

## PRINCIPAL VALUES FOR THE CAUCHY INTEGRAL AND RECTIFIABILITY

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ABSTRACT. We give a geometric characterization of those positive finite measures  $\mu$  on  $\mathbb{C}$  with the upper density  $\limsup_{r \rightarrow 0} \frac{\mu(\{\xi: |\xi-z| \leq r\})}{r}$  finite at  $\mu$ -almost every  $z \in \mathbb{C}$ , such that the principal value of the Cauchy integral of  $\mu$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{|\xi-z|>\varepsilon} \frac{1}{\xi-z} d\mu(\xi),$$

exists for  $\mu$ -almost all  $z \in \mathbb{C}$ . This characterization is given in terms of the curvature of the measure  $\mu$ . In particular, we get that for  $E \subset \mathbb{C}$ ,  $\mathcal{H}^1$ -measurable (where  $\mathcal{H}^1$  is the Hausdorff 1-dimensional measure) with  $0 < \mathcal{H}^1(E) < \infty$ , if the principal value of the Cauchy integral of  $\mathcal{H}^1|_E$  exists  $\mathcal{H}^1$ -almost everywhere in  $E$ , then  $E$  is rectifiable.

### 1. INTRODUCTION

Given a complex Radon measure  $\nu$  on  $\mathbb{C}$ , the *Cauchy integral* of  $\nu$  is defined, for  $z \notin \text{supp}(\nu)$ , by

$$\mathcal{C}\nu(z) = \int \frac{1}{\xi-z} d\nu(\xi).$$

Notice that the definition above does not make sense, in general, for  $z \in \text{supp}(\nu)$ . This is the reason why one considers the *truncated Cauchy integral* of  $\nu$ , which is defined as

$$\mathcal{C}_\varepsilon\nu(z) = \int_{|\xi-z|>\varepsilon} \frac{1}{\xi-z} d\nu(\xi),$$

for any  $\varepsilon > 0$  and  $z \in \mathbb{C}$ . Also, one can define the *maximal Cauchy integral* of  $\nu$ ,

$$\mathcal{C}_*\nu(z) = \sup_{\varepsilon>0} |\mathcal{C}_\varepsilon\nu(z)|,$$

which may be finite or infinite at each point  $z \in \mathbb{C}$ , and the *principal value* of the Cauchy integral of  $\nu$ , if it exists, is

$$\text{p.v. } \mathcal{C}\nu(z) = \lim_{\varepsilon \rightarrow 0} \mathcal{C}_\varepsilon\nu(z).$$

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If  $\mu$  is a positive Radon measure on  $\mathbb{C}$ , the *upper density of  $\mu$  at  $x \in \mathbb{C}$*  is defined as

$$\Theta_\mu^*(x) = \limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r},$$

where  $B(x, r)$  stands for the open disk centered at  $x$  with radius  $r$ , and the *lower density of  $\mu$  at  $x \in \mathbb{C}$*  is

$$\Theta_{*,\mu}(x) = \liminf_{r \rightarrow 0} \frac{\mu(B(x, r))}{r}.$$

For  $A \subset \mathbb{C}$  we set

$$\mathcal{H}_\varepsilon^1(A) = \inf \left\{ \sum_{i=1}^\infty \text{diam}(E_i) : A \subset \bigcup_{i=1}^\infty E_i, \text{diam}(E_i) \leq \varepsilon \right\}.$$

Then, the *1-dimensional Hausdorff measure* of  $A$  is  $\mathcal{H}^1(A) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^1(A)$ . Also,  $A$  is said to be *rectifiable* if it is  $\mathcal{H}^1$ -measurable and it is  $\mathcal{H}^1$ -almost all contained in a countable union of rectifiable curves.

In this paper we give a geometric characterization of those positive finite Radon measures  $\mu$  on  $\mathbb{C}$  with  $\Theta_\mu^*(x) < \infty$  for a.e. ( $\mu$ )  $x \in \mathbb{C}$  (i.e., for almost every, with respect to  $\mu$ ,  $x \in \mathbb{C}$ ) such that p.v.  $\mathcal{C}\mu(x)$  exists for a.e. ( $\mu$ )  $x \in \mathbb{C}$ . In particular, we prove that for  $E \subset \mathbb{C}$ ,  $\mathcal{H}^1$ -measurable with  $0 < \mathcal{H}^1(E) < \infty$ , if p.v.  $\mathcal{C}(\mathcal{H}^1|_E)(x)$  exists for a.e. ( $\mathcal{H}^1$ )  $x \in E$ , then  $E$  is rectifiable. Until now, this was known under the additional hypothesis  $\Theta_{*,\mathcal{H}^1|_E}(x) > 0$  for a.e. ( $\mathcal{H}^1$ )  $x \in E$ , by a result of [Ma]. Now we show that this assumption is not necessary, which solves a question raised by P. Mattila.

