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TWIN SOLUTIONS TO SINGULAR BOUNDARY VALUE PROBLEMS

RAVI P. AGARWAL AND DONAL O'REGAN

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Abstract. In this paper we establish the existence of two nonnegative solutions to singular (n,p) and singular $(p,n-p)$ focal boundary value problems. Our nonlinearity $f(t,y)$ may be singular at $y=0$, $t=0$ and/or $t=1$.

1. Introduction

This paper discusses the existence of multiple nonnegative solutions to the singular (n,p) boundary value problem

$$(1.1)\qquad \begin{cases} y^{(n)}(t)+\phi(t)\,f(t,y(t))=0, & 0<t<1,\\ y^{(i)}(0)=0, & 0\le i\le n-2,\\ y^{(p)}(1)=0 \end{cases}$$

and the singular $(p,n-p)$ focal boundary value problem

$$(1.2)\qquad \begin{cases} (-1)^{n-p}\,y^{(n)}(t)=\phi(t)\,f(t,y(t)), & 0<t<1,\\ y^{(i)}(0)=0, & 0\le i\le p-1,\\ y^{(i)}(1)=0, & p\le i\le n-1; \end{cases}$$

here $n\ge 2$ and $1\le p\le n-1$ is fixed. All the papers in the literature on singular problems (see [1]–[7] and their references) are devoted to establishing the existence of one solution to singular boundary value problems. This is the first paper, to our knowledge, that establishes the existence of more than one solution to (1.1) and (1.2) even in the case when $n=2$. The technique presented to guarantee the existence of twin nonnegative solutions to (1.1) and (1.2) is new and involves combining (i) an existence result from the literature (which relies on a Leray–Schauder alternative), (ii) Krasnoselskii's fixed point theorem in a cone, and (iii) lower type inequalities.

For the remainder of this introduction we present some results from the literature which will be needed in Section 2. Let $k(t,s)$ be the Green's function for

$$(1.3)\qquad \begin{cases} -y^{(n)}=0 \quad \text{on } [0,1],\\ y^{(i)}(0)=0, \quad 0\le i\le n-2,\\ y^{(p)}(1)=0; \end{cases}$$

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see [4] for an explicit representation. It is well known [4] that

$$k^{(i)}(t,s) \geq 0 \quad \text{for} \quad (t,s) \in [0,1]\times[0,1] \quad \text{and} \quad 0 \leq i \leq p; \quad \text{here} \quad k^{(i)} = \frac{\partial^i}{\partial t^i}\, k.$$

In [3] we proved the following lower type inequality.

Theorem 1.1. *Suppose* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *satisfies*

$$\begin{cases} y^{(n)}(t) \leq 0 \quad \textit{for} \quad t \in [0,1], \\ y(0) = a \geq 0, \\ y^{(i)}(0) = 0, \ \ 1 \leq i \leq n-2, \\ y^{(p)}(1) = 0. \end{cases}$$

Then

$$y(t) \geq t^{n-1}\,|y|_0 \ \ (= t^{n-1}\, y(1)) \quad \textit{for} \quad t \in [0,1]; \tag{1.4}$$

here $|y|_0 = \sup_{t\in[0,1]} |y(t)|$.

Next let $G(t,s)$ be the Green's function for

$$\begin{cases} y^{(n)} = 0 \quad \text{on} \quad [0,1], \\ y^{(i)}(0) = 0, \ \ 0 \leq i \leq p-1, \\ y^{(i)}(1) = 0, \ \ p \leq i \leq n-1; \end{cases} \tag{1.5}$$

see [4] for an explicit representation. It is well known [4] that for $(t,s) \in [0,1]\times[0,1]$,

$$(-1)^{n-p}\, G^{(i)}(t,s) \geq 0, \ \ 0 \leq i \leq p-1,$$

and

$$(-1)^{n-i}\, G^{(i)}(t,s) \geq 0, \ \ p \leq i \leq n-1.$$

In [2] we proved the following lower type inequality.

