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LINEAR INDEPENDENCE AND DIVIDED DERIVATIVES OF A DRINFELD MODULE II

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This paper is dedicated to the memory of Bernard Dwork

Abstract. In this note we extend our previous results on the linear independence of *values of the divided derivatives* of exponential and quasi-periodic functions related to a Drinfeld module to *divided derivatives of values* of identity and quasi-periodic functions evaluated at the logarithm of an algebraic value. The change in point of view enables us to deal smoothly with divided derivatives of arbitrary order. Moreover we treat a full complement of quasi-periodic functions corresponding to a basis of de Rham cohomology.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, $A := \mathbb{F}_q[T]$, $k := \mathbb{F}_q(T)$, and ϕ an A-Drinfeld module of rank d defined over $\bar{k}^{sep}$, a separable closure of $\mathbb{F}_q(T)$:

$$\phi(T) = TF^0 + a_1F^1 + \cdots + a_dF^d, \quad a_i \in \bar{k}^{sep},\ a_d \neq 0. \tag{1.1}$$

Here F denotes the q-power Frobenius $F : x \mapsto x^q$. There exists a unique exponential function

$$e(z) = \sum_{h\geq 0} c_h z^{q^h} \tag{1.2}$$

corresponding to ϕ, characterized by the initial condition $c_0 = 1$ together with the functional equation

$$e(Tz) = \phi(T)e(z). \tag{1.3}$$

The function $e(z)$ is an entire function; that is, it converges on all of $\mathbf{C}$, a completion of an algebraic closure of $k_\infty := \mathbb{F}_q((1/T))$. Moreover by (1.3), the coefficients c_h lie in $\bar{k}^{sep}$.

The divided derivatives $D_i, i = 0, 1, 2 \ldots$, defined by

$$D_iT^n = \binom{n}{i}T^{n-i}, \quad n \geq 0,$$

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extend uniquely to all of $\bar{k}_\infty^{sep}$, the separable closure of k_∞, to define a family of hyperderivatives satisfying

$$D_i(ab) = \sum_{r+s=i} D_r(a)D_s(b).$$

(The first author warns of typos in the expressions given in [3].)

For $a \in \bar{k}_\infty^{sep}$, we set $a^{[i]} := D_i(a)$ and we let $e^{[i]}(z) := \sum c_h^{[i]} z^{q^h}$, $i \in \mathbb{N}$.

Elements of the non-commutative ring $\mathbf{C}\{F\}$ are called *twisted polynomials*. For every twisted polynomial P with scalar term zero, i.e. $P \in F\mathbf{C}\{F\}$, the functional equation

$$\begin{aligned} Q(Tz) &= TQ(z) + P(e(z)), \\ Q(z) &\equiv 0 \pmod{z^q} \end{aligned} \tag{1.4}$$

determines a unique entire function $Q(z)$, which is said to be *quasi-periodic* with respect to the biderivation δ determined by $\delta(T) = P$. The space of quasi-periodic functions generates a field of transcendence degree d over $\mathbf{C}(z)$ (see [2] for the facts claimed here), and $e(z) - z$ is quasi-periodic with respect to $P_0 = \phi(T) - TF^0$. *For the remainder of the paper we fix $d-1$ quasi-periodic functions $Q_1(z), \ldots, Q_{d-1}(z) \in \bar{k}^{sep}[[z]]$ which are algebraically independent over $\mathbf{C}(z, e(z))$.* We denote by $Q_j^{[i]}(z)$ the entire function whose power series has coefficients which are the ith divided derivatives of the corresponding coefficients of $Q_j(z)$.

Theorem 1.1. *Let $u \in \mathbf{C}\backslash\{0\}$ such that $e(u) \in \bar{k}^{sep}$. Then the values*

$$\begin{array}{cccccc} 1, & u, & u^{[1]}, & \ldots & u^{[i]}, & \ldots \\ & Q_1(u), & Q_1(u)^{[1]}, & \ldots & Q_1(u)^{[i]}, & \ldots \\ & \vdots & \vdots & & \vdots & \\ & Q_{d-1}(u), & Q_{d-1}(u)^{[1]}, & \ldots & Q_{d-1}(u)^{[i]}, & \ldots \end{array}$$

are linearly independent over $\bar{k}$, the algebraic closure of k.

