PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 6, Pages 1853–1857
S 0002-9939(99)05250-8

AN INDUCTIVE EXPLICIT CONSTRUCTION OF ∗-PRODUCTS ON SOME POISSON MANIFOLDS

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(Communicated by Peter Li)

Abstract. We extend the Cahen Gutt coboundary construction on cotangent bundles of n-dimensional parallelisable manifolds to manifolds which admit n global vector fields defining a parallelisation on a dense open set. This result is used to give an inductive explicit construction of ∗-products on certain Poisson manifolds.

Introduction

The theory of formal deformation quantization of Poisson manifolds was introduced by Bayen et al. in [2]. The main notion of this theory is the concept of a ∗-product. The general question of the existence of such a product for symplectic manifolds has been completely solved by several authors, using various techniques [5],[14],[11]. Recently, M. Kontsevitch has proved the existence of ∗-products on arbitrary finite-dimensional Poisson manifolds [9]. Nevertheless, since Kontsevitch's result is not given by a simple geometrical construction, it has increased the interest of having a simple geometrical of ∗-products on non-regular Poisson manifolds.

Since every Poisson manifold splits into a collection of symplectic submanifolds, known as the *leaves of the symplectic foliation*, one naturally asks whether a ∗-product on a Poisson manifold restricts to give a ∗-product on the symplectic leaves. Lately, in [4],[13] it has been proved that such ∗-products do not always exist. When they exist we called them *tangential.* In particular, the dual of the so-called "book algebra" with the *Lie Poisson structure* admits a tangential ∗-product [1]. Furthermore, this example provides the basic idea to construct explicit ∗-products on some other Poisson manifolds (see Theorem 1).

The goal of this paper will be to give such a construction. In order to do this, we first generalize a coboundary construction due to Cahen and Gutt [3], and use this result to construct explicitly the ∗-product.

Using a different method a similar result has been obtained in [7]. However, the approach used there does not provide an explicit construction of a ∗-product.

Received by the editors July 21, 1998.
1991 *Mathematics Subject Classification.* Primary 58F06, 53Z05; Secondary 81Q99.
Key words and phrases. ∗-product, deformation quantization.
The author was supported by a grant from the University of Warwick.

$*$-PRODUCTS

A *Poisson structure* on a manifold M is a Lie algebra structure $\{\cdot,\cdot\}$ on $C^\infty(M)$ which satisfies the derivation property

$$\{fg,h\}=f\{g,h\}+\{f,h\}g,\quad \forall f,g,h\in C^\infty(M).$$

The operation $\{\cdot,\cdot\}$ determines a contravariant skew-symmetric 2-tensor Λ such that $\{f,g\}=\Lambda(df,dg)$. A Poisson structure may also be defined by such a tensor (the Poisson tensor); the Jacobi identity for the Poisson structure is equivalent to the vanishing of the so-called *Schouten bracket* $[\Lambda,\Lambda]_s=0$ (see [2]).

If (M,Λ) is a Poisson manifold we set $N=C^\infty(M)$. Let $N[[\lambda]]$ be the space of formal power series in a parameter λ, with coefficients in N.

Definition 1 ([2])**.** A $*$-product on (M,Λ) is a bilinear map $N^2\longrightarrow N[[\lambda]]$ defined by

$$(f,g)\longrightarrow f*g=\sum_{n=0}^{\infty}\lambda^n C_n(f,g), \tag{1}$$

where the so-called *cochains* C_n, are bilinear maps with values in N and satisfy the following axioms:

1. $C_0(f,g)=fg,\quad C_1(f,g)=\{f,g\},\quad \forall f,g\in C^\infty(M)$;
2. $C_n(f,g)=(-1)^nC_n(g,f),\quad \forall\ f,g\in C^\infty(M),\quad \forall n\geq 1$;
3. $C_n(f,k)=0,\quad \forall\ f\in C^\infty(M),\quad \forall\ k\in\mathbb{R},\quad \forall\ n\geq 1$;
4. $\sum_{r+s=k}C_r(C_s(f,g),h)=\sum_{r+s=k}C_r(f,C_s(g,h)),\ k\geq 0$.