Theorem 1.2. *Suppose* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *satisfies*

$$\begin{cases} (-1)^{n-p} y^{(n)}(t) \geq 0 \quad \textit{for} \quad t \in (0,1), \\ y(0) = a \geq 0, \\ y^{(i)}(0) = 0, \ \ 1 \leq i \leq p-1, \\ y^{(i)}(1) = 0, \ \ p \leq i \leq n-1. \end{cases}$$

Then

$$y(t) \geq t^{p}\,|y|_0 \ \ (= t^{p}\, y(1)) \quad \textit{for} \quad t \in [0,1]. \tag{1.6}$$

In [2] we proved the following existence result for the singular $(p, n-p)$ focal problem (1.2).

Theorem 1.3. *Suppose the following conditions are satisfied:*

$$\phi \in C(0,1) \quad \textit{with} \quad \phi > 0 \quad \textit{on} \quad (0,1) \quad \textit{and} \quad \phi \in L^1[0,1], \tag{1.7}$$

$$f : [0,1]\times(0,\infty) \to (0,\infty) \quad \textit{is continuous}, \tag{1.8}$$

$$\begin{cases} f(t,y) \leq g(y) + h(y) & \textit{on} \ \ [0,1]\times(0,\infty) \ \textit{with} \ g > 0 \ \textit{continuous} \\ & \textit{and nonincreasing on} \ (0,\infty), \ \ h \geq 0 \ \textit{continuous} \\ & \textit{on} \ \ [0,\infty) \ \ \textit{and} \ \ \frac{h}{g} \ \ \textit{nondecreasing on} \ \ (0,\infty), \end{cases} \tag{1.9}$$

(1.10)
$$\begin{cases} \textit{for each constant } H > 0 \textit{ there exists } \psi_H \textit{ continuous on } [0,1] \\ \textit{and positive on } (0,1) \textit{ such that } f(t,y) \geq \psi_H(t) \textit{ on } [0,1] \times (0,H], \end{cases}$$

$$\int_0^1 \phi(s)\, g(s^p)\, ds < \infty, \tag{1.11}$$

(1.12)
there exists a constant $K_0 > 0$ *with* $g(a\,b) \leq K_0\, g(a)\, g(b)$ *for all* $a \geq 0,\ b \geq 0$,

and

$$\textit{there exists a constant } r > 0 \textit{ with } \frac{r}{g(r) + h(r)} > b_0\, K_0; \tag{1.13}$$

here

$$b_0 = \sup_{t \in [0,1]} \int_0^1 (-1)^{n-p}\, G(t,s)\, \phi(s)\, g(s^p)\, ds. \tag{1.14}$$

Then (1.2) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y > 0$ *on* $(0,1]$ *and* $|y|_0 < r$.

Essentially the same reasoning as in [2] establishes the corresponding result for the singular (n,p) problem (1.1).

Theorem 1.4. *Suppose* (1.7)–(1.10) *and* (1.12) *hold. In addition assume*

$$\int_0^1 \phi(s)\, g\left(s^{n-1}\right)\, ds < \infty \tag{1.15}$$

and

$$\textit{there exists a constant } r > 0 \textit{ with } \frac{r}{g(r) + h(r)} > c_0\, K_0 \tag{1.16}$$

are satisfied; here

$$c_0 = \sup_{t \in [0,1]} \int_0^1 k(t,s)\, \phi(s)\, g\left(s^{n-1}\right)\, ds. \tag{1.17}$$

Then (1.1) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y > 0$ *on* $(0,1]$ *and* $|y|_0 < r$.

Finally we state for completeness Krasnoselskii's fixed point theorem in a cone.