When $d = 1$, there are no Q_j. In that case, the above result asserts the $\bar{k}$-linear independence of $1, u, u^{[1]}, \ldots, u^{[i]}, \ldots$.

Corollary 1.2. *If ω is a non-zero period of $e(z)$ and $\eta_1, \ldots, \eta_{d-1}$ are the related quasi-periods $\eta_i := Q_i(\omega)$, then 1, together with these numbers and their divided derivatives of all orders, are $\bar{k}$-linearly independent.*

It is a fundamental fact of divided derivatives (see Lemma 2.1 below) that $D_i(a^{q^h}) = 0$ unless $q^h | i$. By the continuity of divided derivatives in finite separable extensions of k_∞ (see Lemma 4.1 below), it then follows that when $i < q$, $(Q_j(u))^{[i]} = Q_j^{[i]}(u)$. So our theorem contains that of [3]. In fact, the approach there goes through without essential change to the above situation as long as we are interested only in divided derivatives of order $i < q^{h_0}$, where h_0 is the lowest degree in F of the non-zero terms of P in (1.4). For then

$$Q_j^{[i]}(Tz) = TQ_j^{[i]}(z) + Q_j^{[i-1]}(z),$$

and the T-module is only marginally more complicated than in [3].

However the T-module involved for higher order divided derivatives becomes so unwieldy that it is almost intractable. Even if we, by advantage of hindsight, now see in principle the broad outline of a general proof in terms of these $Q_j^{[i]}(z)$, the complications in carrying out the necessary details would be quite severe. Luckily they are also unnecessary. For we are able to use our knowledge [2] of de Rham cohomology and quasi-periodic functions to reduce the desired statement for general $Q_j(u)^{[i]}$ to the same statement for related $R_j(u)^{[i]}$, where the lowest degree term in $R_j(z)$ is larger than i. Then, as indicated in the preceding paragraph, the continuity of divided derivatives allows us to deduce the desired statement for $R_j(u)^{[i]}$ from the corresponding one for $R_j^{[i]}(u)$, to which the standard transcendence machinery applies as in [3].

Our development of this latter approach arose out of our efforts to deal with a stipulation of the referee to explain just a bit more about the messy general case. Thus we are indebted to him for motivating us to re-think this material.

2. Construction of two basic T-modules

As in [3], whose notation we preserve, we define a T-module using the functional equations satisfied by the functions $e^{[i]}(z)$ and $Q_j^{[i]}(z)$. For that, we differentiate the functional equation (1.3) for $e(z)$, keeping in mind the following fundamental lemma from [3]:

Lemma 2.1.

$$D_i(a^p) = \begin{cases} \left(a^{[\frac{i}{p}]}\right)^p, & \textit{if } p|i, \\ 0, & \textit{otherwise.} \end{cases}$$

Starting from the power series expansion $e(z) = \sum c_h z^{q^h}$, we see that the ith divided derivative of the related power series $e(Tz)$ may be given as

$$e^{[i]}(Tz) + \sum_{q^h \le i} c_h^{[i-q^h]} z^{q^h}.$$

Setting this quantity equal to the ith divided derivative of the right-hand side of (1.3), viz.

$$e(Tz) = \phi(T)e(z) = Te(z) + \cdots + a_d(e(z))^{q^d},$$

then gives

$$\begin{aligned} (2.1) \quad & e^{[i]}(Tz) + \sum_{q^h \le i} c_h^{[i-q^h]} z^{q^h} \\ & \qquad = Te^{[i]}(z) + e^{[i-1]}(z) + \sum_{j=1}^{d} \sum_{m} a_j^{[i-q^j m]} (e^{[m]}(z))^{q^j}. \end{aligned}$$

For every positive integer s we have a T-module of dimension $s+2$ determined by the following lower triangular matrix, which we denote by $\Phi_s(T)$:

$$\begin{pmatrix} T & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \phi(T) & 0 & 0 & 0 & \dots & 0 \\ -1 & \phi^{[1]}(T) & T & 0 & 0 & \dots & 0 \\ 0 & \phi^{[2]}(T) & 1 & T & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \vdots \\ -\sum_h c_h^{[i-q^h]} F^h & \phi^{[i]}(T) & & \dots & \sum_{j=1}^d a_j^{[i-q^j m]} F^j & & \\ \vdots & \vdots & & & & & \vdots \\ -\sum_h c_h^{[s-q^h]} F^h & \phi^{[s]}(T) & & \dots & \sum_{j=1}^d a_j^{[s-q^j m]} F^j & \dots \quad 1 & T \end{pmatrix},$$

where the free variables i and m occur in the $i+2$nd row and $m+2$nd column. The reader is invited to verify that all the diagonal terms but the second equal T and that all the subdiagonal terms but the first two equal 1.