Let us remark that in [Ma] a result is proved which applies to measures more general than  $\mathcal{H}^1|_E$ . Indeed it is proved that if  $\mu$  is such that for a.e. ( $\mu$ )  $x \in \mathbb{C}$ ,  $\Theta_{*,\mu}(x) > 0$  and p.v.  $\mathcal{C}\mu(x)$  exists, then there is a countable collection of rectifiable curves  $\Gamma_1, \Gamma_2, \dots$  such that  $\mu(\mathbb{C} \setminus \bigcup_i \Gamma_i) = 0$ . This result was obtained following an approach based on tangent measures.

Another fact that shows the strong relation between the existence of principal values for the Cauchy integral on the one hand, and rectifiability on the other hand, is the result obtained by P. Mattila and M. Melnikov in [MM]. This result goes in the converse direction: For  $E \subset \mathbb{C}$  with  $\mathcal{H}^1(E) < \infty$ , if  $E$  is rectifiable, then p.v.  $\mathcal{C}(\mathcal{H}^1|_E)(x)$  exists for a.e. ( $\mathcal{H}^1$ )  $x \in E$ .

Thanks to the introduction of the notion of curvature of a measure  $\mu$ , many new results related to the Cauchy integral have been obtained. Given three pairwise different points  $x, y, z \in \mathbb{C}$ , its *Menger curvature* is

$$c(x, y, z) = \frac{1}{R(x, y, z)},$$

where  $R(x, y, z)$  is the radius of the circumference passing through  $x, y, z$  (with  $R(x, y, z) = \infty$ ,  $c(x, y, z) = 0$  if  $x, y, z$  lie on a same line). If two among these points coincide, we let  $c(x, y, z) = 0$ . Then we set

$$c_\mu^2(x) = \iint c(x, y, z)^2 d\mu(y)d\mu(z),$$

and the *curvature of  $\mu$*  is defined as

$$c^2(\mu) = \int c_\mu^2(x)d\mu(x) = \iiint c(x, y, z)^2 d\mu(x)d\mu(y)d\mu(z).$$

Also, we denote

$$c_\varepsilon^2(\mu) = \iiint_{\substack{|x-y| > \varepsilon \\ |x-z| > \varepsilon \\ |y-z| > \varepsilon}} c(x, y, z)^2 d\mu(x)d\mu(y)d\mu(z).$$

The concept of curvature of a measure was introduced by M. Melnikov in [Me] when he was studying a discrete version of analytic capacity. M. Melnikov and J. Verdera showed in [MV] the precise relation of the Cauchy kernel with curvature of measures. They showed that if there is some constant  $C_0$  such that

$$(1.1) \quad \mu(B(x, r)) \leq C_0 r \text{ for all } x \in \mathbb{C} \text{ and all } r > 0,$$

then

$$(1.2) \quad \left| 6 \int |\mathcal{C}_\varepsilon \mu|^2 d\mu - c_\varepsilon^2(\mu) \right| \leq C \|\mu\|,$$

where the constant  $C$  depends only on  $C_0$ .

Curvature of measures is also closely related to rectifiability. So David and Léger [L] have proved that if  $F \subset \mathbb{C}$  is  $\mathcal{H}^1$ -measurable,  $\mathcal{H}^1(F) < \infty$  and  $c^2(\mathcal{H}^1_F) < \infty$ , then  $F$  is rectifiable.

This connection of curvature of measures with the Cauchy kernel on the one hand, and with rectifiability on the other hand, plays an important role in recent results about analytic capacity. See [MMV], [DM], [D] and [NTV], for example.

