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NONVANISHING OF SYMMETRIC SQUARE L-FUNCTIONS OF CUSP FORMS INSIDE THE CRITICAL STRIP

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Abstract. We shall give a certain nonvanishing result for the symmetric square L-function of an elliptic cuspidal Hecke eigenform w.r.t. the full modular group inside the critical strip.

1. Introduction

Let f be a normalized cuspidal Hecke eigenform of integral weight k on the full modular group $SL_2(\mathbf{Z})$ and denote by $D_f^*(s)\,(s \in \mathbf{C})$ the symmetric square L-function of f completed with its archimedean Γ-factors. As is well-known [7], [8], $D_f^*(s)$ has a holomorphic continuation to $\mathbf{C}$ and is invariant under $s \mapsto 2k-1-s$. Note that, by [3], $D_f^*(s)$ (up to a variable shift) also is the standard zeta function of a cuspidal automorphic representation of $GL(3)$, and so by [4] zeros of $D_f^*(s)$ can occur only inside the critical strip $k-1 < \mathrm{Re}(s) < k$. According to the generalized Riemann hypothesis, the zeros of $D_f^*(s)$ should all lie on the critical line $\mathrm{Re}(s) = k - \frac{1}{2}$.

The last statement of course is far from being settled. On the other hand, it turns out to be comparatively easy to prove nonvanishing results for $D_f^*(s)$ on the average. For example, in [6] Xian-Jin Li used an approximate functional equation for an average sum of the $D_f^*(s)$ to show that for any given s with $k-1 < \mathrm{Re}(s) < k$, $s \neq k - \frac{1}{2}$, $\zeta(s-k+1) \neq 0$, there are infinitely many different f such that $D_f^*(s)$ is not zero.

In the present note, using a different approach we will prove that given any s with $k-1 < \mathrm{Re}(s) < k$, $\mathrm{Re}(s) \neq k - \frac{1}{2}$, then for all k large enough there exists a Hecke eigenform f of weight k such that $D_f^*(s) \neq 0$. For the proof we use a "kernel function" for $D_f^*(s)$ as given by Zagier in [8] and then proceed in a similar way as in [5], where a corresponding result for Hecke L-functions was proved.

2. Notation

For $s \in \mathbf{C}$ we usually write $s = \sigma + it$ with $\sigma, t \in \mathbf{R}$.

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3. Statement of result

Let k be an even integer ≥ 12 and let S_k be the space of cusp forms of weight k w.r.t. the full modular group $\Gamma_1 = SL_2(\mathbf{Z})$, equipped with the usual Petersson scalar product $\langle\ ,\ \rangle$. For $f(z) = \sum_{n\geq 1} a(n)e^{2\pi inz}$ ($z \in \mathcal{H} =$ upper half plane) a normalized Hecke eigenform in S_k (recall that normalized means $a(1) = 1$), we denote by

$$D_f(s) = \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1} \quad (\sigma > k)$$

the symmetric square L-function of f, where the product is taken over all rational primes p and α_p, β_p are defined by

$$\alpha_p + \beta_p = a(p),\ \alpha_p\beta_p = p^{k-1}.$$

By [7], [8], $D_f(s)$ has a holomorphic continuation to $\mathbf{C}$, and the function

$$D_f^*(s) = 2^{-s}\pi^{-3s/2}\Gamma(s)\Gamma(\frac{s-k+2}{2})D_f(s)$$

satisfies the functional equation

$$D_f^*(2k-1-s) = D_f^*(s). \tag{1}$$

Let $\{f_{k,1}, \ldots, f_{k,g_k}\}$ ($g_k = \dim S_k$) be the basis of normalized Hecke eigenforms of S_k.