Corollary 2.2. *The exponential function of $\Phi_s(T)$ is given by*

$$\operatorname{Exp}_s \begin{pmatrix} x \\ y \\ z_1 \\ \vdots \\ z_i \\ \vdots \\ z_s \end{pmatrix} := \begin{pmatrix} x \\ e(y) \\ e^{[1]}(y) + z_1 \\ \vdots \\ e^{[i]}(y) + \sum_{q^h m \le i} c_h^{[i-q^h m]} z_m^{q^h} \\ \vdots \\ e^{[s]}(y) + \sum_{q^h m \le s} c_h^{[s-q^h m]} z_m^{q^h} \end{pmatrix}.$$

Proof. According to the uniqueness of the exponential function of a T-module (see [1]), we must verify the following identity:

$$\operatorname{Exp}_s(d\Phi_s(T)\mathbf{z}) = \Phi_s(T)\operatorname{Exp}_s(\mathbf{z}).$$

Let us consider the $i+2$nd coordinate, $1 < i \le s$. (The first three coordinates are equal because the corresponding identities are those for ordinary multiplication by T and for the Drinfeld action for $e(z)$ and $e^{[1]}(z)$; the latter was verified in [3].)

The left-hand term is

$$e^{[i]}(Ty) + \sum_{q^h \le i} c_h^{[i-q^h]} (Tz_1 + y - x)^{q^h} + \sum_{\substack{m \ge 2 \\ q^h m \le i}} c_h^{[i-q^h m]} (z_{m-1} + Tz_m)^{q^h}. \tag{2.2}$$

The right-hand term is

$$\begin{aligned} -\sum c_h^{[i-q^h]} x^{q^h} &+ \sum_{q^j m \le i} a_j^{[i-q^j m]} \left(e^{[m]}(y) + \sum_{q^h l \le m} c_h^{[m-q^h l]} z_l^{q^h} \right)^{q^j} \\ &+ e^{[i-1]}(y) + \sum_{q^h m \le i-1} c_h^{[i-1-q^h m]} z_m^{q^h} + Te^{[i]}(y) + T \sum_{q^h m \le i} c_h^{[i-q^h m]} z_m^{q^h}. \end{aligned} \tag{2.3}$$

The terms in x are the same in (2.2) and (2.3), and the equality of the terms in y follows directly from the equation (2.1).

Therefore let us consider the coefficients of $z_m^{q^h}$. In (2.2), we have:

$$T^{q^h} c_h^{[i-q^h m]} + c_h^{[i-q^h(m+1)]}.$$

In (2.3), we have

$$c_h^{[i-1-q^h m]} + T c_h^{[i-q^h m]} + \sum_{\substack{u+j=h \\ l}} a_j^{[i-q^j l]} \left(c_u^{[l-q^u m]}\right)^{q^j}.$$

In order to prove that these terms are identical, we consider the following identity:

$$c_h^{[i]} T^{q^h} + c_h^{[i-q^h]} = T c_h^{[i]} + c_h^{[i-1]} + \sum_{\substack{u+j=h \\ l}} a_j^{[i-q^j l]} \left(c_u^{[l]}\right)^{q^j},$$

which comes from equating the coefficients of z^{q^h} in (2.1). Replacing i by $i - q^h m$ in this equality gives

$$c_h^{[i-q^h m]} T^{q^h} + c_h^{[i-q^h(m+1)]} = T c_h^{[i-q^h m]} + c_h^{[i-q^h m-1]} + \sum_{\substack{u+j=h \\ l}} a_j^{[i-q^h m-q^j l]} \left(c_u^{[l]}\right)^{q^j}.$$

Replacing l by $l - q^u m$, one obtains the desired conclusion, thus establishing Corollary 2.2. □

For the time being, we fix s:

Convention. Fix $s \in \mathbb{N}$ such that each

$$Q_j(z) \equiv 0 \pmod{z^{s+1}}, \quad 1 \le j \le d-1.$$

Note that, by the functional equation (1.4) , $s \ge q-1$.