Concerning principal values for the Cauchy integral, the notion of curvature of measures has been essential to obtain new results, too. In [T, Theorem 2], it is proved that if  $\mu$  is any positive finite Radon measure on  $\mathbb{C}$ , then the following three statements are equivalent.

a) The measure  $\mu$  satisfies

$$(1.3) \quad \Theta_\mu^*(x) < \infty \text{ for a.e. } (\mu) x \in \mathbb{C},$$

and there exists a countable family of Borel sets  $E_n$  such that

$$(1.4) \quad \text{supp}(\mu) = \bigcup_{n=0}^\infty E_n \text{ and } c_{\mu|_{E_n}}^2(x) < \infty \text{ for a.e. } (\mu) x \in E_n.$$

b) For any finite complex Radon measure  $\nu$  on  $\mathbb{C}$ , the principal value

$$(1.5) \quad \lim_{\varepsilon \rightarrow 0} \mathcal{C}_\varepsilon \nu(x)$$

exists for a.e.  $(\mu) x \in \mathbb{C}$ .

c) For any finite complex Radon measure  $\nu$  on  $\mathbb{C}$ ,

$$(1.6) \quad \mathcal{C}_* \nu(x) < +\infty$$

for a.e.  $(\mu) x \in \mathbb{C}$ .

In the present paper, in Corollary 3.1 we try to give a result stronger than the implication c)  $\Rightarrow$  a) of this result of [T]. Instead of assuming

$$(1.7) \quad \mathcal{C}_* \nu(x) < +\infty \text{ for a.e. } (\mu) x \in \mathbb{C} \text{ and for all finite complex Radon measures } \nu,$$

we will assume only

$$(1.8) \quad \mathcal{C}_* \mu(x) < +\infty \text{ for a.e. } (\mu) x \in \mathbb{C}.$$

That is, we only take  $\nu = \mu$  in (1.7). Then, if we assume also  $\Theta_\mu^*(x) < \infty$  for a.e.  $(\mu) x \in \mathbb{C}$ , we will get that (1.4) holds. We do not know if the condition (1.3) in a)

can be derived from (1.8). Notice also that if  $E \subset \mathbb{C}$ ,  $\mathcal{H}^1(E) < \infty$ , and we choose  $\mu = \mathcal{H}^1|_E$ , then (1.4) implies that  $E$  is rectifiable, by the result of David and Léger [L].

The paper is organized as follows. In Section 2 we prove a theorem of type  $T(1)$  inspired by the Nazarov-Treil-Volberg  $T(b)$  theorem of [NTV], which may have some independent interest. Let us remark that this theorem also holds for other Calderón-Zygmund operators different from the Cauchy integral. In Section 3 we obtain several corollaries from this theorem, and we state precisely and prove the results about principal values mentioned above. The techniques involved to prove these latter results use the notion of curvature of measures and, as a consequence, they cannot be extended easily to other Calderón-Zygmund operators.

## 2. A THEOREM INSPIRED BY NAZAROV-TREIL-VOLBERG RESULTS

The theorem that we will prove in this section will be obtained modifying slightly the ideas of Nazarov, Treil and Volberg in [NTV].

First we introduce some notation. We say that  $B(x, r)$  is a non-Ahlfors disk with respect to some constant  $M > 0$  if  $\mu(B(x, r)) > Mr$ , and  $x \in \mathbb{C}$  is a non-Ahlfors point if there exists *some* non-Ahlfors disk centered at  $x$ .

Let  $\Theta \geq 0$  be some Lipschitz function with Lipschitz constant  $\leq 1$ . Then we denote

$$K_\Theta(x, y) = \frac{\overline{x - y}}{|x - y|^2 + \Theta(x)\Theta(y)},$$

and

$$\mathcal{C}_{\Theta, \varepsilon} \mu(x) = \int_{|x-y|>\varepsilon} K_\Theta(x, y) d\mu(y).$$

Now we will state the  $T(b)$  theorem of [NTV] in the particular case  $T = \mathcal{C}$  and  $b \equiv 1$ , which is a simpler case.

