

UNIQUENESS OF THE LEAST-ENERGY SOLUTION FOR A SEMILINEAR NEUMANN PROBLEM

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ABSTRACT. We prove that the least-energy solution of the problem

$$\begin{cases} -d\Delta u + u = u^p & \text{in } B, \\ u > 0 & \text{in } B, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B, \end{cases}$$

where B is a ball, $d > 0$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$, is unique (up to rotation) if d is small enough.

INTRODUCTION

In this paper we consider the following problem:

$$(0.1) \quad \begin{cases} -d\Delta u + u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in \mathbb{R}^N , ν is the unit outer normal to $\partial\Omega$, $d > 0$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $p > 1$ if $N = 2$.

To our knowledge the first existence result to (0.1) is due to Lin, Ni and Takagi (see [LNT]). In this paper the authors consider the functional

$$(0.2) \quad J_d(u) = \frac{1}{2} \int_{\Omega} (d|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\Omega} (u^+)^{p+1}$$

and

$$(0.3) \quad c_d = \inf_{h \in \Gamma} \max_{t \in [0,1]} J_d(h(t))$$

where $\Gamma = \{f \in C([0,1], H^1(\Omega)) \mid f(0) = 0, f(1) = e\}$ and $e \neq 0$ is a non-negative function of $H^1(\Omega)$ such that $J_d(e) = 0$. From the mountain-pass theorem (see [AR]) we have that c_d is a positive critical value of J_d . Moreover, it turns out that c_d is the least positive critical value for J_d (see [LNT] or [NT]).

Therefore, we define a critical point u_d of J_d satisfying $J_d(u_d) = c_d$ a *least-energy solution* of (0.1).

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Clearly (0.1) admits the trivial solutions $u \equiv 1$ and $u \equiv 0$. In [LNT] it was proved that $c_d = O(d^{N/2})$ as $d \rightarrow 0$ and from this $0 < J_d(u_d) < J_d(1)$, i.e. u_d is a nonconstant solution provided d is small enough.

Another important result concerns the shape of the least-energy solution u_d . In [NT] it was shown that u_d has only one local maximum in $\overline{\Omega}$, and it is achieved at exactly one point which lies on the boundary of Ω , provided d is small enough.

In this paper we will prove the uniqueness of the least-energy solution if Ω is a ball. Of course, in this case, if $u(x)$ is a solution of (0.1), then also $u(Tx)$ solves (0.1) for any T belonging to the orthogonal group $O(N)$; therefore by uniqueness we mean that any two solutions can be obtained from each other using the action of $O(N)$.

Theorem 1. *Let us consider the problem (0.1) when Ω is a ball. Then there exists a $d_0 > 0$ such that for any $d < d_0$ the least-energy solution of (0.1) is unique.*

We want to point out that uniqueness results for (0.1) cannot be obtained if we do not restrict the class of the solutions of (0.1). Indeed, in [DY] Dancer and Yan prove the existence of k -peak solutions to (0.1) (solutions with more than one local maximum point). Clearly the energy of these solutions is strictly greater than c_d . We mention here the papers of [BDS], [Gu], [W1], and [W2] for the existence of multipeak solutions in other domains.

The paper is organized as follows: in Section 1 we recall some important results which we use in the proof of Theorem 1. In Section 2 we prove Theorem 1.

1. KNOWN RESULTS

In this section we recall some known facts which will be useful in the proof of Theorem 1. Here Ω is an arbitrary bounded smooth domain of \mathbb{R}^N and p is a real number such that $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $p > 1$ if $N = 2$. We start by recalling some results on the existence and the shape of the least-energy solution u_d .

Theorem 1.1. *Let $p > 1$ if $N = 1$ or 2 and let $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. Then there exists a $d_0 > 0$ such that, for any $d < d_0$, (0.1) has a positive nonconstant solution u_d satisfying*

$$(1.1) \quad J_d(u_d) = c_d,$$

$$(1.2) \quad \int_{\Omega} u_d^q \leq C_1(q) d^{N/2} \quad \text{for any } q \geq 1,$$

and

$$(1.3) \quad \sup_{\Omega} u_d(x) \leq C_2$$

with $C_1(q), C_2$ independent of d .