For each of the quasi-periodic functions Q_j it is straightforward to deduce from its functional equation (1.4) and from Lemma 2.1, as in [3] (where it was done for $q-1$ instead of s), that

$$Q_j^{[i]}(Tz) = T Q_j^{[i]}(z) + Q_j^{[i-1]}(z), \quad 1 \le i \le s.$$

Assume that the functional equation of Q_j is

$$Q_j(Tz) = T Q_j(z) + P_j(e(z)), \quad 1 \le j \le d-1.$$

If we regroup the equations of the preceding exponential function and the quasi-periodic functions, we obtain the following:

Theorem 2.3. *The T-module Ψ_s determined by*

$$\Psi_s(T) := \left(\begin{array}{c:c} \Phi_s(T) & 0 \\ \hdashline \begin{array}{cc:c} 0 & P_1 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \\ 0 & P_2 & \\ 0 & 0 & \\ \vdots & \vdots & 0 \\ 0 & 0 & \\ 0 & P_{d-1} & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{array} & TI_{(d-1)(s+1)} + S_{(d-1)(s+1)} \end{array}\right),$$

where $I_{(d-1)(s+1)}$ denotes the $(d-1)(s+1)\times(d-1)(s+1)$ unit diagonal matrix and $S_{(d-1)(s+1)}$ the corresponding unit subdiagonal matrix, has as exponential function

$$\mathrm{Exp}(\mathbf{z}) = \mathrm{Exp}\begin{pmatrix} x \\ y \\ z_1 \\ \vdots \\ z_s \\ z_{1,0} \\ z_{1,1} \\ \vdots \\ z_{1,s} \\ \vdots \\ z_{d-1,0} \\ z_{d-1,1} \\ \vdots \\ z_{d-1,s} \end{pmatrix} = \begin{pmatrix} x \\ e(y) \\ \vdots \\ e^{[s]}(y) + \sum_{q^h m \le s} c_h^{[s-q^h m]} z_m^{q^h} \\ Q_1(y) + z_{1,0} \\ Q_1^{[1]}(y) + z_{1,1} \\ \vdots \\ Q_1^{[s]}(y) + z_{1,s} \\ \vdots \\ Q_{d-1}(y) + z_{d-1,0} \\ Q_{d-1}^{[1]}(y) + z_{d-1,1} \\ \vdots \\ Q_{d-1}^{[s]}(y) + z_{d-1,s} \end{pmatrix}.$$

Proof. Corollary 2.2 establishes the part of the functional equation involving only the top $s+2$ functions. The proof of the remaining functional equations was done in [3]. □

For our proof of Theorem 1.1, we will need to establish the algebraic independence of the above coordinate functions.

3. Independence of functions

When we deal with the independence of the functions, we do not need to restrict the order of the derivatives to be at most s.

Theorem 3.1. *For every $i > 0$, the functions*

$$\begin{array}{cccc} z, & e(z), & \dots & e^{[i]}(z) \\ & Q_1(z), & \dots & Q_1^{[i]}(z) \\ & \vdots & & \vdots \\ & Q_{d-1}(z), & \dots & Q_{d-1}^{[i]}(z) \end{array}$$

are algebraically independent over $\mathbf{C}$.