**Theorem 2.1** ([NTV]). *Let  $\mu$  be a positive Radon measure on  $\mathbb{C}$  and let  $M > 0$ ,  $B > 0$ , an open set  $H \subset \mathbb{C}$  with  $\mu(H^c) > 0$ , and  $\Phi : \mathbb{C} \rightarrow [0, \infty)$  a Lipschitz function with constant 1 such that*

- 1) *Every non-Ahlfors disk is contained in  $H$ .*
- 2)  *$\Phi(x) \geq \text{dist}(x, \mathbb{C} \setminus H)$ .*
- 3)  *$\sup_{\varepsilon > 0} |\mathcal{C}_{\Theta, \varepsilon} \mu(x)| \leq B$  for every Lipschitz function  $\Theta \geq \Phi$  with Lipschitz constant 1 and for all  $x \in \mathbb{C}$ .*

*Then,  $\mathcal{C}_{\Phi, \varepsilon}$  is bounded on  $L^2(\mu)$  uniformly on  $\varepsilon > 0$ . In particular, if  $E = \{x : \Phi(x) = 0\}$ , the Cauchy transform is bounded on  $L^2(\mu|_E)$ .*

Notice that in Theorem 2.1, *no* doubling conditions are assumed for the measure  $\mu$ .

We will prove the following result:

**Theorem 2.2.** *Let  $\mu$  be a positive finite Radon measure on  $\mathbb{C}$  satisfying  $\Theta_\mu^*(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$  and such that  $\mathcal{C}_* \mu(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ . Then, for all  $\delta > 0$  there exists a compact set  $E \subset \mathbb{C}$  with  $\mu(\mathbb{C} \setminus E) \leq \delta$  such that the Cauchy integral operator is bounded in  $L^2(\mu|_E)$ .*

We need the following lemma, which is a variation of an estimate of [NTV] concerning the kernel  $K_\Theta$ .

**Lemma 2.3.** *Let  $x \in \mathbb{C}$  and  $r_0 \geq 0$  be such that  $\mu(B(x, r)) \leq C_0 r$  for  $r \geq r_0$  and  $|\mathcal{C}_\varepsilon \mu(x)| \leq C_1$  for  $\varepsilon \geq r_0$ . If  $\Theta(x) \geq r_0$ , then*

$$(2.1) \quad |\mathcal{C}_{\Theta, \varepsilon} \mu(x)| \leq C_2$$

for all  $\varepsilon > 0$ , with  $C_2$  depending only on  $C_0$  and  $C_1$ .

*Proof.* If  $\varepsilon \geq \Theta(x)$ , then

$$\begin{aligned} |\mathcal{C}_{\Theta, \varepsilon} \mu(x) - \mathcal{C}_\varepsilon \mu(x)| &\leq \int_{|x-y|>\varepsilon} \left| \frac{\overline{x-y}}{|x-y|^2 + \Theta(x)\Theta(y)} - \frac{\overline{x-y}}{|x-y|^2} \right| d\mu(y) \\ &= \int_{|x-y|>\varepsilon} \left| \frac{(\overline{x-y})\Theta(x)\Theta(y)}{|x-y|^2(|x-y|^2 + \Theta(x)\Theta(y))} \right| d\mu(y) \\ &\leq \int_{|x-y|>\varepsilon} \frac{\Theta(x)\Theta(y)}{|x-y|^3} d\mu(y) \\ &\leq \int_{|x-y|>\varepsilon} \frac{\Theta(x)(\Theta(x) + |x-y|)}{|x-y|^3} d\mu(y) \\ &= \Theta(x)^2 \int_{|x-y|>\varepsilon} \frac{1}{|x-y|^3} d\mu(y) \\ &\quad + \Theta(x) \int_{|x-y|>\varepsilon} \frac{1}{|x-y|^2} d\mu(y). \end{aligned}$$

Since  $\mu(B(x, r)) \leq C_0 r$  for  $r \geq \varepsilon$ , it is easily checked that

$$\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^3} d\mu(y) \leq C \frac{1}{\varepsilon^2}$$

and

$$\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^2} d\mu(y) \leq C \frac{1}{\varepsilon},$$

where  $C$  depends only on  $C_0$ . Therefore

$$|\mathcal{C}_{\Theta, \varepsilon} \mu(x) - \mathcal{C}_\varepsilon \mu(x)| \leq C \frac{\Theta(x)^2}{\varepsilon^2} + C \frac{\Theta(x)}{\varepsilon} \leq 2C,$$

and so (2.1) holds for  $\varepsilon \geq \Theta(x)$ .