Proof. See [LNT]. □

Theorem 1.2. *Let u_d be a least-energy solution to (1.3). Then u_d has at most one local maximum in Ω and it is achieved exactly at one point which must lie on the boundary, provided d is sufficiently small.*

Proof. See [NT], Theorem 2.1. □

Theorem 1.3. *There is exactly one solution (up to translation) of the problem*

$$(1.4) \quad \begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

Proof. See [K]. □

Next let us introduce a diffeomorphism which straightens a boundary portion of Ω as follows. We denote by B_R the ball of \mathbb{R}^N centered at the origin and radius R . Through translation and rotation of the coordinate system we may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N -axis. Then there exists a smooth function $\psi(x'), x' = (x_1, \dots, x_{N-1})$, defined for $|x| < \delta_0$ such that $\psi(0) = 0, \nabla\psi(0) = 0$ and

$$\partial\Omega \cap \mathcal{N} = \{x = (x', x_N) \mid x_N = \psi(x')\}$$

and

$$\Omega \cap \mathcal{N} = \{x = (x', x_N) \mid x_N > \psi(x')\}.$$

For $|y| < \delta'$ let us define $x = \Phi(y)$, with $\Phi(y) = (\Phi_1(y), \dots, \Phi_N(y))$, by

$$(1.5) \quad \Phi_j(y) = \begin{cases} y_j - y_N \frac{\partial\psi}{\partial x_j}(y') & \text{for } j = 1, \dots, N-1, \\ y_N + \psi(y') & \text{if } j = N. \end{cases}$$

Since $\nabla\psi(0) = 0$ the differential map $D\Phi$ of Φ satisfies $D\Phi(0) = I$, the identity map. Thus there exists the inverse map $y = \Phi^{-1}(x)$ between $C \subset \mathbb{R}^N$ and $B(0, \delta)$ with $\delta < \delta'$.

Theorem 1.4. *Let u_d be a least-energy solution to (0.1) and let us denote by P_d the point where the maximum of u_d is achieved. Then for any $\epsilon > 0$ there exist a positive constant d_0 and a subdomain $\Omega_d^{(i)} \subset \Omega$ such that for $d \in]0, d_0[$ the following statements hold:*

$$(1.6) \quad P_d \in \partial\Omega_d \quad \text{and} \quad \text{diam}(\Omega_d^{(i)}) \leq C\sqrt{d},$$

$$(1.7) \quad u_d(\Phi(\sqrt{d}y)) \rightarrow U(y) \text{ in } C_{\text{loc}}^2(\mathbb{R}_+^N) \text{ as } d \rightarrow 0,$$

$$(1.8) \quad |u_d(x)| \leq C_1\epsilon e^{-\frac{\mu_1\delta(x)}{\sqrt{d}}} \quad \text{for } x \in \Omega \setminus \Omega_d^{(i)},$$

where $\delta(x) = \min\{\text{dist}(x, \partial\Omega_d), \eta_0\}$, C, C_1, μ_1, η_0 are positive constants depending only on Ω and $U(x)$ is the unique solution of (1.4) with maximum achieved at $P_0 = \lim_{d \rightarrow 0} P_d$.

