PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1655–1664
S 0002-9939(00)05100-5
Article electronically published on February 16, 2000

# ON THE COMPLEXITY OF DESCRIPTION OF REPRESENTATIONS OF $*$-ALGEBRAS GENERATED BY IDEMPOTENTS

STANISLAV KRUGLIAK AND YURIĬ SAMOĬLENKO

(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper, we introduce a quasiorder $\succ$ (majorization) on $*$-algebras with respect to the complexity of description of their representations. We show that $C^*(\mathcal{F}_2) \succ \mathfrak{A}$ for any finitely generated $*$-algebra $\mathfrak{A}$ (algebras $\mathfrak{B}$ such that $\mathfrak{B} \succ C^*(\mathcal{F}_2)$ are called $*$-wild). We show that the $*$-algebra generated by orthogonal projections $p$, $p_1$, $p_2$, … , $p_n$ ($p_ip_j = 0$ for $i \neq j$) is $*$-wild if $n \geq 2$. We also prove that $*$-algebras generated by a pair of idempotents and an orthogonal projection, or by a pair of idempotents $q_1$, $q_2$ ($q_1q_2 = q_2q_1 = 0$), etc., are $*$-wild.

## Introduction

In Section 2 of this article, following [1]–[4], we introduce a quasiorder (majorization) $\succ$ of $*$-algebras with respect to the complexity of the structure of their $*$-representations by bounded operators. If $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-agebras, then $\mathfrak{A} \succ \mathfrak{B}$ means (see Section 2) that there is a $C^*$-ideal $\mathfrak{J}$ in $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{J} \approx M_n(\mathbb{C}) \otimes \mathfrak{B}$ for some $n \in \mathbb{N} \cup \{\infty\}$ ($M_\infty(\mathbb{C}) = \mathcal{K}$ is the $C^*$-algebra of compact operators).

In Section 2, following [1], we show that for any finitely generated $*$-algebra $\mathfrak{A}$, we have that $C^*(\mathcal{F}_2) \succ \mathfrak{A}$ (here $\mathcal{F}_2$ is the free group with two generators), and so that if $\mathfrak{B} \succ C^*(\mathcal{F}_2)$, then the problem of describing, up to a unitary equivalence, all $*$-representations of the algebra is extremely difficult (in the article such problems are called $*$-wild).

Then in Sections 3 and 4, we study, from the point of view developed in Section 2, the complexity of representations of some $*$-algebras generated by idempotents or, which is the same thing, the complexity of a unitary description of some families of idempotents $\{Q_i\}_{i=1}^n$, $Q_i \in \mathcal{L}(H)$ ($i = 1, \dots, n$), on a separable Hilbert space $H$ (for the utility of solution of such a problem see, for example, [5] and the bibliography therein). In particular, in Section 3 we give a simpler proof of the fact that the $*$-algebra generated by orthogonal projections (self-adjoint idempotents) $p$, $q_1$, … , $q_n$ such that $q_iq_j = 0$, $i \neq j$ ("all but one" mutually orthogonal projections), is $*$-wild if $n \geq 2$ (see also [6]).

---

Received by the editors February 5, 1997 and, in revised form, May 17, 1998.
2000 *Mathematics Subject Classification.* Primary 46K10, 46L05; Secondary 16G60.
*Key words and phrases.* Involutive algebras, idempotents, orthogonal projections, $*$-representations, irreducible representations, majorizing of representations, $*$-wildness.
This work has been supported in part by the Ukrainian Committee for Fundamental Studies and by CRDF grant no. UM1-311.

In Section 4, we prove that the $*$-algebra is $*$-wild if it is generated by one of the following families of operators:

(i) an idempotent and an orthogonal projection (the problem of unitary description of pairs consisting of an idempotent and an orthogonal projection);
(ii) a pair of idempotents $q_1$, $q_2$ such that $q_1q_2 = q_2q_1 = 0$;
(iii) a resolution of the identity into a sum of idempotents $\{q_i\}_{i=1}^n$, $q_1+\dots+q_n = e$ ($n \geq 3$), or a resolution of the identity into a weighted sum of orthogonal projections $\{p_k\}_{k=1}^n$, $\frac{1}{2}(p_1 + \dots + p_n) = e$ ($n \geq 5$).

For corresponding enveloping $C^*$-algebras, if they exist, this means that their factor algebras are isomorphic to $M_n(C^*(\mathcal{F}_2))$ for some $n \in \mathbb{N}$.

Complexity of description of $*$-representations of group $C^*$-algebras, their generalizations, $*$-algebras that correspond to classes of not self-adjoint operators, Wick $*$-algebras, etc., will be studied in a forthcoming paper.

It is noted that the problems close to the subject of this article are considered in [7].