Proof. We use recurrence on j to prove that, for every finite i, the functions

$$z, e(z), \dots, e^{[i]}(z); Q_1(z), \dots, Q_1^{[i]}(z); \dots; Q_j(z), \dots, Q_j^{[i]}(z)$$

are algebraically independent over $\mathbf{C}$, $j = 1, \dots, d-1$. From the structure of the vector space of quasi-periodic functions (Hodge Theory for Drinfeld Modules), we know (cf. §5 of [7] and Theorem 4.1 of [2]) that it is sufficient to prove the present claim for the specific quasi-periodic functions satisfying

$$Q_j(Tz) = TQ_j(z) + (e(z))^{q^j}, \quad j = 1, \dots, d-1.$$

For $j = 1$, this result is proved as Theorem 2 in [3]. Let us now assume the claim for $j-1$ with $j \leq d-1$ and deduce it for j quasi-periodic functions. As all the functions are $\mathbb{F}_q$-linear, any non-trivial algebraic relation on the functions which has minimal total degree will also be $\mathbb{F}_q$-linear (see the Appendix):

$$\begin{array}{rlll} R(z) = & P_{-1}(z) & + P_{0,0}(e(z)) & + \dots \quad + P_{i,0}(e^{[i]}(z)) \\ & & \vdots & \qquad\qquad\qquad \vdots \\ & & + P_{0,j}(Q_j(z)) & + \dots \quad + P_{i_j,j}(Q_j^{[i_j]}(z)) \quad = 0. \end{array}$$

Take such a relation with i_j minimal; in particular $P_{i_j,j}(x) \neq 0$. Since

$$Q_k^{[l]}(Tz) = TQ_k^{[l]}(z) + Q_k^{[l-1]}(z) + \left(e^{[j]}(z)\right)^{q^k}$$

for $l = jq^k$, the relation

$$T^{\deg P_{i_j,j}} R(z) - R(Tz) = 0 \tag{3.1}$$

is of degree less than $\deg P_{i_j,j}$ in $Q_j^{[i_j]}(z)$. Therefore it must be identically zero. It follows that

$$P_{i_j,j}(x) = \alpha x^{q^v},$$

where $\alpha \in \mathbf{C}$ and $q^v = \deg P_{i_j,j}$. If $i_j \geq 1$, an examination of the role of $P_{i_j-1,j}(x)$ in (3.1) shows that this polynomial can contain only one monomial, say $\alpha_{i_j-1} x^{q^v}$, and that $\alpha = 0$. This contradicts our assumption that $P_{i_j,j} \neq 0$.

Therefore $i_j = 0$. Since this is true no matter what the indexing of the Q_i, we see that we are reduced to showing the algebraic independence of

$$z, e(z), e^{[1]}(z), \dots, e^{[i]}(z), Q_1(z), \dots, Q_j(z).$$

If we suppose that i is minimal for algebraic dependence, we can consider an $\mathbb{F}_q$-linear relation of minimal degree having the form:

$$R(z) = P_{-1}(z) + \dots + P_s(e^{[i]}(z)) + P_{0,1}(Q_1(z)) + \dots + P_{0,j}(Q_j(z)) = 0.$$

If $i \geq 1$, the same principles as above lead to a contradiction. Moreover the case $i = 0$ is treated in Theorem 5.1′ of [2]. Theorem 3.1 follows. □

4. The basic intermediate result

We begin by establishing an intermediate version of Theorem 1.1, in which the order of the divided derivatives is restricted to be at most s. The proof will involve the following property established in the Appendix of [3]:

Lemma 4.1. *The divided derivatives D_i are continuous maps on every finite separable extension of k_∞.*

Theorem 4.2. *Let $u \in \mathbf{C} \setminus \{0\}$ such that $e(u) \in \bar{k}^{sep}$. Then the values*

$$\begin{array}{ccccc} 1, & u, & u^{[1]}, & \dots & u^{[s]} \\ & Q_1(u), & Q_1^{[1]}(u), & \dots & Q_1^{[s]}(u) \\ & \vdots & \vdots & & \vdots \\ & Q_{d-1}(u), & Q_{d-1}^{[1]}(u), & \dots & Q_{d-1}^{[s]}(u) \end{array}$$

are linearly independent over $\bar{k}$, the algebraic closure of k.