Theorem. *Let $t_0 \in \mathbf{R}$ and $0 < \epsilon < \frac{1}{2}$. Then there exists a positive constant $C(t_0, \epsilon)$ depending only on t_0 and ϵ such that for $k > C(t_0, \epsilon)$ the function*

$$\sum_{\nu=1}^{g_k} \frac{1}{\langle f_{k,\nu}, f_{k,\nu}\rangle} D^*_{f_{k,\nu}}(s)$$

does not vanish at any point $s = \sigma + it_0$, $k-1 < \sigma < k - \frac{1}{2} - \epsilon$, $k - \frac{1}{2} + \epsilon < \sigma < k$.

Corollary. *Let $s \in \mathbf{C}$ be fixed with $k-1 < \sigma < k$, $\sigma \neq k - \frac{1}{2}$. Then for all k large enough there exists a normalized Hecke eigenform f in S_k such that $D_f^*(s) \neq 0$.*

4. Proof

The proof proceeds along similar lines as in [5]. We consider the cusp forms dual w.r.t. the Petersson scalar product to the values $D_f^*(s)$ ($2-k < \sigma < k-1$) where f is any normalized Hecke eigenform in S_k. These have been constructed by Zagier [8] and will be denoted Φ_s as in [8] in what follows. To state the relevant properties of Φ_s, we need to introduce several notations.

Let Δ be a discriminant, i.e. $\Delta \in \mathbf{Z}$ and $\Delta \equiv 0, 1 \pmod 4$. Put

$$L(s,\Delta) = \begin{cases} \zeta(2s-1), & \text{if } \Delta = 0, \\ L_D(s)\sum_{d|f, d>0}\mu(d)(\frac{D}{d})d^{-s}\sigma_{1-2s}(\frac{f}{d}), & \text{if } \Delta \neq 0, \end{cases}$$

where if $\Delta \neq 0$ we have written $\Delta = Df^2$ with $f \in \mathbf{N}$ and D the discriminant of $\mathbf{Q}(\sqrt{\Delta})$, $(\frac{D}{\cdot})$ is the Kronecker symbol, $L_D(s)$ the associated L-function defined by analytic continuation of the series $\sum_{n\geq 1}(\frac{D}{n})n^{-s}$ ($\sigma > 1$), μ is the Möbius function and $\sigma_\nu(m) = \sum_{d|m, d>0} d^\nu$ ($m \in \mathbf{N}$, $\nu \in \mathbf{C}$).

Furthermore, for t an integer with $\Delta < t^2$ and $s \in \mathbf{C}$ with $\frac{1}{2} < \sigma < k$ we define

$$I_k(\Delta, t; s) = \int_0^\infty \int_{-\infty}^\infty \frac{y^{k+s-2}}{(x^2 + y^2 + ity - \frac{1}{4}\Delta)^k} dxdy$$

$$= \frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)} \int_0^\infty \frac{y^{k+s-2}}{(y^2 + ity - \frac{1}{4}\Delta)^{k-\frac{1}{2}}} dy \tag{2}$$

where the second integral converges absolutely for $1 - k < \sigma < k$ if $\Delta \neq 0$ [8, Proposition 4]. We are now in a position to state Zagier's theorem.

Theorem ([8])**.** *Let $k \geq 4$ be an even integer. For $m \in \mathbf{N}$, $s \in \mathbf{C}$ set*

$$c_m(s) = m^{k-1} \sum_{t \in \mathbf{Z}} \big(I_k(t^2 - 4m, t; s) + I_k(t^2 - 4m, -t; s)\big) L(s, t^2 - 4m)$$

$$+ \begin{cases} \frac{(-1)^{\frac{k}{2}} \Gamma(s+k-1)\zeta(2s)}{2^{2s+k-3}\pi^{s-1}\Gamma(k)} u^{k-s-1}, & \text{if } m = u^2,\ u > 0, \\ 0, & \text{if } m \text{ is not a perfect square.} \end{cases}$$

Then the following assertions hold:

i) *The series converges absolutely and uniformly for $2 - k < \sigma < k - 1$.*

ii) *The function*

$$\Phi_s(z) = \sum_{m \geq 1} c_m(s) e^{2\pi i m z} \quad (z \in \mathcal{H},\ 2 - k < \sigma < k - 1)$$

is in S_k.

