homeomorphic to R^2, there is a tangent plane Π to S such that for orthogonal projection $\pi : E^3 \to \Pi$ with $C' = \pi(C)$, the portion of S in $\pi^{-1}(Int(C'))$ is the graph of a C^2 function over $Int(C')$. (See* [20]*.)*

(iii) *If C contains no lines, $cl(G_\epsilon(S))$ is convex on S^2. (See* [20]*.)*

Lemma 2. *Suppose a non-compact, oriented surface S is C^2 imbedded in E^3 as the boundary ∂C of a convex subset C of E^3 with non-empty interior. Let P be a plane through the origin that intersects $Int(G_\epsilon(S)) \neq \emptyset$. Let Σ_1 and Σ_2 be the intersections of $G_\epsilon(S)$ with the closed half spaces of E^3 bounded by P, and let $C' = \pi(C)$ for orthogonal projection $\pi : E^3 \to P$. Then*

$$\pi^{-1}(Int(C')) \cap S = S_1 \cup S_2$$

where S_1 and S_2 are graphs of C^2 functions on $Int(C') \neq \emptyset$, with $G_\epsilon(S_1) \subset \Sigma_1$ and $G_\epsilon(S_2) \subset \Sigma_2$.

Proof. Since $Int(G_\epsilon(S)) \neq \emptyset$, the convex surface $S = \partial C$ is not a cylinder, and C contains no lines. Thus $\pi^{-1}(p_0) \cap S \neq \emptyset$ for any p_0 in $Int(C')$. Since the interior of the convex set C in E^3 lies to one side of any tangent plane to $S = \partial C$, there are at most two points in $\pi^{-1}(p_0) \cap S$ for any p_0 in $Int(C')$.

If $\pi^{-1}(p) \cap S$ contains exactly one point for a particular p in $Int(C')$, then $\pi^{-1}(p_0)$ contains exactly one point for every p_0 in $Int(C')$. This follows since if C contains the half line starting at $\pi^{-1}(p) \cap S$ in the direction of a vector ξ perpendicular to P, then C also contains the half line starting at any point in $\pi^{-1}(p_0) \cap S$ in the direction of ξ for every p_0 in $Int(C')$.

But if $\pi^{-1}(p_0) \cap S$ contains exactly one point for every p_0 in $Int(C')$, then $S_0 = \pi^{-1}(Int(C'))$ is the graph of a C^2 function over $Int(C')$ in P, so that $G_\epsilon(S_0)$ lies in one of the two open hemispheres on S^2 bounded by P. This contradicts the assumption that P intersects $Int(G_\epsilon(S))$, since ν_ϵ lies on $P \cap S^2$ at any point on $\pi^{-1}(\partial C') \cap S$.

It follows that $\pi^{-1}(p_0) \cap S$ contains exactly two points for every p_0 in S. Thus $\pi^{-1}(Int(C')) \cap S = S_1 \cup S_2$, with S_1 and S_2 as described in the lemma.

3. Main results

The next lemma is at the heart of our arguments.

Lemma 3. *Suppose that S is a timelike surface C^2 immersed in E_1^3 with $K \leq 0$. Let α be the acute Euclidean angle between ν_ϵ and the x, y-plane, so that $0 \leq \alpha < \pi/4$. Then*

$$K_\epsilon = -K cos^2 2\alpha.$$

If S is also E^3 complete with $K \not\equiv 0$ (making S convex as a surface in E^3), then

$$|H_\epsilon| \geq |H| cos^{3/2} 2\alpha \geq 0.$$

Proof. Let p be a point on the timelike surface S. With no loss of generality, assume that a neighborhood $\mathcal{U}$ of p on S is the graph of a C^2 function $x = f(y, z)$ over an open set U in the y, z-plane, with $p = (f(y_0, z_0), y_0, z_0)$. If β is the angle between ν_ϵ and the positive z-axis, (2) gives

$$(4) \qquad cos^2\beta = {f_z}^2/(1 + {f_y}^2 + {f_z}^2), \quad cos^2\alpha = (1 + {f_y}^2)/(1 + {f_y}^2 + {f_z}^2)$$

since $\alpha = \pm(\frac{\pi}{2} - \beta)$, and

$$cos 2\alpha = 2cos^2\alpha - 1 = (1 + {f_y}^2 - {f_z}^2)/(1 + {f_y}^2 + {f_z}^2).$$