Proof. As mentioned above, this proof is very similar to that of [3]. The main difference is that we have several Q_j here. We evaluate the exponential function of the t-module Ψ_s at the point

$$\mathbf{u} = (1, u, \dots, u^{[s]}, -Q_1(u), \dots, -Q_1^{[s]}(u); \dots; -Q_{d-1}(u), \dots, -Q_{d-1}^{[s]}(u))^{tr}.$$

Since $e(u) \in \bar{k}^{sep}$, the same is true of $e(u)^{[1]}, \dots, e(u)^{[s]}$. By the basic properties recorded in Lemma 2.1 and Lemma 4.1, we see that

$$\mathrm{Exp}(\mathbf{u}) = (1, e(u), \dots, e(u)^{[s]}, 0, \dots, 0),$$

since

$$e(u)^{[i]} = e^{[i]}(u) + \sum_{i \geq q^h m} c_h^{[i-q^h m]} \left(u^{[m]}\right)^{q^h}.$$

The result now follows on applying the following criterion, which was deduced in [3] from Yu's Sub-T-module Theorem [10], [4], [6].

Theorem 4.3 (Linear Independence Criterion)**.** *Let $G = (\mathbb{G}_a^n, \Phi)$ be a T-module defined over $\bar{k}$. Let $\mathbf{u}$ be a point in $\mathrm{Lie}\, G(\bar{k}_\infty)$ such that $\mathrm{Exp}_\Phi(\mathbf{u}) \in G(\bar{k})$. If the coordinates of $\mathbf{u}$ are $\bar{k}$-linearly dependent, then the coordinate functions of the one-parameter analytic subgroup $z \longmapsto \exp_\Phi(z\mathbf{u})$ are algebraically dependent.*

If the assertion of linear independence were false, then the above criterion shows that the coordinates

$$\begin{pmatrix} z \\ e(zu) \\ \vdots \\ e^{[s]}(zu) + \sum_{q^h m \le s} c_h^{[s-q^h m]}(zu^{[m]})^{q^h} \\ Q_1(zu) - zQ_1(u) \\ Q_1^{[1]}(zu) - zQ_1^{[1]}(u) \\ \vdots \\ Q_1^{[s]}(zu) - zQ_1^{[s]}(u) \\ \vdots \\ Q_{d-1}(zu) - zQ_{d-1}(u) \\ Q_{d-1}^{[1]}(zu) - zQ_{d-1}^{[1]}(u) \\ \vdots \\ Q_{d-1}^{[s]}(zu) - zQ_{d-1}^{[s]}(u) \end{pmatrix}$$

of $\mathrm{Exp}(z\mathbf{u})$ would be algebraically dependent functions of z. However, since the second terms in the sums are simply polynomials in z, it would follow that functions in

$$z, e(zu), \ldots, e^{[s]}(zu); Q_1(zu), Q_1^{[1]}(zu), \ldots, Q_1^{[s]}(zu); \\ \ldots; Q_{d-1}(zu), Q_{d-1}^{[1]}(zu), \ldots, Q_{d-1}^{[s]}(zu)$$

would be algebraically dependent, in contradiction to Theorem 3.1. This completes the proof of Theorem 4.2. □

5. de Rham representative with high order of vanishing

As pointed out in the Introduction, the passage to the general case is carried out by replacing the quasi-periodic functions by suitable representatives from the same de Rham cohomology class. This passage is based on the fundamental properties of biderivations and their corresponding quasi-periodic functions. See, for example, [2] and [7] for proofs. For this discussion F_δ denotes the quasi-periodic function corresponding to the biderivation δ satisfying the functional equation (1.4):

$$\delta \longleftrightarrow F_\delta.$$

Then this relationship is additive:

$$\delta_1 + \delta_2 \longleftrightarrow F_{\delta_1} + F_{\delta_2}. \tag{5.1}$$

If δ is *inner*, i.e. there is a twisted polynomial $P \in \mathbf{C}\{F\}$ such that $\delta(T) = P\phi(T) - TP$, then

$$F_\delta(z) = P(e(z)) - \alpha z, \tag{5.2}$$

where $P = \alpha F^0 +$ higher terms. We say that δ_1 and δ_2 are *in the same de Rham cohomology class* if the difference $\delta_1 - \delta_2$ is inner. (See [7], [2] for background.)