Theorem 1.5. *Let* $E = (E, \|\,.\,\|)$ *be a Banach space and let* $K \subset E$ *be a cone in* E. *Assume* Ω_1 *and* Ω_2 *are open subsets of* E *with* $0 \in \Omega_1$ *and* $\overline{\Omega_1} \subset \Omega_2$ *and let* $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ *be continuous and completely continuous. In addition suppose either*

$$\|A\,u\| \leq \|u\| \textit{ for } u \in K \cap \partial\Omega_1 \textit{ and } \|A\,u\| \geq \|u\| \textit{ for } u \in K \cap \partial\Omega_2$$

or

$$\|A\,u\| \geq \|u\| \textit{ for } u \in K \cap \partial\Omega_1 \textit{ and } \|A\,u\| \leq \|u\| \textit{ for } u \in K \cap \partial\Omega_2$$

hold. Then A *has a fixed point in* $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

2. Singular problems

In this section we begin by discussing the singular (n,p) problem

$$\left\{ \begin{array}{l} y^{(n)}(t) + \phi(t)\,[g(y(t)) + h(y(t))] = 0, \;\; 0 < t < 1, \\ y^{(i)}(0) = 0, \;\; 0 \le i \le n-2, \\ y^{(p)}(1) = 0 \end{array} \right. \tag{2.1}$$

with $n \ge 2$ and $1 \le p \le n-1$ is fixed.

Theorem 2.1. *Assume the following conditions are satisfied:*

$$\phi \in C(0,1) \;\; \textit{with} \;\; \phi > 0 \;\; \textit{on} \;\; (0,1) \;\; \textit{and} \;\; \phi \in L^1[0,1], \tag{2.2}$$

$$g > 0 \;\; \textit{is continuous and nonincreasing on} \;\; (0,\infty), \tag{2.3}$$

$$h \ge 0 \;\; \textit{is continuous on} \;\; [0,\infty) \;\; \textit{with} \;\; \frac{h}{g} \;\; \textit{nondecreasing on} \;\; (0,\infty), \tag{2.4}$$

$$\int_0^1 \phi(s)\, g\left(s^{n-1}\right)\, ds < \infty, \tag{2.5}$$

$$\textit{there exists a constant} \;\; K_0 > 0 \;\; \textit{with} \;\; g(a\,b) \le K_0\, g(a)\, g(b) \;\; \textit{for all} \;\; a \ge 0,\; b \ge 0 \tag{2.6}$$

and

$$\textit{there exists a constant} \;\; r > 0 \;\; \textit{with} \;\; \frac{r}{g(r) + h(r)} > c_0\, K_0; \tag{2.7}$$

here

$$c_0 = \sup_{t \in [0,1]} \int_0^1 k(t,s)\, \phi(s)\, g\left(s^{n-1}\right)\, ds \tag{2.8}$$

where $k(t,s)$ is the Green's function for (1.3). *Then* (2.1) *has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ with $y > 0$ on $(0,1]$ and $|y|_0 < r$.*

Proof. The result follows from Theorem 1.4 with $f(t,u) = g(u) + h(u)$. Notice (1.10) is clearly satisfied with $\psi_H(t) = g(H)$. □

Theorem 2.2. *Assume* (2.2)–(2.7) *hold. In addition suppose*

$$\textit{there exists} \;\; R > r \;\; \textit{with} \;\; \frac{R\, g\left(\frac{R}{4^{n-1}}\right)}{g(R)\, g\left(\frac{R}{4^{n-1}}\right) + g(R)\; h\left(\frac{R}{4^{n-1}}\right)} \le \int_{\frac{1}{4}}^1 k(\sigma,s)\, \phi(s)\, ds; \tag{2.9}$$

here $0 \le \sigma \le 1$ is such that

$$\int_{\frac{1}{4}}^1 k(\sigma,s)\, \phi(s)\, ds = \sup_{t \in [0,1]} \int_{\frac{1}{4}}^1 k(t,s)\, \phi(s)\, ds. \tag{2.10}$$

Then (2.1) *has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ with $y > 0$ on $(0,1]$ and $r < |y|_0 \le R$.*