Note that 2α is the angle between ν and ν_ϵ at any point on S. By (1) and Lemma 3, one has

$$(5) \qquad K_\epsilon = \frac{(f_{yy} f_{zz} - {f_{yz}}^2)}{(1 + {f_y}^2 - {f_z}^2)^2} \cdot \frac{(1 + {f_y}^2 - {f_z}^2)^2}{(1 + {f_y}^2 + {f_z}^2)^2} = -K cos 2\alpha$$

with

$$2H_\epsilon = \frac{(({f_y}^2 + 1) f_{zz} - 2 f_y f_z f_{yz} + ({f_z}^2 - 1) f_{yy})}{(1 + {f_y}^2 - {f_z}^2)^{3/2}} \cdot \frac{(1 + {f_y}^2 - {f_z}^2)^{3/2}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}} + \frac{2 f_{yy}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}}$$

giving

$$(6) \qquad H_\epsilon = H cos^{3/2} 2\alpha + \frac{f_{yy}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}}.$$

Note that (5) establishes the first claim in the lemma, since the choice of p was arbitrary.

Suppose now that S is also E^3 complete with $K \not\equiv 0$, so that S is convex as a surface in E^3. Then $\mathcal{U}$ is concave up or concave down with respect to the positive x-axis. With no loss of generality, assume the $\mathcal{U}$ is concave up, so that $f_{yy} \geq 0$, $f_{zz} \geq 0$ and $H_\epsilon \geq 0$ on U.

Let γ be the connected curve in $\mathcal{U} \cap \{z = z_0\}$ given by $\gamma = \gamma(y) = (f(y, z_0), y, z_0)$ with $p = \gamma(y_0)$. The curvature of $\gamma(y)$ as a curve in E^3 is

$$k(\gamma(y)) = \frac{f_{yy}}{(1 + {f_y}^2)^{3/2}} \geq 0, \tag{7}$$

and the principal unit normal vector to $\gamma(y)$ is

$$\mathbf{N} = \mathbf{N}(y) = (1, -f_y, 0)/(1 + {f_y}^2)^{1/2}.$$

If θ is the angle between ν_ϵ and $\mathbf{N}(y)$, then

(8)

$$\begin{aligned} cos\,\theta = \langle \mathbf{N}, \nu_\epsilon \rangle_\epsilon &= \langle (1, -f_y, 0), (1, -f_y, -f_z) \rangle_\epsilon / (1 + {f_y}^2)^{1/2} (1 + {f_y}^2 + {f_z}^2)^{1/2} \\ &= (1 + {f_y}^2)^{1/2} (1 + {f_y}^2 + {f_z}^2)^{1/2} \geq 0. \end{aligned}$$

Thus (7) and (8) give

$$\begin{aligned} \frac{f_{yy}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}} &= \frac{f_{yy}}{(1 + {f_y}^2)^{3/2}} \cdot \frac{(1 + {f_y}^2)^{3/2}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}} \\ &= k(\gamma(y)) cos^3(\theta) \geq 0. \end{aligned} \tag{9}$$

By Meusnier's theorem,

$$k(\gamma(y)) cos\theta = k_n(\gamma'(y)) \geq 0, \tag{10}$$

where $k_n(\gamma'(y))$ is the normal curvature of S at $\gamma(y)$ in the $\gamma'(y)$ direction. Thus (9) becomes

$$\frac{f_{yy}}{(1 + {f_y}^2 + {f_z}^2)^{3/2}} = k_n(\gamma'(y)) cos^2\theta \geq 0,$$

with (6) giving

$$H_\epsilon = H cos^{3/2} 2\alpha + k_n(\gamma'(y)) cos^2\theta \geq 0. \tag{11}$$

If $H \geq 0$, then (10) and (11) yield

$$|H_\epsilon| \geq |H| cos^{3/2} 2\alpha. \tag{12}$$

If $H \leq 0$, then (10) and (11) yield

$$|H| cos^{3/2} 2\alpha = -H cos^{3/2} 2\alpha = k_n(\gamma'(y)) cos^2\theta - H_\epsilon. \tag{13}$$

But $2H_\epsilon$ is the sum of normal curvatures in orthogonal tangent directions at any point on S. Thus at any point on $\gamma(y)$,

$$2H_\epsilon = k_n(\gamma'(y)) + k_n(\mathbf{w}(y)) \tag{14}$$

for $\mathbf{w}(y) = \nu_\epsilon(\gamma(y)) \times \gamma'(y)$, with both terms non-negative on the right side of (14). When $H \leq 0$, (13) and (14) give

$$2|H| cos^{3/2} 2\alpha = 2k_n(\gamma'(y)) cos^2\theta - 2H_\epsilon \leq 2k_n(\gamma'(y)) - 2H_\epsilon$$

$$= k_n(\gamma'(y)) - k_n(\mathbf{w}(y)) \leq k_n(\gamma'(y)) + k_n(\mathbf{w}(y)) = 2H_\epsilon = 2|H_\epsilon|.$$

This shows that (12) holds at any point on $\gamma(y)$, and hence at p. Since p was arbitrary, the second claim in the lemma is established.