Lemma 5.1. *Let δ be a ϕ-biderivation over $\bar{k}^{sep}$ and $i \in \mathbb{N}$. Then there is a ϕ-biderivation δ^* over $\bar{k}^{sep}$ in the same de Rham cohomology class as δ such that*

$$F_{\delta^*}(z) \equiv 0 \pmod{z^{q^i}}. \tag{5.3}$$

Proof. The inner biderivation δ_j corresponding to any F^j is given by

$$\begin{aligned}\delta_j(T) &= F^j\phi(T) - TF^j \\ &= (T^{q^j} - T)F^j + \text{higher terms.}\end{aligned}$$

In the case that $j = \deg_F \delta(T)$, say

$$\delta(T) = \kappa_j F^j + \text{higher terms},$$

we therefore have $\deg_F(\delta(T) - \kappa_j\delta_j(T)) \geq j + 1$. Repeating this process with increasing j, we find, after at most $i - \deg_F \delta(T)$ steps, a ϕ-biderivation

$$\delta^* = \delta - \sum_{j=\deg_F \delta(T)}^{i-1} \kappa_j \delta_j$$

defined over $\bar{k}^{sep}$ such that

1. δ^* and δ are in the same de Rham cohomology class, and
2. $\deg_F \delta^*(T) \geq i$.

Comparing coefficients on both sides of the functional equation (1.4) for $Q = F_{\delta^*}, P = \delta^*(T)$, one sees immediately that the first non-zero term of $F_{\delta^*}(z)$ has degree equal to

$$q^{\deg_F \delta^*(T)}.$$

The lemma follows. □

6. Proof of Theorem 1.1

We are now in a position to establish Theorem 1.1. It will be enough to show that the expressions involving derivatives of order at most any arbitrary i are linearly independent over $\bar{k}$, so fix i. Now consider the quasi-periodic functions $R_1, \ldots, R_{d-1}$ produced when we apply the procedure of Lemma 5.1 to the quasi-periodic functions $Q_1, \ldots, Q_{d-1}$, but with congruences (5.3) now taken modulo $z^{q^{i+1}}$ instead of z^{q^i}.

By construction, for $j = 1, \ldots, d-1$ there is a twisted polynomial

$$P_j = \alpha_j F^0 + \text{ higher terms}$$

over $\bar{k}_\infty^{sep}$ (α_j might be 0) such that

$$\delta_j(T) - \delta_j^*(T) = P_j\phi(T) - TP_j.$$

So by properties (5.1) and (5.2), we know that

$$Q_j(z) - R_j(z) = P_j(e(z)) - \alpha_j z. \tag{6.1}$$

Thus, since the functions $Q_1(z), \ldots, Q_{d-1}(z), e(z), z$ are algebraically independent, the functions $R_1(z), \ldots, R_{d-1}(z), e(z), z$ are likewise independent.

Therefore the intermediate Theorem 4.2 can be applied to the $R_j(z)$ to see that the values

$$\begin{array}{ccccc} 1, & u, & u^{[1]}, & \dots & u^{[i]} \\ & R_1(u), & R_1^{[1]}(u), & \dots & R_1^{[i]}(u) \\ & \vdots & \vdots & & \vdots \\ & R_{d-1}(u), & R_{d-1}^{[1]}(u), & \dots & R_{d-1}^{[i]}(u) \end{array}$$

are $\bar{k}$-linearly independent.

Evaluating (6.1) at $z = u$ and applying $D_m,\ m \leq i$, shows that each

$$Q_j(u)^{[m]} = R_j(u)^{[m]} + (P_j(e(u)))^{[m]} - \sum_{r+s=m} \alpha_j^{[r]} u^{[s]}. \tag{6.2}$$

However, since $\phi(T)$ has coefficients from $\bar{k}^{sep}$, we conclude that $e(z)$ has coefficients from a finite separable extension of k. In addition, since each P_j has coefficients from $\bar{k}^{sep}$ by construction, one sees that the coefficients of $(P_j \circ e)^{[m]}(z)$ lie in a finite separable extension L of k. From the functional equation for $R_j(z)$, it now follows that the coefficients of $R_j(z)$ lie in L.