## 1. Definitions and notations

In this article, we consider the problem of giving a description, up to a unitary equivalence, of families of idempotents $Q_1, \dots, Q_n$. As usual, two families of operators $\{X_\alpha\}_{\alpha\in\Lambda}$ in $H$ and $\{\tilde{X}_\alpha\}_{\alpha\in\Lambda}$ in $\tilde{H}$ are unitarily equivalent if there exists a unitary operator $U \colon H \to \tilde{H}$ such that

$$UX_\alpha = \tilde{X}_\alpha U \qquad (\alpha \in \Lambda).$$

It is natural to make such a description within the framework of the theory of representations of $*$-algebras so as to associate to idempotents $\{Q_k\}_{k=1}^n$ a representation of the $*$-algebra $\mathfrak{Q}_n$. This is the factor algebra of the free $*$-algebra, with generators $q_1, \dots, q_n, q_1^*, \dots, q_n^*$, with respect to the two-sided $*$-ideal, generated by the relations $q_k^2 = q_k$, $(q_k^*)^2 = q_k^*$ ($k = 1, \dots, n$). In the sequel, the $*$-algebra $\mathfrak{A}$ defined by generators $x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots x_n^*$ and relations $P_l(x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots, x_n^*) = 0$, $l = 1, 2, \dots, m$, will be denoted by $\mathbb{C}\langle x_1, \dots, x_n \mid P_l(x_1, x_2, \dots, x_n, x_1^*, x_2^*, \dots x_n^*) = 0, l = 1, \dots, m\rangle$. It will also be assumed that, besides these relations, all relations obtained from them by taking $*$ are valid as well.

A representation of a $*$-algebra $\mathfrak{A}$ is a $*$-homomorphism $\pi \colon \mathfrak{A} \to L(H)$ into the $*$-algebra $L(H)$ of bounded operators in a complex separable Hilbert space $H$. By $\operatorname{Rep}\mathfrak{A}$ we denote the category, objects of which are representations of the algebra $\mathfrak{A}$ and morphisms (intertwining operators). Each representation $\pi$ of the $*$-algebra $\mathfrak{A} = \mathbb{C}\langle x_1, \dots, x_n \mid P_l(x_1, \dots, x_n, x_1^*, \dots, x_n^*) = 0, l = 1, \dots, m\rangle$ determines the family of bounded operators $\{X_k\} = \{\pi(x_k)\}$, $k = 1, \dots, n$, such that

$$P_l(X_1, \dots, X_n, X_1^*, \dots, X_n^*) = 0, \quad l = 1, \dots, m. \tag{1}$$

Conversely, a given family of operators $\{X_k\}$, $k = 1, \dots, n$, such that

$$P_l(X_1, \dots, X_n, X_1^*, \dots, X_n^*) = 0, \qquad l = 1, \dots, m,$$

uniquely defines a representation of the whole $*$-algebra $\mathfrak{A}$. Thus the problem of a unitary description of families of operators $\{X_k\}$, $k = 1, \dots, n$, satisfying relations (1) is a problem of description, up to a unitary equivalence, of representations of the $*$-algebra $\mathfrak{A}$.

In the sequel, we will be considering the unitary classification problems for representations of the following $*$-algebras (and, correspondingly, the unitary classification problems for the following families of operators):

1) $\mathfrak{S}_n = \mathbb{C}\langle a_1, \dots, a_n \mid a_i = a_i^*,\ i = 1, \dots, n\rangle$ (classification problem for $n$ self-adjoint operators);

2) $\mathfrak{U}_n = \mathbb{C}\langle u_1, \dots, u_n \mid u_i u_i^* = u_i^* u_i = e,\ i = 1, \dots, n\rangle$ (classification problem for $n$ unitary operators);

3) $\mathfrak{P}_n = \mathbb{C}\langle p_1, \dots, p_n \mid p_i^2 = p_i = p_i^*, i = 1, \dots, n\rangle$ (classification problem for $n$ orthogonal projections);

4) $\mathfrak{C} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^2 = p_i = p_i^*,\ i = 1, 2, 3;\ p_1 p_2 = p_2 p_1 = 0\rangle$ (classification problem for a triple of orthogonal projections, two of which are orthogonal);

5) $\mathfrak{Q}_n = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i,\ i = 1, \dots, n\rangle$ (classification problem for $n$ idempotents);

6) $\mathfrak{D} = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^*\rangle$ (classification problem for a pair of operators, one of which is idempotent and the other is an orthogonal projection);

7) $\mathfrak{Q}_{n,\perp} = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i,\ i = 1, \dots, n;\ q_i q_j = 0$ for $i \neq j\rangle$ (classification problem for $n$ mutually orthogonal idempotents).

## 2. Majorization of $C^*$-algebras and $*$-algebras with respect to complexity of their representations

Before passing to the problem of unitary description of representations of $*$-algebras generated by idempotents, we give definitions and some results concerning the ideology and methodology of $*$-wildness. In the theory of representations of algebras, it was suggested ([8]) that the representation problem be considered wild if it contains a standard difficult problem of the representation theory, e. g. the problem to describe, up to similarity, a pair of matrices without relations. To define an analogue of wildness for $*$-algebras ($*$-wildness), it was suggested in [1] to choose, for a standard difficult problem in the theory of $*$-representations, the problem of describing pairs of self-adjoint (or unitary) operators up to a unitary equivalence (free $*$-algebra $\mathfrak{S}_2$ (or $\mathfrak{U}_2$) generated by a pair of self-adjoint (or unitary) generators); and there were indications that suggested that the problems, which contain the standard $*$-wild problems, be regarded as $*$-wild. One can prove that these problems contain as a subproblem the problem of describing $*$-representations of any affine $*$-algebra.