Remark 2.1. It is possible to replace (2.9) with

$(2.9)^\star$

there exists $R > r$ with $\dfrac{R\, g\left(R\, a^{n-1}\right)}{g(R)\, g\left(R\, a^{n-1}\right) + g(R)\ h\left(R\, a^{n-1}\right)} \le \displaystyle\int_a^1 k(\eta, s)\, \phi(s)\, ds;$

here $0 < a < \frac{1}{2}$ is fixed and $\eta \in [0,1]$ is such that

$$\int_a^1 k(\eta, s)\, \phi(s)\, ds = \sup_{t\in[0,1]} \int_a^1 k(t,s)\, \phi(s)\, ds.$$

Proof. To show the existence of the solution described in the statement of Theorem 2.2 we will apply Theorem 1.5. First however choose $\epsilon > 0$ and $\epsilon < r$ with

$$\frac{r}{\epsilon + c_0\, K_0\, [g(r) + h(r)]} > 1. \tag{2.11}$$

Let $m_0 \in \{1, 2, ...\}$ be chosen so that $\frac{1}{m_0} < \epsilon$ and $\frac{1}{m_0} < \frac{R}{4^{n-1}}$ and let $N_0 = \{m_0, m_0 + 1, ...\}$. We first show that

$$(2.12)^m \qquad \begin{cases} y^{(n)}(t) + \phi(t)\, [g(y(t)) + h(y(t))] = 0, \ \ 0 < t < 1, \\ y(0) = \frac{1}{m}, \\ y^{(i)}(0) = 0, \ \ 1 \le i \le n-2, \\ y^{(p)}(1) = 0 \end{cases}$$

has a solution y_m for each $m \in N_0$ with $y_m > \frac{1}{m}$ on $(0,1]$ and $r \le |y_m|_0 \le R$.

To show $(2.12)^m$ has such a solution for each $m \in N_0$, we will look at

$$(2.13)^m \qquad \begin{cases} y^{(n)}(t) + \phi(t)\, [g^\star(y(t)) + h(y(t))] = 0, \ \ 0 < t < 1, \\ y(0) = \frac{1}{m}, \\ y^{(i)}(0) = 0, \ \ 1 \le i \le n-2, \\ y^{(p)}(1) = 0 \end{cases}$$

with

$$g^\star(u) = \begin{cases} g(u), \ \ u \ge \frac{1}{m}, \\ g\left(\frac{1}{m}\right), \ \ 0 \le u \le \frac{1}{m}. \end{cases}$$

Remark 2.2. Notice $g^\star(u) \le g(u)$ for $u > 0$.

Fix $m \in N_0$. Let $E = (C[0,1], |\,.\,|_0)$ and

(2.14)

$$K = \left\{u \in C[0,1] : \ u(t) \ge 0 \ \text{ for } \ t \in [0,1] \ \text{ and } \ u(t) \ge t^{n-1}\, |u|_0 \ \text{ for } \ t \in [0,1]\right\}.$$

Clearly K is a cone of E. Let $A : K \to C[0,1]$ be defined by

$$A\, y(t) = \frac{1}{m} + \int_0^1 k(t,s)\, \phi(s)\, [g^\star(y(s)) + h(y(s))]\, ds. \tag{2.15}$$

A standard argument [7] implies $A : K \to C[0,1]$ is continuous and completely continuous. Next we show $A : K \to K$. If $u \in K$, then clearly $A\, u(t) \ge 0$ for $t \in [0,1]$. Also notice that

$$\begin{cases} (A\, u)^{(n)}(t) \le 0 \ \ \text{ on } \ (0,1), \\ A\, u(0) = \frac{1}{m}, \\ (A\, u)^{(i)}(0) = 0, \ \ 1 \le i \le n-2, \\ (A\, u)^{(p)}(1) = 0, \end{cases}$$

and so Theorem 1.1 implies $A\,u(t) \ge t^{n-1}\,|A\,u|_0$ for $t \in [0,1]$. Consequently $A\,u \in K$ so $A : K \to K$. Let

$$\Omega_1 = \{u \in C[0,1] : \ |u|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in C[0,1] : \ |u|_0 < R\}.$$