iii) *Let f be a normalized Hecke eigenform in S_k. Then the Petersson scalar product of Φ_s and f is given by*

$$\langle \Phi_s, f \rangle = c_k \frac{\pi^{\frac{1}{2}(s+k-1)}}{2^{s+k-1}\Gamma(\frac{1+s}{2})} D_f^*(s + k - 1)$$

where

$$c_k = \frac{(-1)^{\frac{k}{2}}\pi}{2^{k-3}(k-1)}.$$

From the theorem, taking $m = 1$ we deduce

$$c_1(s) = c_k \frac{\pi^{\frac{1}{2}(s+k-1)}}{2^{s+k-1}\Gamma(\frac{1+s}{2})} \sum_{\nu=1}^{g_k} \frac{1}{\langle f_{k,\nu}, f_{k,\nu} \rangle} D_{f_{k,\nu}}^*(s + k - 1) \tag{3}$$

$$(2 - k < \sigma < k - 1).$$

In view of the functional equation (1), it is sufficient to prove the theorem in the range $k - \frac{1}{2} + \epsilon < \sigma < k$. Suppose that the right-hand side of (3) vanishes at $s = \frac{1}{2} + \delta + it_0$ where $\epsilon < \delta < \frac{1}{2}$. Then from the definition of $c_1(s)$ we obtain

$$\sum_{t \in \mathbf{Z}} \big(I_k(t^2 - 4, t; \frac{1}{2} + \delta + it_0) + I_k(t^2 - 4, -t; \frac{1}{2} + \delta + it_0)\big) L(\frac{1}{2} + \delta + it_0, t^2 - 4)$$

$$+ (-1)^{\frac{k}{2}} \frac{\Gamma(k - \frac{1}{2} + \delta + it_0)\zeta(1 + 2\delta + 2it_0)}{2^{k-2+2\delta+2it_0}\pi^{-\frac{1}{2}+\delta+it_0}\Gamma(k)} = 0,$$

or

$$(4)\quad \frac{(-1)^{\frac{k}{2}-1}2^k\Gamma(k)}{\Gamma(k-\frac{1}{2}+\delta+it_0)}\sum_{t\in\mathbf{Z}}\big(I_k(t^2-4,t;\frac{1}{2}+\delta+it_0)+I_k(t^2-4,-t;\frac{1}{2}+\delta+it_0)\big)$$

$$\cdot L(\frac{1}{2}+\delta+it_0,t^2-4)=\frac{\zeta(1+2\delta+2it_0)}{2^{2\delta-2+2it_0}\pi^{-\frac{1}{2}+\delta+it_0}}.$$

Clearly the right-hand side of (4) does not depend on k and is never zero for $\epsilon\le\delta\le\frac{1}{2}$. Therefore in absolute value it is bounded from below by a positive absolute constant depending only on ϵ.

We will show that the left-hand side of (4) goes to zero uniformly for $\epsilon<\delta<\frac{1}{2}$ as $k\to\infty$, thereby arriving at a contradiction.

We first look at the terms $I_k(t^2-4,t;\frac{1}{2}+\delta+it_0)+I_k(t^2-4,-t;\frac{1}{2}+\delta+it_0)$ in (4).

If $t=0$, we obtain from (2)

$$2I_k(-4,0;\frac{1}{2}+\delta+it_0)=2\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)}\int_0^\infty\frac{y^{k-\frac{3}{2}+\delta+it_0}}{(y^2+1)^{k-\frac{1}{2}}}dy$$

$$=\frac{\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(k)}B(\frac{k-\frac{1}{2}+\delta+it_0}{2},\frac{k-\frac{1}{2}-\delta-it_0}{2})$$

where $B(z,w)$ is the Beta function and in the last line we have used [1, 6.2.1]. Since $B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ we deduce

$$(5)\qquad 2I_k(-4,0;\frac{1}{2}+\delta+it_0)=\sqrt{\pi}\frac{\Gamma(\frac{k-\frac{1}{2}+\delta+it_0}{2})\Gamma(\frac{k-\frac{1}{2}-\delta-it_0}{2})}{\Gamma(k)}.$$