We give exact definitions, examples, and statements necessary for what follows.

**Definition 1.** Let $\mathfrak{A}$ be a $*$-algebra. A pair $(\tilde{\mathfrak{A}}; \phi\colon \mathfrak{A} \to \tilde{\mathfrak{A}})$, where $\tilde{\mathfrak{A}}$ is a $*$-algebra and $\phi$ is a $*$-homomorphism, is called an enveloping $*$-algebra of the algebra $\mathfrak{A}$ if, for any $*$-representation $\pi\colon \mathfrak{A} \to L(H)$ of the algebra $\mathfrak{A}$, there exists a unique $*$-representation $\tilde{\pi}\colon \tilde{\mathfrak{A}} \to L(H)$ such that the diagram

$$\begin{tikzcd}
\tilde{\mathfrak{A}} \arrow[dr, "\tilde{\pi}"] & \\
\mathfrak{A} \arrow[u, "\phi"] \arrow[r, "\pi"] & L(H)
\end{tikzcd}$$

is commutative, and any operator $X : H_1 \to H_2$ which intertwines two representations $\pi_1 : \mathfrak{A} \to L(H_1)$, $\pi_2 : \mathfrak{A} \to L(H_2)$ of the algebra $\mathfrak{A}$ is also an intertwining operator for the representations $\tilde{\pi}_1$, $\tilde{\pi}_2$ of the algebra $\tilde{\mathfrak{A}}$.

The following examples of enveloping $*$-algebras will be used in the sequel.

1) $\tilde{\mathfrak{A}} = \mathfrak{A}$, $\phi$ is the identity mapping;

2) Let $\Sigma$ be any set of elements of an algebra $\mathfrak{A}$, the images of which are invertible operators for any representation $\pi\colon \mathfrak{A} \to L(H)$. Let $\tilde{\mathfrak{A}} = \mathfrak{A}[\Sigma^{-1}]$ be the quotient algebra (see [9]) of the algebra $\mathfrak{A}$ with respect to the set $\Sigma$, and let $\phi$ be the natural imbedding of $\mathfrak{A}$ into $\mathfrak{A}[\Sigma^{-1}]$;

3) Let $\mathfrak{A}$ be a star-bounded $*$-algebra, $\tilde{\mathfrak{A}}$ its enveloping $C^*$-algebra, and $\phi$ its canonical $*$-homomorphism of $\mathfrak{A}$ into $\tilde{\mathfrak{A}}$, defined by a faithful representation (see, for example, [10]).

Let $M_n(\tilde{\mathfrak{A}})$ be the matrix algebra over $\tilde{\mathfrak{A}}$ with the naturally given $*$-structure. Any representation $\pi\colon \mathfrak{A} \to L(H)$ induces the representation

$$\pi_n\colon M_n(\mathfrak{A}) \to L(H \oplus H \oplus \cdots \oplus H)$$

and, hence, the representation

$$\tilde{\pi}_n\colon M_n(\tilde{\mathfrak{A}}) \to L(H \oplus H \oplus \cdots \oplus H)$$

of the algebra $\tilde{M}_n(\mathfrak{A})$ enveloping the algebra $M_n(\mathfrak{A})$. If $\psi\colon \mathfrak{B} \to M_n(\tilde{\mathfrak{A}})$ is a $*$-homomorphism of the algebras, then there is a natural way to construct the functor $F_\psi\colon \mathrm{Rep}(\mathfrak{A}) \to \mathrm{Rep}(\mathfrak{B})$. By definition, $F_\psi(\pi) = \tilde{\pi}_n \circ \psi$ and, if $\alpha\colon \pi_1 \to \pi_1$ is a morphism of representations, then $F_\psi(\alpha) = \mathrm{diag}(\alpha, \alpha, \ldots, \alpha)$.

**Definition 2.** A $*$-algebra $\mathfrak{B}$ majorizes a $*$-algebra $\mathfrak{A}$ ($\mathfrak{B} \succ \mathfrak{A}$) if there exist $n \in \mathbb{N}$, an enveloping algebra $\tilde{\mathfrak{A}}$, and a $*$-homomorphism $\psi\colon \mathfrak{B} \to M_n(\tilde{\mathfrak{A}})$ such that the functor $F_\psi\colon \mathrm{Rep}(\mathfrak{A}) \to \mathrm{Rep}(\mathfrak{B})$ is full and faithful.