Thus, since $R_j(z) \equiv 0 \pmod{z^{q^{i+1}}}$,

$$R_j(z) = \sum_{h=i+1}^{\infty} \rho_{jh} z^{q^h} \in L[[z]].$$

Moreover it was shown in [5] that $u \in \bar{k}_\infty^{sep}$. Finally, the composite $Lk_\infty(u)$ is a finite separable extension of k_∞ and, in particular, is complete. Therefore

$$R_j(u) = \sum_{h=i+1}^{\infty} \rho_{jh} u^{q^h} \in Lk_\infty(u).$$

By Lemma 4.1 on the continuity of D_m in $Lk_\infty(u)$, we know that

$$\begin{aligned} D_m(R_j(u)) &= \sum_{h=i+1}^{\infty} D_m(\rho_{jh} u^{q^h}) \\ &= \sum_{r+s=m} \sum_{h=i+1}^{\infty} D_r(\rho_{jh}) D_s(u^{q^h}). \end{aligned}$$

However, from Lemma 2.1 we know that all $D_s(u^{q^h}) = 0$ when $0 < s \leq m$, since $m < i+1 \leq h$. Thus

$$D_m(R_j(u)) = \sum_{h=i+1}^{\infty} \rho_{jh}^{[m]}(u^{q^h}) = R_j^{[m]}(u).$$

Combining this with (6.2) shows that

$$Q_j(u)^{[m]} - R_j^{[m]}(u) \in \bar{k}^{sep} + \bar{k}^{sep} u + \cdots + \bar{k}^{sep} u^{[m]},$$

since by hypothesis we know that $e(u) \in \bar{k}^{sep}$ and we know that P_j has coefficients from $\bar{k}^{sep}$. Consequently, showing the $\bar{k}$-linear independence of the values appearing in Theorem 1.1 is equivalent to showing the linear independence of the values appearing in the already established Theorem 4.2. This proves Theorem 1.1.

7. Appendix: Algebraic dependence of additive functions

Although the following lemma (due essentially to E. Artin) has been implicitly invoked in many transcendence considerations in positive characteristic at least since Yu's earliest papers (cf. [9]), the referee thought it advisable to state and prove it here explicitly for ease of reference.

Lemma 7.1. *Let the $\mathbb{F}_q$-linear power series $f_1(z), \dots, f_r(z) \in \mathbf{C}[[z]]$, $f_i(z) = \sum a_{ih} z^{q^h}$, be algebraically dependent over $\mathbf{C}$. Then any algebraic relation they satisfy of minimal degree will be an $\mathbb{F}_q$-linear relation, i.e. of the form*

$$R_1(f_1(z)) + \cdots + R_r(f_r(z)) = 0,$$

$R_i(X_i) = \sum \rho_{ih} X_i^{q^h}$, $i = 1, \dots, r$.

Proof. Let R be a non-zero polynomial of minimal degree, say d, such that

$$R(f_1(z), \dots, f_r(z)) = 0. \tag{7.1}$$

The proof of Theorem 12.1 in Chapter VI, §12 of [8], due to Artin, shows that R is in fact an additive polynomial. Thus, as explained on p. 310 of [8], R has the form

$$R(X_1, \dots, X_r) = \sum_i R(0, \dots, 0, X_i, 0, \dots, 0) = \sum_i R_i(X_i),$$

where each $R_i(X_i) = \sum a_{ij} X_i^{p^j}$. Since $\mathbf{C}$ is perfect, at least one $a_{i0} \neq 0$, else the pth root of R applied to the $f_i(z)$ would give a relation over $\mathbf{C}$ of degree d/p.

Since the f_i are $\mathbb{F}_q$-linear, when we replace z by αz for any $\alpha \in \mathbb{F}_q$, we find from (7.1) that $R(\alpha f_1(z), \dots, \alpha f_r(z)) = 0$. Since then

$$R(\alpha X_1, \dots, \alpha X_r) - \alpha^d R(X_1, \dots, X_r)$$

is a polynomial of degree less than d vanishing at $f_1(z), \dots, f_r(z)$, we see by the minimality of d that

$$R(\alpha X_1, \dots, \alpha X_r) = \alpha^d R(X_1, \dots, X_r).$$

Thus each $a_{ij}(\alpha^d - \alpha^{p^j}) = 0$. The terms involving non-zero a_{i0} show that $\alpha^d = \alpha^{p^0} = \alpha$ for all $\alpha \in \mathbb{F}_q$. Hence $a_{ij}(\alpha - \alpha^{p^j}) = 0$, $\forall \alpha \in \mathbb{F}_q$ and so, when p^j is not a power of q, then $a_{ij} = 0$. This establishes the $\mathbb{F}_q$-linearity of the R_i. □

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