In the remainder of this paper, let Π^+ and Π^- denote the planes $z = \sqrt{2}/2$ and $z = -\sqrt{2}/2$ respectively, with

$$\begin{aligned} &\Sigma = S^2 \cap \{-\sqrt{2}/2 \leq z \leq \sqrt{2}/2\}, \\ &\Sigma^+ = S^2 \cap \Pi^+, \quad \Sigma^- = S^2 \cap \Pi^-, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &R = \Sigma \cap \{x \geq 0\}, R^+ = R \cap \Pi^+, R^- = R \cap \Pi^-, \\ &R_0^+ = R^+ \cap \{x = 0\}, R_0^- = R^- \cap \{x = 0\}. \end{aligned} \tag{16}$$

Suppose p lies in R_0^+, so its antipodal point q lies in R_0^-. There is a unique plane $\Pi(p,q)$ in E^3 containing the parallel lines tangent to Σ^+ at p and Σ^- at q. Moreover,

$$\Gamma_{p,q} = \Pi(p,q) \cap S^2$$

is the only great circle on S^2 through p and q that is contained in Σ.

Lemma 4. *Suppose that $\Omega \subset R$ with $cl(\Omega)$ convex on S^2 and $Int(\Omega) \neq \emptyset$. Let $X = cl(\Omega) \cap \{\Pi^+ \cup \Pi^-\}$. Then there is a plane P through the origin intersecting $Int(\Omega)$ that bounds an open half space of E^3 containing X, and contains no pair of antipodal points from $cl(\Omega)$.*

Proof. Because $cl(\Omega)$ is convex on S^2, $cl(\Omega) \cap \Pi^+$ and $cl(\Omega) \cap \Pi^-$ each contains at most one point, so that X contains at most two points. Suppose that X contains two points, p in $cl(\Omega) \cap \Pi^+$ and q in $cl(\Omega) \cap \Pi^-$. We claim that p and q are not antipodal. Otherwise, $x = 0$ at p and q since $cl(\Omega) \subset cl(R) \subset \{x \geq 0\}$. Thus p lies in R_0^+, q lies in R_0^- and $\Gamma_{p,q} = \Pi(p,q) \cap S^2$ is the only great circle on S^2 through p and q contained in Σ. Since $Int(\Omega) \neq \emptyset$, there is a point m in $Int(\Omega)$ that does not lie on the plane $\Pi(p,q)$. Thus the great circle $\tilde{\Gamma}_{p,q}$ through p, q and m is not $\Gamma_{p,q}$, and is not contained in Σ. But the convexity of $cl(\Omega)$ on S^2 guarantees that the semicircle $\tilde{\Gamma}_{p,q}^R = \tilde{\Gamma}_{p,q} \cap \{x \geq 0\}$ is contained in $cl(\Omega) \subset \Sigma$, since p, q and m all lie on $\tilde{\Gamma}_{p,q}^R \cap cl(\Omega)$. If $\tilde{\Gamma}_{p,q}^R$ lies in Σ, so does $\tilde{\Gamma}_{p,q}$. This contradiction confirms that p and q are not antipodal.

Let γ_1 be the shorter great circular arc joining p to q in $cl(\Omega)$, and let P_1 be the plane containing γ_1. The line $l_{p,q}$ through p and q does not pass through the origin, since p and q are not antipodal. Since $Int(\Omega) \neq \emptyset$, there is a point m in $Int(\Omega)$ which lies off P_1. Let P be the plane through m containing the line l through the origin that is parallel to $l_{p,q}$. No point of $l_{p,q}$ lies on P, since otherwise the contradiction $P = P_1$ would follow. Thus $X = \{p, q\}$ is contained in one of the open half spaces in E^3 bounded by P. Finally, since $P_1 \cap cl(\Omega) = \gamma_1$ contains no pair of antipodal points, and since P converges to P_1 as m approaches γ_1 in $Int(\Omega)$, we can choose m so close to γ_1 in $Int(\Omega)$ that $P \cap cl(\Omega)$ contains no antipodal points.