*Remark* 1. For a class of $C^*$-algebras, it is possible to consider only homomorphisms $\psi\colon \mathfrak{B} \to M_n(\mathfrak{A})$ in Definition 2, since for a $C^*$-algebra $\mathfrak{A}$, $M_n(\mathfrak{A})$ is also a $C^*$-algebra, and the unique enveloping algebra of a $C^*$-algebra is the algebra itself. Moreover, we have the following. *A $C^*$-algebra $\mathfrak{B}$ majorizes a $C^*$-algebra $A$ if and only if $\mathfrak{B}$ contains an ideal $\mathfrak{I}$ such that $\mathfrak{B}/\mathfrak{I} \approx M_n(\mathfrak{A})$. This isomorphism is defined precisely by the mapping $\psi$.* A proof of this claim is not difficult and will be given elsewhere.

*Remark* 2. It is also possible to show that the majorization is a quasiorder relation: if $\mathfrak{C} \succ \mathfrak{B}$ and $\mathfrak{B} \succ \mathfrak{A}$, then $\mathfrak{C} \succ \mathfrak{A}$.

*Remark* 3. One should also note that if $\mathfrak{B} \succ \mathfrak{A}$, and $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}}$ are enveloping algebras of $\mathfrak{A}$, $\mathfrak{B}$ correspondingly, then $\mathfrak{B} \succ \tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}} \succ \mathfrak{A}$, $\tilde{\mathfrak{B}} \succ \tilde{\mathfrak{A}}$, and $M_n(\tilde{\mathfrak{A}})$ is an enveloping algebra of the algebra $M_n(\mathfrak{A})$ with the natural embedding of the algebra $M_n(\mathfrak{A})$ into $M_n(\tilde{\mathfrak{A}})$.

**Theorem 1.** *$\mathfrak{S}_2 \succ \mathfrak{S}_m$ for any $m = 1, 2, \ldots$.*

*Proof.* For the algebra $\tilde{\mathfrak{S}}_m$, take the algebra

$$\mathfrak{S}_m = \mathbb{C}\langle b_1, \ldots, b_m \mid b_i = b_i^*,\ i = 1, \ldots, m\rangle$$

itself, $n = m+2$. Define a homomorphism $\psi\colon \mathfrak{S}_2 \to M_{m+2}(\mathfrak{S}_m)$ as follows:

$$\psi(a_1) = \begin{bmatrix} e & & & & & & \\ & \frac{1}{2}e & & & & 0 & \\ & & \frac{1}{3}e & & & & \\ & & & \ddots & & & \\ & & & & \frac{1}{m}e & & \\ & & 0 & & & \frac{1}{m+1}e & \\ & & & & & & \frac{1}{m+2}e \end{bmatrix},$$

$$\psi(a_2) = \begin{bmatrix} 0 & e & b_1 & & & & \\ e & 0 & e & b_2 & & 0 & \\ b_1 & e & 0 & e & \ddots & & \\ & b_2 & e & \ddots & & b_{m-1} & \\ & & \ddots & & 0 & e & b_m \\ & 0 & & b_{m-1} & e & 0 & e \\ & & & & b_m & e & 0 \end{bmatrix}.$$

One can directly check that the functor $F_\psi$ is full and faithful. □

Theorem 1 permits us to say that the problem of unitary classification of pairs of self-adjoint operators contains, as a subproblem, the problem of unitary classification of representation of any $*$-algebra with a countable number of generators (because it is always possible to choose these generators to be self-adjoint).

**Corollary 1.** $\mathfrak{S}_2 \succ \mathfrak{U}_m$ *for any* $m = 1, 2, 3, \dots$ .

**Theorem 2.** $\mathfrak{U}_2 \succ \mathfrak{S}_2$.

*Proof.* Choose the enveloping algebra $\tilde{\mathfrak{S}}_2$ to be the quotient algebra of the algebra $\mathfrak{S}_2$ with respect to the set

$$\Sigma = \{a - ie, a + ie, b - ie, b + ie\}, \quad n = 1,$$
$$\psi(u_1) = (a - ie)(a + ie)^{-1}, \quad \psi(u_2) = (b - ie)(b + ie)^{-1}$$

(the Cayley transformation). The rest is obvious. □

Theorems 1 and 2 allow us, as a model of complexity for problems of unitary classification of representations of $*$-algebras, to choose the problem of unitary classification of representations of the algebra $\mathfrak{U}_2$ or, which is the same thing, its enveloping $C^*$-algebra $C^*(\mathcal{F}_2)$, where $\mathcal{F}_2$ is a free group with two generators.

**Definition 3.** A $*$-algebra $\mathfrak{A}$ is called $*$-wild if $\mathfrak{A} \succ C^*(\mathcal{F}_2)$.

*Remark* 4. A $C^*$-algebra $\mathfrak{A}$ is $*$-wild if and only if there exist $n \in \mathbb{N} \cup \{\infty\}$ and a $C^*$-ideal $\mathfrak{J} \subset \mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{J} \approx M_n(C^*(\mathcal{F}_2))$.