If  $\varepsilon < \Theta(x)$ , then

$$(2.2) \quad |\mathcal{C}_{\Theta, \varepsilon} \mu(x)| \leq \int_{|x-y| \leq \Theta(x)} |K_\Theta(x, y)| d\mu(y) + \left| \int_{|x-y| > \Theta(x)} K_\Theta(x, y) d\mu(y) \right|.$$

To estimate the first integral on the right-hand side we use the inequality (see [NTV])

$$|K_\Theta(x, y)| \leq \frac{1}{\Theta(x)}.$$

The second integral on the right-hand side of (2.2) equals  $\mathcal{C}_{\Theta, \Theta(x)} \mu(x)$ . This term is bounded by some constant, as we have shown in the case  $\varepsilon \geq \Theta(x)$ .  $\square$

*Proof of Theorem 2.2.* We will set  $H = H_1 \cup H_2$ . The set  $H_1$  will contain all non-Ahlfors disks (in fact, it will be quite similar to the set  $H$  of [NTV]). The set  $H_2$  will allow us to obtain a uniform estimate for  $|\mathcal{C}_\varepsilon \mu(x)|$  if  $x \notin H_1 \cup H_2$ .

Let us start the construction of  $H_1$ . Let  $M > 0$  be some big constant which will be fixed below. Let  $H_0$  be the set of non-Ahlfors points with respect to the

constant  $M$  (that is,  $H_0 = \{x \in \mathbb{C} : \exists r > 0 \text{ with } \mu(B(x, r)) > Mr\}$ ). For every non-Ahlfors point  $x \in H_0$ , we set

$$r(x) = \sup\{r > 0 : \mu(B(x, r)) > Mr\},$$

and  $r(x) = 0$  if  $x \notin H_0$ . Then, by Vitali's  $5r$  covering theorem, there exists a disjoint family of open disks  $B(x_i, r(x_i))$ ,  $x_i \in H_0$ , such that every non-Ahlfors disk is contained in

$$H_1 := \bigcup_i B(x_i, 5r(x_i)).$$

Observe that  $H_1$  is open. Notice also that  $\mu(B(x_i, r(x_i))) \geq Mr(x_i)$  and so

$$(2.3) \quad \sum_i r(x_i) \leq \sum_i M^{-1} \mu(B(x_i, r(x_i))).$$

Let us check that we can choose  $M$  so that  $\mu(H_1) \leq \delta/2$ . We set

$$E_n = \{x \in \mathbb{C} : \sup_{r>0} \frac{\mu(B(x, r))}{r} < n\}.$$

Since  $\Theta_\mu^*(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ ,

$$\mu(\mathbb{C} \setminus \bigcup_n E_n) = 0.$$

We take  $m$  such that  $\mu(\mathbb{C} \setminus E_m) \leq \delta/4$ . Then,

$$\mu(H_1) \leq \mu(\mathbb{C} \setminus E_m) + \mu(H_1 \cap E_m) \leq \frac{\delta}{4} + \sum_i \mu(B(x_i, 5r(x_i)) \cap E_m).$$

If  $B(x_i, 5r(x_i)) \cap E_m \neq \emptyset$ , there exists  $z_i \in B(x_i, 5r(x_i)) \cap E_m$  and so

$$\mu(B(x_i, 5r(x_i)) \cap E_m) \leq \mu(B(z_i, 10r(x_i))) \leq 10m r(x_i).$$

Therefore, by (2.3),

$$\mu(H_1) \leq \frac{\delta}{4} + 10m \sum_i r(x_i) \leq \frac{\delta}{4} + \frac{10m \|\mu\|}{M}.$$

Thus if  $M$  is big enough,  $\mu(H_1) \leq \delta/2$ .

Now we turn our attention to  $H_2$ . We define

$$G = \{x \in \mathbb{C} \setminus H_1 : \mathcal{C}_* \mu(x) > k\},$$

with  $k$  some big constant (much bigger than  $M$ ). For every  $x \in G$ , consider  $\varepsilon_0(x) > 0$  such that  $|\mathcal{C}_{\varepsilon_0(x)} \mu(x)| > k$  and  $|\mathcal{C}_\varepsilon \mu(x)| \leq k$  for all  $\varepsilon > 2\varepsilon_0(x)$ , and let  $\varepsilon_0(x) = 0$  if  $x \notin G$ . Then, we set

$$H_2 := \bigcup_{x \in G} B(x, \varepsilon_0(x)).$$

Notice that  $\mathcal{C}_* \mu(x) \leq k$  if  $x \notin H_2$ , because  $G \subset H_2$ .