We first show

$$|A\,y|_0 \le |y|_0 \ \text{ for } \ y \in K \cap \partial\Omega_1. \tag{2.16}$$

To see this let $y \in K \cap \partial\Omega_1$ so $y \in K$, $|y|_0 = r$ and $y(t) \ge t^{n-1}\,r$ for $t \in [0,1]$. Also notice

$$g^\star(y(t)) + h(y(t)) \le g(y(t)) + h(y(t)) \quad \text{for} \ \ t \in (0,1)$$

since g is nonincreasing on $(0,\infty)$. Now for $t \in [0,1]$,

$$\begin{aligned} A\,y(t) &= \frac{1}{m} + \int_0^1 k(t,s)\,\phi(s)\,[g^\star(y(s)) + h(y(s))]\,ds \\ &\le \epsilon + \int_0^1 k(t,s)\,\phi(s)\,[g(y(s)) + h(y(s))]\,ds \\ &\le \epsilon + \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^1 k(t,s)\,\phi(s)\,g(y(s))\,ds \\ &\le \epsilon + \left\{1 + \frac{h(r)}{g(r)}\right\} K_0\,g(r) \int_0^1 k(t,s)\,\phi(s)\,g\left(s^{n-1}\right)\,ds \\ &\le \epsilon + [g(r) + h(r)]\,K_0 \sup_{t\in[0,1]} \int_0^1 k(t,s)\,\phi(s)\,g\left(s^{n-1}\right)\,ds \end{aligned}$$

using (2.3), (2.4), (2.6) and the fact that $y(t) \ge t^{n-1}\,r$ for $t \in [0,1]$. Consequently

$$|A\,y|_0 \le \epsilon + [g(r) + h(r)]\,K_0\,c_0 \, < \, r = |y|_0$$

using (2.11). Hence (2.16) is true.

Next we show

$$|A\,y|_0 \ge |y|_0 \ \text{ for } \ y \in K \cap \partial\Omega_2. \tag{2.17}$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(t) \ge t^{n-1}\,R$ for $t \in [0,1]$. In particular

$$y(t) \in \left[\frac{R}{4^{n-1}}\,,\,R\right] \quad \text{for} \ \ t \in \left[\frac{1}{4},1\right]. \tag{2.18}$$

Also for $s \in \left[\frac{1}{4},1\right]$ we have

$$g^\star(y(s)) + h(y(s)) = g(y(s)) + h(y(s))$$

since $y(s) \geq \frac{R}{4^{n-1}} > \frac{1}{m}$ for $s \in \left[\frac{1}{4}, 1\right]$. With σ as defined in (2.10) we have, using (2.18) and (2.9),

$$\begin{aligned} A\,y(\sigma) &= \frac{1}{m} + \int_0^1 k(\sigma,s)\,\phi(s)\,[g^\star(y(s)) + h(y(s))]\,ds \\ &\geq \int_{\frac{1}{4}}^1 k(\sigma,s)\,\phi(s)\,[g^\star(y(s)) + h(y(s))]\,ds \\ &= \int_{\frac{1}{4}}^1 k(\sigma,s)\,\phi(s)\,[g(y(s)) + h(y(s))]\,ds \\ &= \int_{\frac{1}{4}}^1 k(\sigma,s)\,\phi(s)\,\left\{1 + \frac{h(y(s))}{g(y(s))}\right\}\,g(y(s))\,ds \\ &\geq \left\{1 + \frac{h\left(\frac{R}{4^{n-1}}\right)}{g\left(\frac{R}{4^{n-1}}\right)}\right\}\,g(R)\,\int_{\frac{1}{4}}^1 k(\sigma,s)\,\phi(s)\,ds \\ &\geq R = |y|_0 \end{aligned}$$

and so $|A\,y|_0 \geq |y|_0$. Hence (2.17) is true.