The cases in which at least one of the sets $cl(\Omega) \cap \Pi^+$ or $cl(\Omega) \cap \Pi^-$ is empty can be handled in a similar way, but with easier arguments. □

Theorem. *Suppose that S is an E^3 complete timelike surface C^2 immersed in E^3 with $0 \not\equiv K < 0$ (so that S is convex as a surface in E^3). Then neither H nor K can be bounded away from zero on S.*

Proof. Since $K \not\equiv 0$ on S, $K_\epsilon \not\equiv 0$ by (3), so that S is not a cylinder. This implies that the closed convex set C in E^3 with $S = \partial C$ contains no lines. By Lemma 1(i), S is homeomorphic to R^2. By Lemma 1(ii), S is locally the graph of a C^2 function over one of its tangent planes, which we assume, with no loss of generality, to be the y, z-plane, with S lying in the half space $\{x \geq 0\}$. Then $G_\epsilon(S) \subset R$ for the subset R of S^2 defined in (16).

By Lemma 1(iii), $cl(G_\epsilon(S))$ is a convex subset of S^2. Thus Lemma 4 applies with $\Omega = G_\epsilon(S)$, giving a plane P through the origin with $P \cap Int(G_\epsilon(S)) \neq \emptyset$, with

$$X = cl(G_\epsilon(S)) \cap \{\Pi^+ \cup \Pi^-\}$$

contained in one of the open half spaces H_1 or H_2 in E^3 bounded by P, and with $P \cap cl(G_\epsilon(S))$ containing no pair of antipodal points. Index H_1 and H_2 so that X lies in H_2. Then if $G_1 = G_\epsilon(S) \cap cl(H_1)$ and $G_2 = G_\epsilon(S) \cap cl(H_2)$, one has $X \cap G_1 = \emptyset$. Apply Lemma 2 with $\pi : E^3 \to P$ orthogonal projection onto P and $C' = \pi(C)$, so that $\pi^{-1}(Int(C')) \cap S = S_1 \cup S_2$, with S_1 and S_2 each the graph of a C^2 function on $Int(C')$. Index S_1 and S_2 so that $G_\epsilon(S_1) \subset G_1$.

We claim that C' cannot be contained between parallel lines l_1 and l_2 in P. Otherwise, $S = \partial C$ would be contained in the closed region bounded by the planes P_1 and P_2 through l_1 and l_2 respectively that are perpendicular to P. Since C is convex, $cl(G_\epsilon(S))$ would contain antipodal points perpendicular to P_1 and P_2, and lying in P. This would contradict the fact that $P \cap cl(G_\epsilon(S))$ contains no antipodal points, so our claim is verified.

Since $S \subset \{x \geq 0\}$ is a graph over the y, z-plane, C contains a half line that is not perpendicular to P. Thus $Int(C')$ contains a half line λ. Since the convex set C' in P is not contained between any two lines parallel to λ, one easily argues that for any $r > 0$, $Int(C')$ contains a disk of radius r. Let D be a disk of radius $2r$ contained in $Int(C')$ and centered at a point c. Let l be the line through c perpendicular to P. Take a closed ball B of radius r centered on l, starting on the same side of S_1 as C, with $B \cap S_1 = \emptyset$. Move B toward S_1 keeping its center on l until it first touches $S_1 \cap \pi^{-1}(D)$ at a point p. All of B now lies on S_1 or to the same side of S_1 as C. Thus $0 \leq |H_\epsilon(p)| \leq 1/r$ and $K_\epsilon(p) \leq 1/r^2$. On the other hand, we took $G_\epsilon(S_1) \subset G_1$ with $X \cap G_1 = \emptyset$. Thus there is a constant α_0 satisfying $0 \leq \alpha \leq \alpha_0 < \pi/4$, where α is the angle ν_ϵ on S_1 makes with the x, y-plane. By Lemma 3, the inequalities

$$|H_\epsilon| \geq |H| cos^{3/2} 2\alpha_0 \geq 0, \qquad K_\epsilon \geq |K| cos^2 2\alpha_0 \tag{17}$$

hold on S_1. If H or K is bounded away from zero on S, we obtain a contradiction to (17) at p by taking r sufficiently large. This establishes the theorem.

Corollary. *If $H \equiv constant \neq 0$ and $K \leq 0$ on an E^3 complete timelike surface C^2 immersed in E_1^3, then (up to rigid motions of E_1^2) S is a cylinder over a circle of radius $r = 1/2H$ in the x, y-plane, or over a hyperbola of curvature $2H$ in the y, z-plane.*

Proof. By our theorem, $K \equiv 0$ must hold on S, making $K_\epsilon \equiv 0$ and S a cylinder. The corollary then follows by elementary arguments.

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