Proof. See [NT], (i) and (iii) of Theorem 2.3 and (4.29). □

Theorem 1.5. *Let us consider the problem*

$$(1.9) \quad \begin{cases} -\Delta z + z = pU(x)^{p-1}z & \text{in } \mathbb{R}^N, \\ \|z\|_{L^\infty(\mathbb{R}^N)} \leq C, \end{cases}$$

where $U(x)$ is the unique solution of (1.4) symmetric with respect to the origin. Then the only nontrivial solutions of (1.9) are given by

$$(1.10) \quad z = \sum_{i=1}^N \alpha_i \frac{\partial U}{\partial x_i}, \quad \alpha_i \in \mathbb{R}.$$

Proof. This result is contained in the proof of Proposition 1 of [D]. □

2. THE UNIQUENESS RESULT

In this section we denote by $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ a generic point of \mathbb{R}^N . Let us assume that $\Omega = B = \{x \in \mathbb{R}^N \mid |x'|^2 + (x_N - 1)^2 < 1\}$. From Theorem 1.2 the maximum of a least-energy solution u of (0.1) is achieved on the boundary of B if d is small enough. So, since the problem (0.1) is invariant with respect to the orthogonal group of \mathbb{R}^N , by a suitable rotation we can assume that u attains its maximum at the origin. Hence $\nabla u(0) = 0$ for any least-energy solution of (0.1).

We suppose by contradiction that there exist a sequence $d_n \searrow 0$ and two distinct least-energy solutions $u_{1,n}$ and $u_{2,n}$ which solve (0.1) with $d = d_n$. So we can suppose that $u_{1,n}$ and $u_{2,n}$ achieve their maximum at the origin for any $n \in \mathbb{N}$. Since $u_{1,n} - u_{2,n} \not\equiv 0$ we can consider

$$(2.1) \quad z_n(x) = \frac{u_{1,n}(x) - u_{2,n}(x)}{\|u_{1,n} - u_{2,n}\|_{L^\infty}}.$$

By assumption on $u_{1,n}$ and $u_{2,n}$ we have

$$(2.2) \quad \nabla z_n(0) = 0.$$

Note that z_n satisfies

$$(2.3) \quad \begin{cases} -d_n \Delta z_n + z_n = c_n(x) z_n & \text{in } B, \\ \frac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial B, \\ \|z_n\|_{L^\infty(B)} = 1, \end{cases}$$

where

$$(2.4) \quad c_n(x) = p \int_0^1 (t u_{1,n}(x) + (1-t) u_{2,n}(x))^{p-1} dt.$$

Let us set

$$(2.5) \quad \tilde{z}_n(x) = z_n(\sqrt{d_n}x), \quad \tilde{z}_n : B/\sqrt{d_n} \mapsto \mathbb{R}.$$

We remark that for any compact set $K \subset \mathbb{R}_+^N$, where $\mathbb{R}_+^N = \{x \in \mathbb{R}^N \mid x_N > 0\}$, we have that $K \subset B/\sqrt{d_n}$ for n large enough. In the following proposition we study the asymptotic behavior of \tilde{z}_n .

Proposition 2.1. *We have that*

$$(2.6) \quad \tilde{z}_n \rightarrow z \quad \text{in } C_{\text{loc}}^1(\mathbb{R}_+^N),$$

where z is a bounded solution of

$$-\Delta z + z = pU^{p-1}z \quad \text{in } \mathbb{R}^N$$

and U is the unique solution of (1.4) symmetric with respect to the origin.

Proof. Let us consider the diffeomorphism Φ between $B(0, \delta)$ and $C \subset \mathbb{R}^N$ which straightens the boundary of B (see (1.5)). Now, as in [NT], let us set

$$(2.7) \quad v_n(y) = z_n(\Phi(y)), \quad y \in \overline{B_{2k}^+},$$

where $B_{2k}^+ = \{y \in B_{2k} \mid y_N > 0\}$.