Let us check that  $\mu(H_2 \setminus H_1)$  tends to 0 as  $k \rightarrow \infty$ . Observe that if  $y \in H_2 \setminus H_1$ , then  $y \in B(x, \varepsilon_0(x))$  for some  $x \in G$ . First we will show that

$$(2.4) \quad |\mathcal{C}_{\varepsilon_0(x)} \mu(x) - \mathcal{C}_{\varepsilon_0(x)} \mu(y)| \leq 12M.$$

In fact, we have

$$(2.5) \quad |\mathcal{C}_{\varepsilon_0(x)}\mu(x) - \mathcal{C}_{\varepsilon_0(x)}\mu(y)| \leq |\mathcal{C}_{\varepsilon_0(x)}\mu|_{B(y,2\varepsilon_0(x))}(x)| + |\mathcal{C}_{\varepsilon_0(x)}\mu|_{B(y,2\varepsilon_0(x))}(y)| \\ + |\mathcal{C}_{\varepsilon_0(x)}\mu|_{\mathbb{C} \setminus B(y,2\varepsilon_0(x))}(x) - \mathcal{C}_{\varepsilon_0(x)}\mu|_{\mathbb{C} \setminus B(y,2\varepsilon_0(x))}(y)|.$$

Notice that the first two terms on the right-hand side are bounded by

$$\frac{\mu(B(y, 2\varepsilon_0(x)))}{\varepsilon_0(x)} \leq 2M,$$

since  $y \notin H_1$ . The last term on the right-hand side of (2.5) is estimated as follows:

$$\int_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))} \left| \frac{1}{z-x} - \frac{1}{z-y} \right| d\mu(z) = \int_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))} \frac{|x-y|}{|z-x||z-y|} d\mu(z) \\ \leq 2\varepsilon_0(x) \int_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))} \frac{1}{|z-y|^2} d\mu(z),$$

where we have applied that  $|x-y| \leq \varepsilon_0(x)$  and  $|z-x| \geq |z-y|/2$  in the last inequality. As  $y \notin H_1$ , we have the following standard estimate:

$$2\varepsilon_0(x) \int_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))} \frac{1}{|z-y|^2} d\mu(z) \\ = 2\varepsilon_0(x) \sum_{k=0}^{\infty} \int_{2^k\varepsilon_0(x) \leq |z-y| < 2^{k+1}\varepsilon_0(x)} \frac{1}{|z-y|^2} d\mu(z) \\ \leq 2\varepsilon_0(x) \sum_{k=0}^{\infty} \frac{\mu(B(y, 2^{k+1}\varepsilon_0(x)))}{2^{2k}\varepsilon_0(x)^2} \\ \leq 8M.$$

So we get

$$|\mathcal{C}_{\varepsilon_0(x)}\mu|_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))}(x) - \mathcal{C}_{\varepsilon_0(x)}\mu|_{\mathbb{C} \setminus B(y, 2\varepsilon_0(x))}(y)| \leq 8M,$$

and (2.4) holds. As a consequence,

$$H_2 \setminus H_1 \subset \{y \in \mathbb{C} : \mathcal{C}_*\mu(y) > k - 12M\}.$$

Since,  $\mathcal{C}_*\mu(y) < \infty$  for a.e.  $(\mu) y \in \mathbb{C}$ , we get that  $\mu\{y \in \mathbb{C} : \mathcal{C}_*\mu(y) > k - 12M\}$  tends to 0 as  $k \rightarrow \infty$ . Thus  $\mu(H_2 \setminus H_1) \leq \delta/2$  if  $k$  is big enough.

We define

$$\Phi(x) = \text{dist}(x, \mathbb{C} \setminus H).$$

As  $H$  is open,  $\Phi(x) > 0$  if and only if  $x \in H$ . Notice also that  $\Phi(\cdot)$  is a Lipschitz function with constant 1 such that  $\Phi(x) \geq \max\{\varepsilon_0(x), r(x)\}$ . Moreover,

$$(2.6) \quad |\mathcal{C}_\varepsilon\mu(x)| \leq k + 2M \text{ if } \varepsilon > \Phi(x).$$

Indeed, if  $\varepsilon > \Phi(x)$ , from the definition of  $\varepsilon_0(x)$ , we get

$$(2.7) \quad |\mathcal{C}_{2\varepsilon}\mu(x)| \leq k.$$

Since  $\varepsilon > r(x)$ , we have

$$(2.8) \quad |\mathcal{C}_\varepsilon\mu(x) - \mathcal{C}_{2\varepsilon}\mu(x)| \leq \frac{\mu(B(x, 2\varepsilon))}{\varepsilon} \leq 2M.$$

Now, from (2.7) and (2.8), we obtain (2.6).

