

## THE FURUTA INEQUALITY IN BANACH \*-ALGEBRAS

KÔTARÔ TANAHASHI AND ATSUSHI UCHIYAMA

(Communicated by David R. Larson)

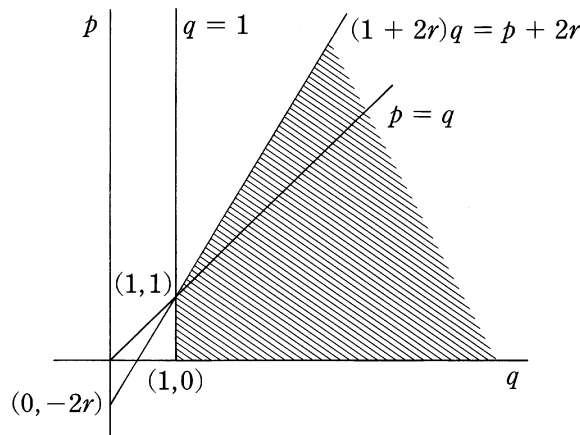
ABSTRACT. Let  $0 < p, q, r \in \mathbb{R}$  be real numbers with  $p + 2r \leq (1 + 2r)q$  and  $1 \leq q$ . Furuta (1987) proved that if bounded linear operators  $A, B \in B(H)$  on a Hilbert space  $H$  satisfy  $0 \leq B \leq A$ , then  $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ . This inequality is called the Furuta inequality and has many applications. In this paper, we prove that the Furuta inequality holds in a unital hermitian Banach \*-algebra with continuous involution.

Let  $A, B$  be bounded linear operators on a Hilbert space  $H$ . The celebrated Löwner-Heinz inequality states the following;

**Theorem A** (Löwner-Heinz inequality [4], [5]). *Let  $A, B \in B(H)$  satisfy  $0 \leq B \leq A$ . If  $0 < p < 1$ , then  $B^p \leq A^p$ .*

For an extension of Theorem A, Furuta obtained the following interesting inequality in [1] and one page elementary proof in [2].

**Theorem B** (Furuta inequality [1], [2]). *Let  $0 \leq p, q, r \in \mathbb{R}$  and  $A, B \in B(H)$  satisfy  $0 \leq B \leq A$ . If  $p + 2r \leq (1 + 2r)q$  and  $1 \leq q$  as shown in the figure, then  $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ .*



Received by the editors February 12, 1998 and, in revised form, July 13, 1998.

1991 *Mathematics Subject Classification*. Primary 47A05, 47B15.

*Key words and phrases*. The Löwner-Heinz inequality, the Furuta inequality.

This research is partially supported by Grant-in-Aid Scientific Research (K. Tanahashi, No. 10640185).

One of the authors, Tanahashi [8], proved that the domain drawn for  $p, q$  and  $r$  in Figure is the best possible one for Theorem B. Many applications of Theorem B have been developed, but we omit citing them.

Let  $A$  be a unital Banach  $*$ -algebra with unit  $e$  and  $a, b \in A$ .  $A$  is called hermitian if  $a = a^*$  implies  $\sigma(a) \subset \mathbb{R}$  where  $\sigma(a)$  denotes the spectrum of  $a$ .  $a \geq 0$  means that  $a = a^*$  and  $\sigma(a) \subset [0, \infty)$ .  $a > 0$  means  $a \geq 0$  and  $0 \notin \sigma(a)$ .  $a \geq b$  means  $a - b \geq 0$ , and  $a > b$  means  $a - b > 0$ . Let  $\alpha \in \mathbb{R}$ . For  $a > 0$ ,  $a^\alpha$  means  $\exp(\alpha \log a)$  where  $\log$  is the principal branch of the complex logarithm. By the Shirali-Ford theorem [7],  $\sigma(a^*a) \subset [0, \infty)$  for all  $a \in A$ .

Recently, Okayasu [6] proved that the Löwner-Heinz inequality holds in a unital hermitian Banach  $*$ -algebra with continuous involution.

**Theorem C** ([6]). *Let  $A$  be a unital hermitian Banach  $*$ -algebra with continuous involution. Let  $a, b \in A$  and  $p \in (0, 1]$ . Then  $a^p < b^p$  if  $0 < a < b$ , and  $a^p \leq b^p$  if  $0 < a \leq b$ .*

In this paper, we prove that the Furuta inequality holds in this case.