The theory of *deformations* in the sense of [6] relates the deformations of an associative algebra to the corresponding *Hochschild cohomology.*

Definition 2. A (Hochschild) p-cochain is a p-linear map $N^p\to N$. The *Hochschild coboundary* of a p-cochain is the $(p+1)$-cochain ∂C given by

$$\partial C(u_0,\dots,u_p)=u_0C(u_1,\dots,u_p)-C(u_0u_1,u_2,\dots,u_p)+C(u_0,u_1u_2,\dots,u_p)$$
$$+\cdots+(-1)^pC(u_0,u_1,\dots,u_{p-1}u_p)+(-1)^{p+1}C(u_0,\dots,u_{p-1})u_p.$$

A cochain C is called differential if it is defined by multi-differential operators in each argument. A $*$-product is called differential if all its cochains are differential. In [12] it has been proved that if E is a *p-cocycle* (differential and null on the constants), then there exist a skew-symmetric contravariant smooth p-tensor A and a $(p-1)$-cochain C such that

$$E(f_1,...,f_p)=\partial C(f_1,...,f_p)+A(df_1,...,df_p),\quad f_i\in C^\infty(M). \tag{2}$$

A bilinear map (1) is said to be an *associative formal deformation* up to the order k if

$$(f*g)*h-f*(g*h)=0,\quad f,g,h\in C^\infty(M), \tag{3}$$

is satisfied modulo λ^{k+1}. By developing (3) into powers of λ, the coefficients of λ^t will vanish if

$$E_t(f,g,h):=\sum_{r+s=t,\ r,s\geq 1}C_r(C_s(f,g),h)-C_r(f,C_s(g,h))=\partial C_t(f,g,h). \tag{4}$$

Thus, an associative formal deformation up to the order k can be extended to one of order $k+1$ provided that the cocycle E_{k+1} is a 3-coboundary.

Tangential $*$-products

Let (M, Λ) be a Poisson manifold, and let O be a symplectic leaf.

Definition 3. Let $x \in O$. A *differential operator* D on M is *tangential* to O at x, if there exist a neighbourhood V of x in O and a neighbourhood U of V in M, such that when $\varphi_1, \varphi_2 \in C^\infty(U)$ with ${\varphi_1}_{|_V} = {\varphi_2}_{|_V}$, then

$$D(\varphi_1)_{|_V} = D(\varphi_2)_{|_V}.$$

A *bi-differential* operator C on M is said to be *tangential* to O, if for any function $f \in C^\infty(M)$, the differential operators $C(f, \cdot)$ and $C(\cdot, f)$ are tangential to O, at x for all $x \in O$.

Definition 4. A differential $*$-product is called tangential to O, if all its cochains C_n, $n \geq 1$, are tangential.

A coboundary construction

In what follows, we shall use the summation convention on pairs of upper and lower indices. Let (M, Λ) be a Poisson manifold of dimension n, and let $T^1, ..., T^n$ be smooth vector fields on M such that they are pointwise linearly independent on a dense open set of M. The following proposition is a simple generalization of Proposition 2 in [3]. The argument given in [3] is combinatorial, and is based on 3 lemmas which in fact only require independence of the vector fields T^i on a dense open set.

Proposition 1 ([3]). *Let E be a differential 3-cocycle (null on the constants), of the form*

$$E(f, g, h) = \sum_{0<a,b,c \leq K} E_{i_1 \ldots i_a, j_1 \ldots j_b, k_1 \ldots k_c} T^{i_1} \cdots T^{i_a} f T^{j_1} \cdots T^{j_b} g T^{k_1} \cdots T^{k_c} h,$$

where $f, g, h \in C^\infty(M)$, and $E_{i_1 \ldots i_a, j_1 \ldots j_b, k_1 \ldots k_c}$ are smooth functions on M symmetric in the i's, in the j's and in the k's. Then, there is a 2-cochain C completely determined by E of the form

$$C(f, g) = \sum_{0<p,q \leq K} C_{i_1 \ldots i_p, j_1 \ldots j_q} T^{i_1} \cdots T^{i_p} f T^{j_1} \cdots T^{j_q} g, \tag{5}$$

such that $E = \partial C + A$, where A is the completely antisymmetric part of E, i.e., a 3-contravariant smooth tensor. Moreover, the coefficients $C_{i_1 \ldots i_p, j_1 \ldots j_q}$ are constant (rational) linear combinations of the coefficients $E_{k_1 \ldots k_a, l_1 \ldots l_b, m_1 \ldots m_c}$ of E.

Remark 1. Note that if the C_r's ($r \leq k$) satisfy the symmetry properties of Definition 1, then the cocycle E_{k+1} satisfies $E_{k+1}(f, g, h) = (-1)^k E_{k+1}(h, g, f)$. Thus, if $E_{k+1} = \partial C_{k+1}$ we can always assume that $C_{k+1}(f, g) = (-1)^{k+1} C_{k+1}(g, f)$ just by replacing C_{k+1} by its symmetrization or antisymmetrization.