Step 1. In this step we prove that

$$(2.8) \quad v_n(\sqrt{d_n}y) \rightarrow z(y) \quad \text{in } C_{\text{loc}}^2(\mathbb{R}_+^N).$$

First of all we extend v_n to $B(0, 2k)$ by reflection,

$$\tilde{v}_n(y) = \begin{cases} v_n(y) & \text{if } y \in \overline{B_{2k}^+}, \\ v_n(y', -y_N) & \text{if } y \in \overline{B_{2k}^-}, \end{cases}$$

with $\overline{B_{2k}^-} = \{y \in B_{2k} \mid y_N < 0\}$. Moreover we define a scaled function $w_n(z)$ by

$$(2.9) \quad w_n(y) = \tilde{v}_n(\sqrt{d_n}y) \quad \text{for } y \in \overline{B_{\frac{k}{\sqrt{d_n}}}}.$$

It is easily seen that

$$w_n \in C^2(\overline{B_{\frac{k}{\sqrt{d_n}}}} \setminus \{y_N = 0\}) \cap C^1(\overline{B_{\frac{k}{\sqrt{d_n}}}})$$

since $\frac{\partial v_n}{\partial y_N}$ on $y_N = 0$ and that w_n satisfies the elliptic equation

$$(2.10) \quad -L_n w_n + w_n = \tilde{c}_n(y) w_n$$

with

$$(2.11) \quad L_n w_n = \sum_{i,j=1}^N a_{ij}^n(y) \frac{\partial^2 w_n}{\partial z_i \partial z_j} + \sqrt{d_n} \sum_{j=1}^N b_j^n(y) \frac{\partial w_n}{\partial z_j}$$

in $B_{\frac{k}{\sqrt{d_n}}} \setminus \{y_N = 0\}$, where a_{ij}^n, b_j^n and $\tilde{c}_n(y)$ are defined as follows. First, put

$$\Psi = \Phi^{-1},$$

$$a_{ij}(y) := \sum_{l=1}^N \frac{\partial \Psi_i}{\partial x_l}(\Phi(y)) \frac{\partial \Phi_j}{\partial x_l}(\Phi(y)), \quad 1 \leq i, j \leq N,$$

$$b_j(y) := (\Delta \Psi_j)(\Phi(y)), \quad j = 1, \dots, N,$$

$$\hat{c}_n(y) = c_n(\Phi(y)).$$

Then set

$$a_{ij}^n(y) := \begin{cases} a_{ij}(\sqrt{d_n}y) & \text{if } y_N \geq 0, \\ (-1)^{\delta_N^i + \delta_N^j} a_{ij}(\sqrt{d_n}y', -\sqrt{d_n}y_N) & \text{if } y_N < 0, \end{cases}$$

$$b_j^n(y) := \begin{cases} b_j(\sqrt{d_n}y) & \text{if } y_N \geq 0, \\ (-1)^{\delta_N^j} b_j(\sqrt{d_n}y', -\sqrt{d_n}y_N) & \text{if } y_N < 0, \end{cases}$$

$$\tilde{c}_n(y) := \begin{cases} \hat{c}_n(\sqrt{d_n}y) & \text{if } y_N \geq 0, \\ (-1)^{\delta_N^j} \hat{c}_n(\sqrt{d_n}y', -\sqrt{d_n}y_N) & \text{if } y_N < 0, \end{cases}$$

where δ_j^i is the Kroneker symbol.

Note that by (1.3) the functions w_n and c_n are uniformly bounded with respect to n . Then, by using the standard L^r -estimate and the interior Schauder estimate as in [NT], pp. 834-836, we obtain that

$$(2.12) \quad w_n \rightarrow z \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^N)$$

and

$$(2.13) \quad -L_n w_n \rightarrow -\Delta z.$$

Hence, since by (1.7) and (2.4) we get $\tilde{c}_n \rightarrow pU^{p-1}$, z is a solution of

$$(2.14) \quad -\Delta z + z = pU^{p-1}z \quad \text{in } \mathbb{R}^N.$$

By the definition of w_n and \tilde{v}_n (2.8) follows.

Step 2. In this step we prove the claim of the proposition.