**Theorem 1.** *Let  $A$  be a unital hermitian Banach  $*$ -algebra with continuous involution and  $a, b \in A$ . Let  $0 \leq p, q, r \in \mathbb{R}$  satisfy  $p + 2r \leq (1 + 2r)q$  and  $1 \leq q$ . Then*

$$b^{\frac{p+2r}{q}} < (b^r a^p b^r)^{\frac{1}{q}} \quad \text{if } 0 < b < a,$$

and

$$b^{\frac{p+2r}{q}} \leq (b^r a^p b^r)^{\frac{1}{q}} \quad \text{if } 0 < b \leq a.$$

To prove Theorem 1, we need following lemmas.

**Lemma 2.** *If  $0 < a$ , then  $0 < a^{-1}$ .*

*Proof.* Let  $0 < a$ . Then  $\sigma(a) \subset (0, \infty)$ , and  $\sigma(a^{-1}) = \sigma(a)^{-1} \subset (0, \infty)$ . This implies  $0 < a^{-1}$ .  $\square$

**Lemma 3.** *If  $0 < a < e$ , then  $e < a^{-1}$ . Also, if  $0 < a \leq e$ , then  $e \leq a^{-1}$ .*

*Proof.* Let  $0 < a < e$ . Then

$$a^{-1} - e = a^{-\frac{1}{2}}(e - a)a^{-\frac{1}{2}} = \{(e - a)^{\frac{1}{2}}a^{-\frac{1}{2}}\}^* \{(e - a)^{\frac{1}{2}}a^{-\frac{1}{2}}\}.$$

This implies  $0 < a^{-1} - e$  and  $e < a^{-1}$ . The rest of the proof is similar.  $\square$

**Lemma 4.** *If  $0 < b < a$ , then  $0 < a^{-1} < b^{-1}$ . Also, if  $0 < b \leq a$ , then  $0 < a^{-1} \leq b^{-1}$ .*

*Proof.* Let  $0 < b < a$ . Then

$$a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = \left(b^{\frac{1}{2}}a^{-\frac{1}{2}}\right)^* \left(b^{\frac{1}{2}}a^{-\frac{1}{2}}\right)$$

and

$$e - a^{-\frac{1}{2}}ba^{-\frac{1}{2}} = a^{\frac{1}{2}}(a - b)a^{-\frac{1}{2}} = \left\{(a - b)^{\frac{1}{2}}a^{-\frac{1}{2}}\right\}^* \left\{(a - b)^{\frac{1}{2}}a^{-\frac{1}{2}}\right\}.$$

This implies  $0 < a^{-\frac{1}{2}}ba^{-\frac{1}{2}} < e$ . Hence, by Lemma 3,

$$e < \left(a^{-\frac{1}{2}}ba^{-\frac{1}{2}}\right)^{-1} = a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}.$$

Then

$$\begin{aligned} b^{-1} - a^{-1} &= a^{-\frac{1}{2}} \left( a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \\ &= \left\{ \left( a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \right\}^* \left\{ \left( a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}} - e \right) a^{-\frac{1}{2}} \right\}. \end{aligned}$$

This implies  $0 < b^{-1} - a^{-1}$ . Hence  $0 < a^{-1} < b^{-1}$  by Lemma 2.

Let  $0 < b \leq a$ . Then  $0 < a$  by Lemma 2 of [6]. The rest of the proof is similar.  $\square$

**Lemma 5.** *If  $0 < b < a$  and  $0 < c$ , then  $cbc < cac$ . Also, if  $0 < b \leq a$  and  $0 < c$ , then  $cbc \leq cac$ .*

*Proof.* Let  $0 < b < a$  and  $0 < c$ . Then

$$cac - cbc = c(a - b)c = \left\{ (a - b)^{\frac{1}{2}} c \right\}^* \left\{ (a - b)^{\frac{1}{2}} c \right\}.$$

This implies  $0 < cac - cbc$  and  $cbc < cac$ . The rest of the proof is similar.  $\square$