Now Theorem 1.5 implies A has a fixed point $y_m \in K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$, i.e. $r \leq |y_m|_0 \leq R$. In fact $|y_m|_0 > r$ (this is a standard argument since (2.11) holds [see the end of this Theorem]) and $y_m(t) \geq \frac{1}{m}$ for $t \in [0,1]$ (see (2.15)). Consequently $(2.13)^m$ (and also $(2.12)^m$) has a solution $y_m \in C^{n-1}[0,1] \cap C^n(0,1)$, $y_m \in K$, with

(2.19)
$$\frac{1}{m} \leq y_m(t) \quad \text{for} \quad t \in [0,1], \quad r < |y_m|_0 \leq R, \quad \text{and} \quad y_m(t) \geq t^{n-1}\,r \quad \text{for} \quad t \in [0,1].$$

Also notice for $t \in (0,1)$ that

(2.20)
$$-\,y_m^{(n)}(t) = \phi(t)\,[g(y_m(t)) + h(y_m(t))] \leq \phi(t)\,g\left(t^{n-1}\,r\right)\,\left\{1 + \frac{h(R)}{g(R)}\right\}.$$

It is immediate from (2.19) and (2.20) (with of course (2.5)) that

(2.21) $\{y_m\}_{m \in N_0}$ is a bounded, equicontinuous family on $[0,1]$.

The Arzela–Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y \in C[0,1]$ with y_m converging uniformly on $[0,1]$ to y as $m \to \infty$ through N. Also $y(0) = 0$, $r \leq |y|_0 \leq R$ and $y(t) \geq t^{n-1}\,r$ for $t \in [0,1]$. In particular $y > 0$ on $(0,1]$. Now y_m, $m \in N$, satisfies

(2.22)
$$y_m(t) = \frac{1}{m} + \int_0^1 k(t,s)\,\phi(s)\,[g(y_m(s)) + h(y_m(s))]\,ds \quad \text{for} \quad t \in [0,1].$$

Also as in (2.20),

$$\phi(s)\,[g(y_m(s)) + h(y_m(s))] \leq \phi(s)\,g\left(s^{n-1}\,r\right)\,\left\{1 + \frac{h(R)}{g(R)}\right\} \in L^1[0,1].$$

Now let $m \to \infty$ through N in (2.22) to obtain (here we use the Lebesgue Dominated Convergence Theorem),

$$y(t) = \int_0^1 k(t,s)\,\phi(s)\,[g(y(s)) + h(y(s))]\,ds \quad \text{for} \quad t \in [0,1]. \tag{2.23}$$

From (2.23) we deduce immediately that $y \in C^{n-1}[0,1] \cap C^n(0,1)$, $y^{(n)}(t) + \phi(t)[g(y(t)) + h(y(t))] = 0$ for $0 < t < 1$, $y^{(i)}(0) = 0$ for $1 \le i \le n-2$ and $y^{(p)}(1) = 0$. Finally $|y|_0 > r$. To see this suppose $|y|_0 = r$. Then for $t \in [0,1]$,

$$\begin{aligned} y(t) &= \int_0^1 k(t,s)\,\phi(s)\,[g(y(s)) + h(y(s))]\,ds \\ &\le \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^1 k(t,s)\,\phi(s)\,g(y(s))\,ds \\ &\le \left\{1 + \frac{h(r)}{g(r)}\right\} K_0\, g(r) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,\phi(s)\,g\left(s^{n-1}\right)\,ds \end{aligned}$$

since $y(t) \ge t^{n-1}\,r$ for $t \in [0,1]$. Thus

$$r = |y|_0 \le [g(r) + h(r)]\,K_0\,c_0,$$

and this contradicts (2.7). □

Remark 2.3. If in (2.9) we have $R < r$, then (2.1) has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1)$ with $y > 0$ on $(0,1]$ and $R \le |y|_0 < r$. The argument is similar to that in Theorem 2.2 except here we use the other half of Theorem 1.5.