The function \tilde{z}_n satisfies the equation

$$(2.15) \quad \begin{cases} -\Delta \tilde{z}_n + \tilde{z}_n = c_n(\sqrt{d_n})\tilde{z}_n & \text{in } B/\sqrt{d_n}, \\ \tilde{z}_n > 0 & \text{in } B/\sqrt{d_n}, \\ \frac{\partial \tilde{z}_n}{\partial \nu} = 0 & \text{on } \partial B/\sqrt{d_n}. \end{cases}$$

From (1.3) we get $\|c_n \tilde{z}_n\|_\infty \leq C$ and then by standard L^p -estimates we deduce, for a subsequence, $\|\tilde{z}_n\|_{C^1(K)} \leq C$ for any compact set $K \subset \mathbb{R}^N$.

By Step 1 we know that $\|z_{d_n}(\Phi(\sqrt{d_n}y)) - U(y)\|_{C^0_{loc}(\mathbb{R}^N_+)} \rightarrow 0$. So if we show that

$$(2.16) \quad \|z_{d_n}(\Phi(\sqrt{d_n}y)) - \tilde{u}_n(y)\|_{C^0_{loc}(\mathbb{R}^N_+)} \rightarrow 0,$$

the claim follows.

Let us fix a compact set $K \subset \mathbb{R}^N$. By the definition of Φ (see (1.5)) we get

$$(2.17) \quad |\Phi(\sqrt{d_n}y) - \sqrt{d_n}y|^2 = |d_n y_N^2 |\nabla_{y'} \psi(\sqrt{d_n}y)|^2 + \psi^2(\sqrt{d_n}y)| \rightarrow \psi^2(0) = 0$$

uniformly in K .

Then, since $|\nabla u_{d_n}| \leq C$ in K , we obtain that

$$(2.18) \quad \|z_{d_n}(\Phi(\sqrt{d_n}y)) - \tilde{z}_n(y)\|_{C^0(K)} \leq C \|\Phi(\sqrt{d_n}y) - \sqrt{d_n}y\|_{C^0(K)} \rightarrow 0,$$

and this finishes the proof. □

Proof of Theorem 1. Let us again consider the functions z_n and \tilde{z}_n defined by (2.1) and (2.5). First of all we remark that since $u_{1,n}$ and $u_{2,n}$ solve (0.1) we have

$$\int_{\Omega} u_{1,n} u_{2,n} (u_{1,n}^{p-1} - u_{2,n}^{p-1}) = 0$$

and then \tilde{z}_n does change sign.

Since we have that $\|\tilde{z}_n\|_{L^\infty(B/\sqrt{d_n})} = 1$ we can assume that there exists a sequence of points $\tilde{x}_n \in B/\sqrt{d_n}$ such that $\tilde{z}_n(\tilde{x}_n) = \max_{x \in B/\sqrt{d_n}} \tilde{z}_n(x) = 1$. Moreover we have that

$$(2.19) \quad \nabla \tilde{z}_n(\tilde{x}_n) = 0 \quad \text{and} \quad \Delta \tilde{z}_n(\tilde{x}_n) \leq 0$$

(this is obvious if $\tilde{x}_n \in B/\sqrt{d_n}$ and it follows by $\frac{\partial \tilde{z}_n}{\partial \nu}(\tilde{x}_n) = 0$ if $\tilde{x}_n \in \frac{\partial B}{\sqrt{d_n}}$).

From Proposition 2.1 the following alternative occurs: either

$$(2.20) \quad \tilde{z}_n \rightarrow 0$$

or

$$(2.21) \quad \tilde{z}_n \rightarrow \sum_{i=1}^N \alpha_i \frac{\partial U}{\partial x_i} \quad \text{for some } (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \setminus \{0\}.$$

We will prove that in any case we reach a contradiction.

Case 1. $|\tilde{z}_n| \rightarrow z \equiv 0$.

Here we have the following alternative:

- i) $|\tilde{x}_n| \leq C$,
- ii) $|\tilde{x}_n| \rightarrow \infty$.

Case i). $|\tilde{x}_n| \leq C$.

