PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1881–1883
S 0002-9939(00)05758-0
Article electronically published on February 25, 2000

ON CONDENSATIONS OF C_p-SPACES ONTO COMPACTA

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(Communicated by Alan Dow)

ABSTRACT. A condensation is a one-to-one onto mapping. It is established that, for each σ-compact metrizable space X, the space $C_p(X)$ of real-valued continuous functions on X in the topology of pointwise convergence condenses onto a metrizable compactum. Note that not every Tychonoff space condenses onto a compactum.

The space $C_p(X)$ of all real-valued continuous functions on a Tychonoff space X in the topology of pointwise convergence is σ-compact only in the trivial case when X is finite (see Theorem 2 below). This makes natural the following question: *when can $C_p(X)$ be condensed onto a compact space or onto a σ-compact space?* *A condensation* is any one-to-one onto continuous mapping. *A compactum* is a compact Hausdorff space. We consider only Tychonoff spaces (observe that *every T_1-space can be condensed onto a compact T_1-space*). S. Banach was probably the first to ask when a (separable metrizable) space can be condensed onto a compact space (see [4]). E.G. Pytkeev solved one of Banach's problems, proving that every separable Banach space can be condensed onto a (metrizable) compactum [7]. The question above appears as Problem 35 in [2], accompanied by several versions of it. In particular, Problem 39 in [2] runs as follows: *Is it possible to condense $C_p(D^\omega)$ onto a compact space* (where D is the discrete two-point space, and D^ω is the Cantor set)? We answer this question below. The main result is:

1. Theorem. *For any σ-compact metrizable space X, the space $C_p(X)$ condenses onto a metrizable compactum.*

To prove Theorem 1, we need a result from [1]. If X is a space and Y is a subspace of X, then $C_p(Y, X)$ is the subspace of $C_p(Y)$ consisting of restrictions to Y of continuous real-valued functions on X. The next theorem was established in [1] (see Theorem 1.2.2):

2. Theorem. *If Y is dense in X, and $C_p(Y, X)$ is σ-countably compact, then X is pseudocompact, and Y is a P-space.*

Recall that *a P-space* is a space in which every G_δ-subset is open, and that a space is σ-countably compact, if it is the union of a countable family of countably compact subspaces.

Received by the editors May 24, 1997 and, in revised form, May 15, 1998.
2000 *Mathematics Subject Classification.* Primary 54A10, 54C35, 54C10.
Key words and phrases. Condensation, compactum, network, topology of pointwise convergence, σ-compact space, Borel space, P-space.

Proof of Theorem 1. It suffices to establish that $C_p(X)$ condenses onto some compactum. Indeed, any compactum with a countable network is metrizable [3], and any network for $C_p(X)$ is a network for any coarser topology. Since X has a countable base, $C_p(X)$ has a countable network [1].

If X is discrete, then $C_p(X)$ is just R^X. Clearly, the space R condeses onto a compactum, since, obviously, every locally compact space condenses onto a compactum [4]. Therefore, R^X condenses onto a compactum.

It remains to consider the case when X is not discrete. We can fix a countable subspace Y dense in X such that $C_p(Y, X)$ is not σ-compact. Indeed, assume the contrary, and take any countable Y dense in X. Then, by Theorem 2, Y is a P-space. Since Y is countable, it follows that Y is discrete. Therefore, X is discrete, since X is separable and every countable dense subspace of X is discrete, a contradiction.

On the other hand, from Theorem 6.2 in [6] it follows that $C_p(Y, X)$ is a Borel subset of the separable complete metric space R^Y (it is here that we use σ-compactness of X). Now, E.G. Pytkeev established in [7] that every non-σ-compact separable metrizable Borel space condenses onto a compactum. (Notice that the space Q of rational numbers does not condense onto any compactum, since each non-empty countable compactum has an isolated point.)

Therefore, $C_p(Y, X)$ condenses onto a compactum K. Finally, since the natural restriction mapping of functions on X to Y is a condensation of $C_p(X)$ onto $C_p(Y, X)$, we conclude that $C_p(X)$ condenses onto the compactum K. $\square$

3. *Remarks.* A result similar to Theorem 1 holds for the space $C_p^b(X)$ of bounded continuous real-valued functions on X in the topology of pointwise convergence: this space also condenses onto a compactum whenever X is a σ-compact metrizable space. A minor change is needed in the proof: we should refer to Proposition 9.2 in [3]. Theorem 1 also implies that, under the same restrictions on X as in Theorem 1, $C(X)$ condenses onto a compactum when $C(X)$ is endowed with any stronger topology than the topology of pointwise convergence (for example, with the compact-open topology).

4. Problem. Is it true that $C_p(X)$ condenses onto a compactum (onto a σ-compact space) for every separable metrizable space X?

I conjecture that it might be impossible to condense $C_p(J)$ onto any compactum, where J is the space of irrational numbers.

5. Problem. Is it true that $C_p(X)$ condenses onto a σ-compact space for every compact space X?

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