Nuclear $C^*$-algebras have only hyperfinite factor representations and, consequently, they cannot be $*$-wild. There also exist non-nuclear $C^*$-algebras which are not wild. For example, the group $C^*$-algebra $C^*(B(m,2))$ of the Burnside group with two generators and sufficiently large odd $n$ is not nuclear ([11] and [12]), and it is also not wild [13].

## 3. A unitary description of orthogonal projections

Following the general ideology of representation theory for $*$-algebras, we will try to solve the problem of unitary description of $*$-representations of the algebra $\mathfrak{A}$ by using a description of its irreducible representations and of all its representations as an integral of irreducible ones; a representation $\pi$ of a $*$-algebra $\mathfrak{A}$ is irreducible if there is no nontrivial subspace in $H$, invariant with respect to the operators $\pi(x)$ for all $x \in \mathfrak{A}$.

For representations of the $*$-algebra $\mathfrak{P}_2 = \mathbb{C}\langle p_1, p_2 \mid p_1^* = p_1 = p_1^2, p_2^* = p_2 = p_2^2\rangle$ (a pair of orthogonal projections $P_1$, $P_2$), there is a structure theorem (see, for example, [14] and others) that gives a decomposition of representations into a direct sum (or integral) of irreducible representations which are either one-dimensional or two-dimensional, and, up to a unitary equivalence, they coincide with one of the following:

1) four one-dimensional: $\pi_{(0,0)}(p_1) = \pi_{(0,0)}(p_2) = 0$; $\pi_{(0,1)}(p_1) = 0$, $\pi_{(0,1)}(p_2) = 1$; $\pi_{(1,0)}(p_1) = 1$, $\pi_{(1,0)}(p_2) = 0$; $\pi_{(1,1)}(p_1) = \pi_{(1,1)}(p_2) = 1$;

2) the family, parametrized by $\phi \in (0, \pi/2)$, of two-dimensional

$$\pi_\phi(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_\phi(p_2) = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}.$$

One possible way to prove this theorem is to directly verify that

$$\mathfrak{P}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*;\ \{a, b\} = ab + ba = 0;\ a^2 + b^2 = e\rangle,$$

where $a = p_1 - p_2$, $b = e - p_1 - p_2$, and then to apply results from [15] about the structure of a pair of anticommuting self-adjoint operators.

The problem is to describe, up to a unitary equivalence, a family of orthogonal projections $P_1, P_2, \ldots, P_n$ for $n \geq 3$ that is $*$-wild. We give a fairly simple proof that this problem is $*$-wild for $n = 3$.

**Theorem 3.** *Let*

$$\mathcal{P} = \mathbb{C}\langle p_1, p_2, p_3 \,|\, p_i^2 = p_i^* = p_i \ (i = 1, 2, 3)\rangle.$$

*Then $\mathcal{P}_3 \succ C^*(\mathcal{F}_2)$, i.e. $\mathcal{P}_3$ is $*$-wild.*

*Proof.* Let us define the homomorphism $\psi\colon \mathcal{P}_3 \to M_4(\mathcal{F}_2)$ as follows:

$$\psi(p_1) = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\psi(p_2) = \begin{pmatrix} \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \\ \frac{1}{2}e & 0 & \frac{1}{2}e & 0 \\ 0 & \frac{1}{2}e & 0 & \frac{1}{2}e \end{pmatrix},$$

$$\psi(p_3) = \begin{pmatrix} \frac{3}{8}e & \frac{\sqrt{3}}{8}uv^8 & \frac{\sqrt{3}}{8}u & \frac{3}{8}e \\ \frac{\sqrt{3}}{8}vu^* & \frac{5}{8}e & \frac{3}{8}v & -\frac{\sqrt{3}}{8}vu^* \\ \frac{\sqrt{3}}{8}u^* & \frac{3}{8}v^* & \frac{1}{4}e & 0 \\ \frac{3}{8}e & -\frac{\sqrt{3}}{8}uv^* & 0 & \frac{3}{4}e \end{pmatrix}.$$

It is easy to check that the corresponding functor $F_\psi\colon \mathrm{Rep}C^*(\mathcal{F}_2) \to \mathrm{Rep}\mathcal{P}_3$ is full and faithful. $\square$

Moreover, the following theorem holds ([1]).

**Theorem 4.** *Let*

$$\mathfrak{C} = \mathbb{C}\langle p_1, p_2, p_3 \mid p_i^* = p_i, p_i^2 = p_i, p_1p_2 = p_2p_1 = 0\rangle.$$

*Then* $\mathfrak{C} \succ C^*(\mathcal{F}_2)$*, i.e.* $\mathfrak{C}$ *is* $*$*-wild.*

*Proof.* The proof given here is simpler than the proof in [1].