Next suppose that $t\ne 0,\pm2$. We will then use the fact that $I_k(t^2-4,t;s)$ can be expressed in terms of standard Legendre functions $P_\nu^\mu(z)$ [8, Proof of Proposition 4]. More precisely, one has

$$I_k(t^2-4,t;s)=\big(\frac{|t^2-4|}{4}\big)^{\frac{s-k}{2}}\frac{\Gamma(k-\frac{1}{2})\sqrt{\pi}}{\Gamma(k)}\cdot\begin{cases}I_{k,s}(\frac{it}{\sqrt{|t^2-4|}}), & \text{if } t^2<4,\\ e^{\frac{\pi i}{2}(s-k)\mathrm{sign}(t)}I_{k,s}(\frac{|t|}{\sqrt{t^2-4}}), & \text{if } t^2>4,\end{cases}$$

where

$$I_{k,s}(z)=\frac{2^{1-k}\sqrt{\pi}}{\Gamma(k-\frac{1}{2})}\Gamma(k-1+s)\Gamma(k-s)(z^2-1)^{-\frac{k-1}{2}}P_{-s}^{1-k}(z)$$

$(1-k<\sigma<k,\ z\in\mathbf{C}\setminus(-\infty,1])$. Also for $|z-1|<2$ the identity

$$P_{-s}^{1-k}(z)=\frac{1}{\Gamma(k)}\Big(\frac{z+1}{z-1}\Big)^{\frac{1-k}{2}}F(s,1-s,k;\frac{1-z}{2})$$

holds, where $F(a,b,c;z)$ $(|z|<1)$ denotes the Gauss hypergeometric series [1, 8.1.2].

Therefore for $t=\pm1$ we easily find that

$$(6)\qquad 2\big(I_k(-3,1;\frac{1}{2}+\delta+it_0)+I_k(-3,-1;\frac{1}{2}+\delta+it_0)\big)$$

$$\ll_{t_0}\frac{|\Gamma(k-\frac{1}{2}+\delta+it_0)\Gamma(k-\frac{1}{2}-\delta-it_0)|}{2^k\Gamma(k)^2},$$

and for $\pm t \geq 3$ we get

$$
\begin{aligned}
&2\big(I_k(t^2-4,t;\frac{1}{2}+\delta+it_0)+I_k(t^2-4,-t;\frac{1}{2}+\delta+it_0)\big) \ll_{t_0} (t^2-4)^{-\frac{1}{4}+\frac{\delta}{2}} \\
&\cdot \Big(\frac{|t|-\sqrt{t^2-4}}{|t|+\sqrt{t^2-4}}\Big)^{\frac{k-1}{2}} \frac{|\Gamma(k-\frac{1}{2}+\delta+it_0)\Gamma(k-\frac{1}{2}-\delta-it_0)|}{2^k\Gamma(k)^2} \qquad (7)
\end{aligned}
$$

where the constants implied in $\ll_{t_0}$ depend only on t_0.

Finally, for $t=\pm 2$ we have by [8, Proposition 4]

$$
\begin{aligned}
2\big(I_k(0,2;\frac{1}{2}+\delta+it_0)&+I_k(0,-2;\frac{1}{2}+\delta+it_0)\big) \\
&= 2\big(e^{\frac{\pi i}{2}(\frac{1}{2}-k+\delta+it_0)}+e^{-\frac{\pi i}{2}(\frac{1}{2}-k+\delta+it_0)}\big)\sqrt{\pi} \\
&\quad \cdot \frac{\Gamma(\delta+it_0)\Gamma(k-\frac{1}{2}-\delta-it_0)}{\Gamma(k)}\, 2^{-k+\frac{1}{2}+\delta+it_0} \qquad (8)\\
&\ll_{t_0,\epsilon} \frac{|\Gamma(k-\frac{1}{2}-\delta-it_0)|}{2^k\Gamma(k)}.
\end{aligned}
$$