Let $x_0 = \lim_n \tilde{x}_n$. If $x_0 \notin \partial\mathbb{R}_+^N$, the claim follows easily. If $x_0 \in \partial\mathbb{R}_+^N$, we again consider the diffeomorphism Φ between $B(0, \delta')$ and $B(0, \delta)$ which straightens the boundary. Since $\tilde{x}_n \in B/\sqrt{d_n}$ we have that $\tilde{x}_n = \frac{x_n}{\sqrt{d_n}}$ with $x_n \in B$ and $x_n \rightarrow 0$. Then there exists $P_n \in B_\delta^+$ such that

$$(2.22) \quad \Phi(P_n) = x_n.$$

Set $y_n = \frac{P_n}{\sqrt{d_n}}$. Then, since $\Phi(0) = 0$ and Φ is a diffeomorphism in B_δ^+ ,

$$(2.23) \quad |\sqrt{d_n}y_n| = |P_n| = |\Phi^{-1}(x_n)| = |\Phi^{-1}(\sqrt{d_n}\tilde{x}_n)| \leq C|\sqrt{d_n}\tilde{x}_n|,$$

so $|y_n| \leq C$. Finally, by (2.7)-(2.9)

$$(2.24) \quad 1 = \tilde{z}_n(\tilde{x}_n) = z_n(x_n) = z_n(\Phi(P_n)) = z_n(\Phi(\sqrt{d_n}y_n)) \rightarrow 0$$

and this gives a contradiction.

Case ii). $|\tilde{x}_n| \rightarrow \infty$.

Let us prove that $c_n(\sqrt{d_n}\tilde{x}_n) \rightarrow 0$. We have

$$c_n(\sqrt{d_n}\tilde{x}_n) = p \int_0^1 (tu_{1,n}(\sqrt{d_n}\tilde{x}_n) + (1-t)u_{2,n}(\sqrt{d_n}\tilde{x}_n))^{p-1} dt.$$

Again since $\tilde{x}_n \in B/\sqrt{d_n}$ there exists $x_n \in B$ such that $x_n = \sqrt{d_n}\tilde{x}_n$ and

$$(2.25) \quad \frac{|x_n|}{\sqrt{d_n}} \rightarrow \infty.$$

Then (1.6) and (1.7) of Theorem 1.4 applies: since by (2.25) $x_n \notin B_{d_n}^{(i)}$ we obtain, for any $\epsilon > 0$,

$$(2.26) \quad u_{1,n}(x_n) < \epsilon \text{ and } u_{2,n}(x_n) < \epsilon \text{ for } n \text{ large.}$$

Hence $c_n(\sqrt{d_n}\tilde{x}_n) = c_n(x_n) \rightarrow 0$. Finally from (2.19) we obtain

$$(2.27) \quad 0 \leq -\Delta \tilde{z}_n(\tilde{x}_n) = c_n(x_n) - 1 \rightarrow -1,$$

and this gives a contradiction.

Case 2. $|\tilde{z}_n| \rightarrow z = \sum_{i=1}^N \alpha_i \frac{\partial U}{\partial x_i}$ for some $(\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N \setminus \{0\}$.

First of all let us prove that

$$(2.28) \quad \frac{\partial^2 U}{\partial x_j^2}(0) < 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_i \partial x_j}(0) = 0 \text{ if } i \neq j, \quad i, j \in \{1, \dots, N\}.$$

This is a well-known fact (see [NT] for example), but we repeat the proof for the reader's convenience. Since the function $U(\rho)$ is radially decreasing for $\rho > 0$ (see [GNN]), we have that $\frac{\partial U}{\partial x_j} = U'(\rho) \frac{x_j}{\rho} > 0$ in $T_j = \{x \in \mathbb{R}^N \mid x_j < 0\}$. Hence $\frac{\partial U}{\partial x_j}|_{\partial T_j} \equiv 0$ and then $\frac{\partial^2 U}{\partial x_i \partial x_j}(0) = 0$ for any $i \neq j$. Moreover $f = \frac{\partial U}{\partial x_j}$ solves

$$\begin{cases} -\Delta f + f = pU(x)^{p-1}f & \text{in } T_j, \\ f > 0 & \text{in } T_j, \\ f = \frac{\partial U}{\partial x_j} = 0 & \text{on } \partial T_j. \end{cases}$$

So, by Hopf's Lemma, $\frac{\partial f}{\partial x_j} = \frac{\partial^2 U}{\partial x_j^2}(0) < 0$ on ∂T_j , and then (2.28) is proved.