Let

$$E_k = \underbrace{\begin{bmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{bmatrix}}_{k \text{ times}}, \qquad e \text{ is the identity in the algebra } C^*(\mathcal{F}_2),$$

$$J_1 = \begin{bmatrix} E_4 \\ 0_{3\times 4} \\ 0_{5\times 4} \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0_{4\times 3} \\ E_3 \\ 0_{5\times 3} \end{bmatrix}, \quad J_3 = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix},$$

where

$$A_1 = \frac{1}{N}\begin{bmatrix} e & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & 2e & 0 & 0 \\ 0 & 0 & 0 & 3e & 0 \end{bmatrix}, \quad A_2 = \frac{1}{N}\begin{bmatrix} e & 0 & e & e & e \\ 0 & 2e & e & u_1 & 0 \\ 0 & 0 & e & 0 & u_2 \end{bmatrix},$$

$u_1$, $u_2$ are generators of the algebra $C^*(\mathcal{F}_2)$, $A_3 = \sqrt{E_5 - A_1^*A_1 - A_2^*A_2}$. $N$ is chosen so that $\|A_1^*A_1 + A_2^*A_2\| < 1$ in $M_5(C^*(\mathcal{F}_2))$. Then $J_1^*J_1 = E_4$, $J_2^*J_2 = E_3$, and $J_3^*J_3 = E_4$. This implies that $(J_iJ_i^*)^2 = J_iJ_i^*$, $i = 1, 2, 3$. Moreover, since $J_1^*J_2 = 0$ and $J_2J_1^* = 0$, we see that

$$(J_1J_1^*)(J_2J_2^*) = (J_2J_2^*)(J_1J_1^*) = 0.$$

Set $\psi(p_i) = J_iJ_i^*$; $\psi$ defines a homomorphism of the $*$-algebra $\mathfrak{C}$ into $M_{14}(C^*(\mathcal{F}_2))$. One can directly check that the functor $F_\psi\colon \mathrm{Rep}(\mathfrak{C}) \to \mathrm{Rep}(C^*(\mathcal{F}_2))$ is full and faithful. $\square$

**Corollary 2.** *The problem of a unitary classification of "all but one" orthogonal projections* $p, p_1, \ldots, p_n$ ($p_ip_j = 0$ *for* $i \neq j$) *is* $*$*-wild if* $n \geq 2$.

**Corollary 3.** *The problem of unitary classification of quadruples of orthogonal projections* $p_1$, $p_2$, $p_3$, $p_4$ *such that*

$$\alpha(p_1 + p_2 + p_3 + p_4) = I, \qquad 0 < \alpha < 1,$$

*for a fixed* $\alpha \neq \frac{1}{2}$*, has only a finite number of irreducible representations, the dimension of which depends on the parameter* $\alpha$ ([16]). *If* $\alpha = \frac{1}{2}$*, then irreducible representations are only in dimensions one and two.*

*It directly follows from Theorem 4 that the problem of unitary classification of five orthogonal projections* $p_1$, $p_2$, $p_3$, $p_4$, $p_5$ *such that* $p_1 + p_2 + p_3 + p_4 + p_5 = 2I$ *is* $*$*-wild.*

## 4. A UNITARY DESCRIPTION OF IDEMPOTENTS

For a single idempotent, the situation is similar to the situation for two orthogonal projections. Consider the $*$-algebra $\mathfrak{Q}_1$ generated by a pair of an idempotent and its adjoint $q_1$, $q_1^*$. Let $q_1 = a_1 + ib_1$, $q_1^* = a_1 - ib_1$, where $a_1^* = a_1$, $b_1^* = b_1$. The $*$-algebra $\mathfrak{Q}_1$ coincides with the algebra

$$\mathbb{C}\langle a, b \mid a = a^*, b = b^*; \{a, b\} = ab + ba = 0, a^2 - b^2 = e\rangle,$$

where $a = 2\bigl(a_1 - \frac{1}{2}e\bigr)$, $b = 2b_1$.

Irreducible representations of the algebra $\mathfrak{Q}_1$, up to a unitary equivalence, coincide with one of the following:

1) two one-dimensional representations given by $\pi_0(q_1) = 0$ and $\pi_1(q_1) = 1$;

2) a family, depending on a parameter $y > 0$, of two-dimensional representations:

$$\pi_{(x,y)}(q_1) = Q_1 = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}.$$

By decomposing a representation of the algebra $\mathfrak{Q}_1$ into the direct sum of irreducible representations on a finite-dimensional space $H$, we obtain a structure theorem (see [17] and [18]) for the unitary description of idempotents in the finite-dimensional case.

There is a structure theorem that gives a description of any bounded idempotent on any separable Hilbert space in the form of an integral of irreducible representations [4].

Consider the problem of a unitary description of pairs of idempotents $Q_1$, $Q_2$ ($Q_1^2 = Q_1$, $Q_2^2 = Q_2$). The fact that the problem of a unitary description of pairs of idempotents is difficult is just mathematical folklore. We will prove a corresponding theorem and show that, even if an additional restriction of self-adjointness is imposed on one of the idempotents (one of the idempotents is an orthogonal projection), the problem does not become easier.