**Lemma 6.** *If  $0 < a$ , then  $(a^2)^{\frac{1}{2}} = a$ .*

*Proof.* Since  $\sigma(a^2) = \{\sigma(a)\}^2$ , there exists a positively oriented circle  $\Lambda$  containing  $\sigma(a^2)$  with  $0 < \Re \lambda$  for all  $\lambda \in \Lambda$ . Let

$$\lambda = |\lambda|e^{i\theta} \in \Lambda \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

and

$$\Gamma = \{ \mu = |\lambda|^{\frac{1}{2}} e^{\frac{1}{2}i\theta} \mid \lambda = |\lambda|e^{i\theta} \in \Lambda \}.$$

Then

$$\begin{aligned} \log a^2 &= \int_{\Lambda} (\log \lambda) (\lambda - a^2)^{-1} d\lambda \\ &= \int_{\Gamma} (\log \mu^2) (\mu^2 - a^2)^{-1} 2\mu d\mu \\ &= \int_{\Gamma} 2(\log \mu) (\mu + a)^{-1} d\mu + \int_{\Gamma} 2(\log \mu) (\mu - a)^{-1} d\mu \\ &= 0 + 2 \log a = 2 \log a. \end{aligned}$$

Thus

$$(a^2)^{\frac{1}{2}} = e^{\frac{1}{2} \log a^2} = e^{\log a} = a.$$

$\square$

**Lemma 7.** *If  $0 < a, b$  and  $\lambda$  is a real number, then*

$$(bab)^\lambda = ba^{\frac{1}{2}} \left( a^{\frac{1}{2}} b^2 a^{\frac{1}{2}} \right)^{\lambda-1} a^{\frac{1}{2}} b.$$

*Proof.* Let

$$F(\lambda) = (bab)^\lambda \quad \text{and} \quad G(\lambda) = ba^{\frac{1}{2}} \left( a^{\frac{1}{2}} b^2 a^{\frac{1}{2}} \right)^{\lambda-1} a^{\frac{1}{2}} b.$$

Since  $bab = \left(a^{\frac{1}{2}}b\right)^* \left(a^{\frac{1}{2}}b\right)$ , we have  $0 < F(\lambda), G(\lambda)$ . By the definitions, we have  $F(1) = G(1)$ . Also,

$$\begin{aligned} \{G(2^{-1})\}^2 &= \left\{ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b\right\} \left\{ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b\right\} \\ &= ba^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b^2a^{\frac{1}{2}} \left(a^{\frac{1}{2}}b^2a^{\frac{1}{2}}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}b \\ &= bab = \{F(2^{-1})\}^2. \end{aligned}$$

Hence  $F(2^{-1}) = G(2^{-1})$  by Lemma 6. Similarly, we have

$$F(2^{-n}) = G(2^{-n}) \quad \text{for } n = 2, 3, \dots$$

Since  $F(\lambda), G(\lambda)$  are analytic on the real line  $\mathbb{R}$  and  $2^{-n} \rightarrow 0 (n \rightarrow \infty)$ , we have

$$F(\lambda) = G(\lambda) \quad \text{for } \lambda \in \mathbb{R}.$$

□

*Remark.* We remark that Lemma 7 has been obtained in [3] for operator case on Hilbert space, but our method in Banach  $*$ -algebra case is completely different from one in [3].

*Proof of Theorem 1.* Let  $0 < b < a$ . We show first the case  $0 < p \leq 1$ . In this case,  $b^p < a^p$  by Theorem C, and  $b^{p+2r} < b^r a^p b^r$  by Lemma 5. Hence  $b^{\frac{p+2r}{q}} < (b^r a^p b^r)^{\frac{1}{q}}$  by Theorem C. We consider the case  $1 < p$ . We may assume  $q = \frac{p+2r}{1+2r}$  by Theorem C. Let  $0 < r \leq \frac{1}{2}$ . Then  $0 < b^{2r} < a^{2r}$  by Theorem C. Hence

$$\begin{aligned} b^{-r} (b^r a^p b^r)^{\frac{1}{q}} b^{-r} &= b^{-r} b^r a^{\frac{p}{2}} \left(a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}}\right)^{\frac{1}{q}-1} a^{\frac{p}{2}} b^r b^{-r} \\ &= a^{\frac{p}{2}} \left(a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}}\right)^{\frac{1-p}{p+2r}} a^{\frac{p}{2}} \end{aligned}$$