If  $\Theta$  is any Lipschitz function with constant 1 such that  $\Theta(x) \geq \Phi(x)$ , by Lemma 2.3 we get

$$(2.9) \quad \sup_{\varepsilon > 0} |\mathcal{C}_{\Theta, \varepsilon} \mu(x)| \leq B$$

for all  $x \in \mathbb{C}$  and for some constant  $B$  (which may depend on  $M$  and  $k$ ).

Now, by the theorem of Nazarov, Treil and Volberg, we get that  $\mathcal{C}_{\Phi, \varepsilon}$  is bounded in  $L^2(\mu)$ , with a uniform bound for  $\varepsilon > 0$ , and so the Cauchy integral is bounded in  $L^2(\mu|_{\mathbb{C} \setminus H})$ .  $\square$

### 3. SOME COROLLARIES

**Corollary 3.1.** *Let  $\mu$  be a positive finite Radon measure on  $\mathbb{C}$  satisfying  $\Theta_\mu^*(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$  and such that  $\mathcal{C}_* \mu(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ . Then, there is a countable collection of compact sets  $E_n$  such that  $\mu(\mathbb{C} \setminus \bigcup_n E_n) = 0$  and  $c^2(\mu|_{E_n}) < \infty$ .*

*Proof.* By the theorem above, there is a countable collection of compact sets  $E_n$  such that  $\mu(\mathbb{C} \setminus \bigcup_n E_n) = 0$ , with the Cauchy integral bounded on  $L^2(\mu|_{E_n})$ . Then, by (1.2),  $c^2(\mu|_{E_n}) < \infty$  (notice that (1.1) holds because the Cauchy integral is bounded on  $L^2(\mu|_{E_n})$ ).  $\square$

Also we have:

**Corollary 3.2.** *Let  $\mu$  be a positive finite Radon measure on  $\mathbb{C}$  satisfying  $\Theta_\mu^*(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ . Then  $\mathcal{C}_* \mu(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$  if and only if p.v.  $\mathcal{C} \mu(x)$  exists for a.e.  $(\mu)$   $x \in \mathbb{C}$ .*

*Proof.* It is obvious that if the principal value  $\lim_{\varepsilon \rightarrow 0} \mathcal{C}_\varepsilon \mu(x)$  exists, then  $\mathcal{C}_* \mu(x) < \infty$ .

If we assume  $\mathcal{C}_* \mu(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ , then by Corollary 3.1 there is a countable collection of compact sets  $E_n$  such that  $\mu(\mathbb{C} \setminus \bigcup_n E_n) = 0$  and  $c^2(\mu|_{E_n}) < \infty$ . Finally, by the result proved in [T], p.v.  $\mathcal{C} \mu(x)$  exists for a.e.  $(\mu)$   $x \in \mathbb{C}$ .  $\square$

Using the result of David and Léger [L], we get

**Corollary 3.3.** *Let  $\mu$  be a positive finite Radon measure on  $\mathbb{C}$  satisfying  $0 < \Theta_\mu^*(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$  and such that  $\mathcal{C}_* \mu(x) < \infty$  for a.e.  $(\mu)$   $x \in \mathbb{C}$ . Then,  $\mu$  is supported on a rectifiable set.*

From Corollary 3.3 and the result of Mattila and Melnikov [MM] we obtain

**Corollary 3.4.** *Let  $E \subset \mathbb{C}$  be  $\mathcal{H}^1$ -measurable with  $\mathcal{H}^1(E) < \infty$ . Then  $E$  is rectifiable if and only if p.v.  $\mathcal{C}(\mathcal{H}^1|_E)(x)$  exists for a.e.  $x \in \mathbb{C}$  with respect to the measure  $\mathcal{H}^1|_E$ .*

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