**Theorem 5.** *Let* $\mathfrak{Q}_2 = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q^2\rangle$, $\mathfrak{D} = \mathbb{C}\langle q, p \mid q^2 = q, p^2 = p = p^*\rangle$, $\mathfrak{S}_2 = \mathbb{C}\langle a_1, a_2 \mid a_1 = a_1^*, a_2 = a_2^*\rangle$. *Then* $\mathfrak{Q}_2 \succ \mathfrak{D} \succ \mathfrak{S}_2$, *so that the* $*$*-algebras* $\mathfrak{Q}_2$, $\mathfrak{D}$ *are* $*$*-wild.*

*Proof.* Because $\mathfrak{D}$ is a factor algebra of the algebra $\mathfrak{Q}_2$, we have that $\mathfrak{Q}_2 \succ \mathfrak{D}$ (we choose an enveloping algebra for $\mathfrak{D}$ to be the algebra $\mathfrak{D}$ itself, $n = 1$, $\psi\colon \mathfrak{Q}_2 \to \mathfrak{D}$ is the natural epimorphism of the algebra onto the factor algebra).

Let us show that $\mathfrak{D} \succ \mathfrak{S}_2$. Construct the homomorphism $\psi\colon \mathfrak{D} \to M_2(\mathfrak{S}_2)$:

$$\psi(q) = \begin{pmatrix} e & a_1 + ia_2 \\ 0 & 0 \end{pmatrix}, \quad \psi(p) = \frac{1}{2}\begin{pmatrix} e & e \\ e & e \end{pmatrix}.$$

It is easy to check that the corresponding functor $F_\psi\colon \operatorname{Rep}\mathfrak{S}_2 \to \operatorname{Rep}\mathfrak{D}$ is full and faithful. □

**Corollary 4.** *The algebra* $\mathfrak{Q}_n$, *for* $n \geq 2$ *(the problem of unitary description of* $n$ *idempotents if* $n \geq 2$*) is* $*$*-wild.*

Finally, we show that the $*$-algebra $\mathfrak{Q}_{n,\perp}$ (the problem of unitary classification of a family of pairwise orthogonal idempotents $Q_1, Q_2, \dots, Q_n$, $Q_iQ_j = 0$ for $i \neq j$) is $*$-wild for $n \geq 2$.

**Theorem 6.** *Let*

$$\mathfrak{Q}_{2,\perp} = \mathbb{C}\langle q_1, q_2 \mid q_1^2 = q_1, q_2^2 = q_2,\ q_1 q_2 = q_2 q_1 = 0\rangle.$$

*Then* $\mathfrak{Q}_{2,\perp} \succ \mathfrak{S}_2$, *i.e.* $\mathfrak{Q}_{2,\perp}$ *is a wild* $*$*-algebra.*

*Proof.* Let us define a homomorphism $\psi\colon \mathfrak{Q}_{2,\perp} \to M_3(\mathfrak{S}_2)$ as follows:

$$\psi(q_1) = \begin{bmatrix} e & e & a_1 + ia_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi(q_2) = \begin{bmatrix} 0 & -e & -e \\ 0 & e & e \\ 0 & 0 & 0 \end{bmatrix}.$$

One can directly check that $[\psi(q_k)]^2 = \psi(q_k)$, $k = 1, 2$, $\psi(q_1)\,\psi(q_2) = \psi(q_2)\,\psi(q_1) = 0$, and that the functor $F_\psi\colon \operatorname{Rep}\mathfrak{S}_2 \to \operatorname{Rep}\mathfrak{Q}_{2,\perp}$ is full and faithful. ☐

**Corollary 5.** *The problem of unitary classification of pairs of commuting idempotents is* $*$*-wild.*

**Corollary 6.** *The* $*$*-algebra* $\mathfrak{Q}_{n,\perp} = \mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i, i = 1, \dots, n;\ q_i q_j = 0 \text{ for } i \neq j\rangle$ *(the problem of unitary classification of* $n$ *pairwise orthogonal idempotents) is* $*$*-wild for* $n \geq 2$.

**Corollary 7.** *The* $*$*-algebra* $\mathbb{C}\langle q_1, \dots, q_n \mid q_i^2 = q_i,\ i = 1, \dots, n;\ q_1 + q_2 + \dots + q_n = e\rangle$ *(the problem of unitary classification of* $n$ *idempotents* $Q_1, \dots, Q_n$ *such that* $Q_1 + \dots + Q_n = I$*) is* $*$*-wild for* $n \geq 3$.

*Proof.* If $m = 3$, the condition $q_1 + q_2 + q_3 = e$ implies that the idempotents $q_1$, $q_2$, $q_3$ are pairwise orthogonal. Then the algebra under consideration coincides with the algebra $\mathfrak{Q}_{2,\perp}$. ☐

**Corollary 8.** *Let* $\mathfrak{A}_{R_3} = \mathbb{C}\langle x \mid R_3(x) \overset{\text{def}}{=} (x - \alpha_1 e)(x - \alpha_2 e)(x - \alpha_3 e) = 0,\ \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C},\ \alpha_k \neq \alpha_l \text{ for } k \neq l\rangle$. *Then* $\mathfrak{A}_{R_3} \succ \mathfrak{Q}_{2,\perp}$, *and consequently, the* $*$*-algebra* $\mathfrak{A}_{R_3}$ *is* $*$*-wild.*

*Proof.* Define the homomorphism $\psi\colon \mathfrak{A}_{R_3} \to \mathfrak{Q}_{2,\perp}$ as follows:

$$\psi(a) = \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3(e - q_1 - q_2).$$

It is easy to check that the functor $F_\psi$ is full and faithful. ☐

*Remark* 5. Corollary 8 is given in [19]. The proof in [19] actually uses the fact that the problem of unitary classification of two orthogonal idempotents is $*$-wild, and implicitly contains this proof.