Theorem 1. *Let (M, Λ) be a Poisson manifold, and let us assume that there exist T^1, T^2 smooth vector fields on M such that they are pointwise linearly independent on a dense open set of M, and such that Λ can be written as $\Lambda = T^1 \wedge T^2$. Then, there is a $*$-product on (M, Λ) with 2-cochains C_r of the form (5).*

Proof. By assumption the Poisson structure C_1 on M admits the expression $C_1(f,g) = T^1 f T^2 g - T^2 f T^1 g$. Let us assume that there exist k ($k \geq 1$) 2-cochains $C_1, \dots, C_k$ constructed recursively (using Proposition 1) from the equations

$$E_t = \partial C_t \quad t = 1, \dots, k,$$

defining a deformation (up to the order k) on M so that

$$C_t(f,g) = \sum_{0<p,q\leq K_t} C_{i_1 \dots i_p, j_1 \dots j_q} T^{i_1} \cdots T^{i_p} f T^{j_1} \cdots T^{j_q} g,$$

where all the T^i's and T^j's are T^1 or T^2.

Since $[T^1 \wedge T^2, T^1 \wedge T^2]_s = 0$, it follows that $[T^1, T^2]_s = f_1 T^1 + f_2 T^2$ ($f_1, f_2 \in C^\infty(M)$), and so that E_{k+1} expressed as in Proposition 1 only includes T^1's and T^2's. Let C_{k+1} and A_{k+1} be the 2-cochains constructed by means of Proposition 1 such that $E_{k+1} = \partial C_{k+1} + A_{k+1}$. Then, since A_{k+1} is a 3-contravariant (skew-symmetric) tensor only including T^1 and T^2, it follows that A_{k+1} vanishes, i.e., $E_{k+1} = \partial C_{k+1}$. Thus, the theorem follows by induction on k. ☐

Remark 2. The $*$-product constructed is tangential to the 2-dimensional symplectic leaves.

Examples

Example 1. Let $\mathfrak{g}$ be the Lie algebra (*book algebra*) with basis (e_1, e_2, e_3), such that $[e_1, e_2] = 0$, $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$. Let (x_1, x_2, x_3) be a coordinate system on $\mathfrak{g}^*$ determined by the dual basis (e^1, e^2, e^3). The Lie-Poisson structure Λ can be expressed in terms of the above global coordinate system as

$$\Lambda = \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \wedge \frac{\partial}{\partial x_3}.$$

Therefore, using Theorem 1 we get an explicit $*$-product (in fact, the *Gutt $*$-product* [8]).

Example 2. Let us consider the Lie group $SU(2)$, and let us choose

$$e_2 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_4 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

as a basis of its Lie algebra $\mathfrak{su}(2)$.

Let $Sp(1)$ be the group of unitary quaternions. We identify $Sp(1)$ and $SU(2)$ as Lie groups by means of

$$\psi : Sp(1) \longrightarrow SU(2),$$

$$(x_1, x_2, x_3, x_4) \longrightarrow \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}.$$

As usual, we denote by R_g (L_g) the right translation (left translation) map. Let $\mathbf{r} = e_3 \wedge e_4 \in \bigwedge^2 \mathfrak{su}(2)$; then the *Iwasawa-Poisson-Lie structure* π on $SU(2)$ is defined by [10]

$$\pi(g) := dR_g \mathbf{r} - dL_g \mathbf{r}, \quad g \in SU(2).$$

The *linearization* (at the identity) of this Poisson structure is *isomorphic* to the book algebra, and therefore this Poisson structure can be considered as the non-linear version of that in Example 1. Let X_i ($i = 2, 3, 4$) be the right invariant vector fields on $SU(2)$ corresponding to e_i. We define two vector fields on $SU(2)$ by setting

$$T^1 := x_2X_2 + x_3X_3 + x_4X_4,$$

$$T^2 := (-2x_1)X_2 + (-2x_4)X_3 + (2x_3)X_4.$$

A straightforward computation shows that the Poisson structure π on $SU(2)$ can be written as $\pi = T^1 \wedge T^2$. Hence, using Theorem 1 one can construct an explicit $*$-product on this Poisson manifold (the non-linear version of the one constructed in the previous example).

Acknowledgments

The author would like to deeply thank J. Rawnsley for his invaluable help throughout this work, S. Gutt for helpful comments on the manuscript, and V. Chloup-Arnould and J. L. Cisneros for important remarks and their help in the organization of the paper.

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