We now look at the quantities $L(\frac{1}{2}+\delta+it_0,t^2-4)$ on the left-hand side of (4). For $|t|=2$, we have by definition

$$L(\frac{1}{2}+\delta+it_0,0)=\zeta(2\delta+2it_0)$$

which is a continuous function in the range $\epsilon\leq\delta\leq\frac{1}{2}$, provided $t_0\neq 0$. If $t_0=0$, the same applies to the function

$$2i^k\cos\big(\frac{\pi}{2}(\frac{1}{2}+\delta+it_0)\big)\cdot\zeta(2\delta+2it_0)$$

where

$$2i^k\cos\big(\frac{\pi}{2}(\frac{1}{2}+\delta+it_0)\big)=e^{\frac{\pi i}{2}(\frac{1}{2}-k+\delta+it_0)}+e^{-\frac{\pi i}{2}(\frac{1}{2}-k+\delta+it_0)}$$

is the second factor on the right-hand side of (8).

On the other hand, if $|t|\neq 2$, then for all $\delta\geq 0$ and all $\epsilon'>0$ one has

$$L(\frac{1}{2}+\delta+it_0,t^2-4)\ll_{t_0,\epsilon'}|t^2-4|^{\frac{1}{2}+\epsilon'}. \qquad (9)$$

In fact, if t^2-4 is a fundamental discriminant, this follows from [2, Chapter 12, Example 22 (b)], and the general case then easily follows from the definitions.

Denote the left-hand side of (4) by L_{k,δ,t_0}. Then from (5)–(9) (fixing any small $\epsilon'>0$ in (9)) and the separate discussion in the case $|t|=2$, $t_0=0$ above we deduce that

$$
\begin{aligned}
|L_{k,\delta,t_0}| &\ll_{t_0,\epsilon} \Big|\frac{2^k\Gamma(\frac{k-\frac{1}{2}+\delta+it_0}{2})\Gamma(\frac{k-\frac{1}{2}-\delta-it_0}{2})}{\Gamma(k-\frac{1}{2}+\delta+it_0)}\Big| + \frac{|\Gamma(k-\frac{1}{2}-\delta-it_0)|}{\Gamma(k)} \qquad (10)\\
&+\Big|\frac{\Gamma(k-\frac{1}{2}-\delta-it_0)}{\Gamma(k-\frac{1}{2}+\delta+it_0)}\Big| + \frac{|\Gamma(k-\frac{1}{2}-\delta-it_0)|}{\Gamma(k)}\sum_{t\geq 3}(t^2-4)^{\frac{1}{4}+\frac{\delta}{2}+\epsilon'}\Big(\frac{t-\sqrt{t^2-4}}{t+\sqrt{t^2-4}}\Big)^{\frac{k-1}{2}}.
\end{aligned}
$$

Elementary considerations show that the sum over $t\geq 3$ converges and is bounded by an absolute constant independent of k.

Note that by Legendre's duplication formula for the Γ-function, the first term on the right-hand side of (10) can also be written in the form

$$2^{\frac{3}{2}-\delta}\sqrt{\pi}\left|\frac{\Gamma(\frac{k}{2}-\frac{1}{4}-\frac{\delta}{2}-\frac{it_0}{2})}{\Gamma(\frac{k}{2}+\frac{1}{4}+\frac{\delta}{2}+\frac{it_0}{2})}\right|.$$

Using the fact that

$$\lim_{x\to\infty} x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 \quad (a,b\in \mathbf{C}\setminus\mathbf{R};\, x\to\infty)$$

[1, 6.1.46, 6.1.47], we now see indeed that

$$L_{k,\delta,t_0} \to 0 \quad (k\to\infty). \tag{11}$$

Moreover, using more precisely the explicit asymptotics of $x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$ for $x\to\infty$ given in [1, 6.1.47], one sees that the convergence in (11) is uniform in δ, since $\delta > \epsilon > 0$ is bounded away from zero.

This proves the theorem.

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