## Acknowledgements

The authors are sincerely grateful to Palle E. T. Jorgensen, S. Popovich, and S. Rabanovich for useful discussions, and to V. L. Ostrovs'kiĭ and Yu. A. Chapovsky for help in preparation of this article.

## References

[1] S. A. Kruglyak and Yu. S. Samoĭlenko. On unitary equivalence of collections of self-adjoint operators. *Funktsional'nyi Analiz i ego Prilozheniya*, 14(1):60–62, 1980. Russian. MR **81k:**47031

[2] S. A. Kruglyak. Representations of involutive quivers. Deposited in VINITI, 7266-84. 62 pages. Russian.

[3] S. Kruglyak and A. Piryatinskya. On "wild" $*$-algebras and the unitary classification of weakly centered operators. Preprint ser. of Mittag-Leffler Inst., no. 11, 1995/96.

[4] S. Kruglyak and S. Samoĭlenko. Structure theorems for families of idempotents. *Ukrainskii Matematicheskii Zhurnal*, 50(4):523–533, 1988. Russian.

[5] A. Böttcher, I. Gohberg, Yu. Karlovich, N. Krupnik, S. Roch, B. Silberman, and I. Spitkovsky. Banach algebras generated by $n$ idempotents and applications. *Operator Theory Adv. and Appl.*, 90:19–54, 1996. MR **97h:**46078

[6] N. L. Vasilevski. $C^*$-algebras generated by orthogonal projections and their applications. Reporte Interno # 220, Departamento de Matematicas CINVESTAV del I.P.N. México, 1997. MR **99e:**46074

[7] V. V. Sergeĭchuk. Unitary and Euclidean representations of a quiver. *Linear Algebra Appl.*, 278:37–62, 1998. MR **99g:**16020

[8] P. Donovan and M. R. Freislich. The representation theory of finite graphs and associated algebras. *Carleton Math. Lect. Notes*, 5:1–119, 1973. MR **50:**9701

[9] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Springer, Berlin, Heidelberg, New York, 1967. MR **35:**1019

[10] A. Ya. Khelemskiĭ. *Banach algebras and polynormed algebras: general theory, representations, homology.* Nauka, Moscow, 1989. Russian. MR **91h:**46001

[11] A. Yu. Ol'shanskiĭ. Infinite simple torsion free Noether group. *Izvestiya Academii Nauk SSSR. Ser. Mat.*, 43(6):1328–1393, 1979. Russian. MR **81i:**20033

[12] S. I. Adyan. Random walks on free periodic groups. *Izvestiya Academii Nauk SSSR. Ser. Mat.*, 46(6):1139–1149, 1982. Russian. MR **84m:**43001

[13] S. A. Kalutskiĭ and Yu. S. Samoĭlenko. Periodic groups are not wild. *Ukrainskii Matematicheskii Zhurnal*, 49(5):729–730, 1997. Russian. MR **98h:**46081

[14] P. Halmos. Two subspaces. *Trans. Amer. Math. Soc.*, 144:381–189, 1969. MR **40:**4746

[15] Yu. S. Samoĭlenko. *Spectral theory of families of self-adjoint operators.* Naukova Dumka, Kiev, 1984. Russian. English translation: *Spectral theory of families of self-adjoint operators.* Kluwer Acad. Publ., 1991. MR **92j:**47038

[16] Yu. N. Bespalov. Collections of orthogonal projections satisfying relations. *Ukrainskii Matematicheskii Zhurnal*, 44(3):309–317, 1992. Russian.

[17] D. Ž. Dokovič. Unitary similarity of projectors. *Aequationes Math.*, 42:220–224, 1991.

[18] Kh. D. Ikramov. On canonical form of projections with respect to a unitary similarity. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, 36(1), 1980. Russian.

[19] Yu. N. Bespalov and Yu. S. Samoĭlenko. Algebraic operators and pairs of self-adjoint operators satisfying an algebraic relation. *Funktsional'nyi Analiz i ego Prilozheniya*, 25(4):72–74, 1991. Russian. MR **93d:**47044

Institute of Mathematics, Ukrainian National Academy of Sciences, vul. Tereshchinkivs'ka, 3, Kiev, 252001, Ukraine

Institute of Mathematics, Ukrainian National Academy of Sciences, vul. Tereshchinkivs'ka, 3, Kiev, 252001, Ukraine
*E-mail address*: `Yurii_Sam@imath.kiev.ua`