Now let us recall that $\nabla \tilde{z}_n(0) = 0$. Since $\Phi(0) = 0$ we have

$$(2.29) \quad \nabla(z_n \circ \Phi)(0) = 0.$$

Then since $(\alpha_1, \dots, \alpha_N) \neq (0, \dots, 0)$ there exists $\alpha_j \neq 0$. Finally, from (2.8), (2.28) and (2.29) we get

$$(2.30) \quad 0 = \frac{\partial(z_n \circ \Phi)}{\partial x_j}(0) \rightarrow \sum_{i=1}^N \alpha_i \frac{\partial^2 U}{\partial x_i \partial x_j}(0) = \alpha_j \frac{\partial^2 U}{\partial x_j^2} \neq 0,$$

a contradiction. The proof of Theorem 1 is finished.

REFERENCES

- [AR] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications", *J. Funct. Anal.* **14**, 349-381, (1973). MR **51**:6412
- [BDS] P. Bates, E.N. Dancer and J. Shi, "Multi-spike stationary solutions on the Cahn-Hilliard equation in higher dimension and instability", preprint.
- [D] E.N. Dancer, "On the uniqueness of the positive solution of a singularly perturbed problem", *Rocky Mountain Journal of Mathematics* **25** (1995), 957-975. MR **96j**:35021
- [DY] E.N. Dancer and S. Yan, "Multipeak solutions for a singularly perturbed Neumann problem" (to appear).
- [GNN] B. Gidas, W.N. Ni and L. Nirenberg, "Symmetry of positive solutions of nonlinear elliptic equation in \mathbb{R}^N ", *Advances in Math. Studies* **7 A**, 369-402, (1981). MR **84a**:35083
- [GT] D. Gilbarg and N. Trudinger, "Elliptic partial differential equations of second order", Springer Verlag (1983). MR **86c**:35035
- [Gu] C. Gui, "Multipeak solutions for a semilinear Neumann problem", *Duke Math. J.* **84**, 739-769, (1996). MR **97i**:35052
- [K] M.K. Kwong, "Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbb{R}^N ", *Arch. Rat. Mech. Anal.*, **105**, 243-266, (1989). MR **90d**:35015
- [LN] C.S. Lin and W.M. Ni, "On the diffusion coefficient of a semilinear Neumann problem, *Calculus of Variations and Partial Differential Equations*", S. Hildebrandt, D. Kinderlehrer and M. Miranda, eds., *Lecture Notes in Math.* 1340, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 160-174, (1988). MR **90d**:35101
- [LNT] C.S. Lin, W.M. Ni and I. Takagi, "Large amplitude stationary solutions to a chemotaxis system", *J. Diff. Eqns.* **72**, 1-27, (1988). MR **89e**:35075
- [NT] W.M. Ni and I. Takagi, "On the shape of Least Energy Solutions to a Semilinear Neumann Problem", *Comm. Pure Math. Appl.*, **Vol XLIV**, 819-851, (1991). MR **92i**:35052
- [W1] Z.Q. Wang, "On the existence of multiple single-peaked solution for a semilinear Neumann problem", *Arch. Rat. Mech. Anal.*, **120**, 375-399, (1992). MR **93k**:35109
- [W2] Z.Q. Wang, "Nonradial solutions of nonlinear Neumann problems in radially symmetric domains", *Topology in Nonlinear analysis* (Warsaw, 1994), 85-96, Banach Center Publications, **35**, Polish Acad. Sci., Warsaw, (1996). MR **98e**:35075

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