by Lemma 7. Since  $a^{\frac{p}{2}} b^{2r} a^{\frac{p}{2}} < a^{p+2r}$  by Lemma 5 and  $-1 < \frac{1-p}{p+2r} < 0$ , we have

$$b^{-r} (b^r a^p b^r)^{\frac{1}{q}} b^{-r} > a^{\frac{p}{2}} (a^{p+2r})^{\frac{1-p}{p+2r}} a^{\frac{p}{2}} = a > b$$

by Lemma 4 and Theorem C. Hence

$$(1) \quad (b^r a^p b^r)^{\frac{1}{q}} > b^{1+2r} \quad \text{for } q = \frac{p+2r}{1+2r}$$

by Lemma 5. The rest of the proof is the same as in [1], but we cite its proof for the sake of importance as follows.

Put  $a_1 = (b^r a^p b^r)^{\frac{1}{q}}$  and  $b_1 = b^{1+2r}$  in (1). Repeating (1) again for  $a_1 > b_1 > 0, 0 < r_1 \leq \frac{1}{2}$  and  $1 < p_1$ ,

$$(b_1^{r_1} a_1^{p_1} b_1^{r_1})^{\frac{1}{q_1}} > b_1^{1+2r_1} \quad \text{for } q_1 = \frac{p_1 + 2r_1}{1 + 2r_1}.$$

Put  $p_1 = q \geq 1$  and  $r_1 = \frac{1}{2}$ ; then

$$(b^{2r+\frac{1}{2}} a^p b^{2r+\frac{1}{2}})^{\frac{1}{q_1}} > b^{2(1+2r)}.$$

Put  $s = 2r + \frac{1}{2}$ . Then  $q_1 = \frac{p_1 + 2r_1}{1 + 2r_1} = \frac{p + 2s}{1 + 2s}$  since  $p_1 = q$  and  $2(1 + 2r) = 1 + 2s$ .

Hence

$$(2) \quad (b^s a^p b^s)^{\frac{1}{q_1}} > b^{1+2s} \quad \text{for } q_1 = \frac{p + 2s}{1 + 2s}.$$

Since  $r \in (0, \frac{1}{2}]$  in (1),  $s = 2r + \frac{1}{2} \in (\frac{1}{2}, \frac{3}{2}]$  in (2). Consequently (1) holds for  $r \in (0, \frac{3}{2}]$  and  $q = \frac{p+2r}{1+2r}$ . Repeating this method, (1) holds for each  $r > 0$  and  $q = \frac{p+2r}{1+2r}$ .

If  $0 < b \leq a$ , then  $a$  is invertible by Lemma 2 of [6] and  $0 < a^{-1} \leq b^{-1}$ . The rest of the proof is similar to the case  $0 < b < a$ .  $\square$

#### ACKNOWLEDGEMENTS

The authors would like to express their thanks to Professor T. Okayasu and the referee for heart-warming suggestions.

#### REFERENCES

- [1] T. Furuta,  $A \geq B \geq O$  assures  $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$  for  $r \geq 0$ ,  $p \geq 0$ ,  $q \geq 1$  with  $(1 + 2r)q \geq (p + 2r)$ , Proc. Amer. Math. Soc., **101** (1987), 85–88. MR **89b**:47028
- [2] T. Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126. MR **90g**:47029
- [3] T. Furuta, *Two operator functions with monotone property*, Proc. Japan Acad., **111** (1991), 511–516. MR **91f**:47023
- [4] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann., **123** (1951), 415–438. MR **13**:471f
- [5] K. Löwner, Über monotone Matrixfunktionen, Math. Z., **38** (1934), 177–216.
- [6] T. Okayasu, *Heinz's inequality in Banach \*-algebras*, ( preprint ).
- [7] S. Shirali and J. W. M. Ford, *Symmetry in complex involutory Banach algebras II*, Duke Math. J., **37** (1970), 275–280. MR **41**:5977
- [8] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146. MR **96d**:47025

DEPARTMENT OF MATHEMATICS, TOHOKU COLLEGE OF PHARMACY, KOMATSUSHIMA, AOBA-KU, SENDAI 981-8558, JAPAN

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, AOBA-KU, SENDAI 980-8578, JAPAN