Remark 2.4. It is also possible to use the ideas in Theorem 2.2 to discuss the more general problem

$$\begin{cases} y^{(n)}(t) + \phi(t)\,f(t,y(t)) = 0, & 0 < t < 1, \\ y^{(i)}(0) = 0, & 0 \le i \le n-2, \\ y^{(p)}(1) = 0. \end{cases}$$

Theorem 2.3. *Assume* (2.2)–(2.7) *and* (2.9) *hold. Then* (2.1) *has two solutions* $y_1,\, y_2 \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y_1 > 0$, $y_2 > 0$ *on* $(0,1]$ *and* $|y_1|_0 < r < |y_2|_0 \le R$.

Proof. The existence of y_1 follows from Theorem 2.1 and the existence of y_2 follows from Theorem 2.2. □

Example 2.1. The singular boundary value problem

$$\begin{cases} y'' + \frac{1}{3}\left(y^{-\alpha} + y^{\beta} + 1\right) = 0 \quad \text{on} \quad (0,1), \\ y(0) = y'(1) = 0, \quad 0 < \alpha < 1 < \beta \end{cases} \tag{2.24}$$

has two solutions $y_1,\, y_2 \in C^1[0,1] \cap C^2(0,1)$ with $y_1 > 0$, $y_2 > 0$ on $(0,1]$ and $|y_1|_0 < 1 < |y_2|_0$.

To see this we will apply Theorem 2.3 with $n = 2$, $p = 1$, $\phi = \frac{1}{3}$, $g(u) = u^{-\alpha}$ and $h(u) = u^{\beta} + 1$. Clearly (2.2), (2.3), (2.4), (2.5) (since $0 < \alpha < 1$) and (2.6) (with $K_0 = 1$) hold. In this situation

$$k(t,s) = \begin{cases} s, & 0 \le s \le t, \\ t, & t \le s \le 1, \end{cases}$$

so

$$c_0 = \frac{1}{3} \sup_{t\in[0,1]} \left[\int_0^t s^{1-\alpha}\, ds + t \int_t^1 s^{-\alpha}\, ds \right] = \frac{1}{3}\, \frac{1}{2-\alpha}.$$

Now (2.7) holds (with $r = 1$) since if $r = 1$,

$$\frac{r}{g(r)+h(r)} = \frac{r}{r^{-\alpha}+r^{\beta}+1} = \frac{1}{3} > \frac{1}{3}\, \frac{1}{2-\alpha}.$$

Finally since (note $\beta > 1$),

$$\lim_{R\to\infty} \frac{R\, g\left(\frac{R}{4}\right)}{g(R)\, g\left(\frac{R}{4}\right) + g(R)\, h\left(\frac{R}{4}\right)} = \lim_{R\to\infty} \left(\frac{R^{\alpha+1}\, 4^{\alpha}}{4^{\alpha} + 4^{-\beta}\, R^{\alpha+\beta} + R^{\alpha}} \right) = 0,$$

there exists $R > 1$ so that (2.9) holds. The result now follows from Theorem 2.3.

Remark 2.5. The result in Theorem 2.2 extends to the nonsingular problem, i.e. when $g \equiv 0$. The proof is a lot easier in this case. In fact one can deal directly here with (2.1) (with $g \equiv 0$), i.e. there is no need to modify the problem as in $(2.12)^m$ (with $g \equiv 0$).

Next we discuss the singular $(p, n-p)$ focal boundary value problem

$$(2.25) \qquad \begin{cases} (-1)^{n-p}\, y^{(n)}(t) = \phi(t)\, [g(y(t)) + h(y(t))], & 0 < t < 1, \\ y^{(i)}(0) = 0, & 0 \le i \le p-1, \\ y^{(i)}(1) = 0, & p \le i \le n-1. \end{cases}$$

Theorem 2.4. *Assume* (2.2)–(2.4) *and* (2.6) *are satisfied. In addition assume*

$$(2.26) \qquad \int_0^1 \phi(s)\, g\left(s^p\right)\, ds < \infty$$

and

$$(2.27) \qquad \text{there exists a constant } r > 0 \text{ with } \frac{r}{g(r)+h(r)} > b_0\, K_0$$

hold; here

$$(2.28) \qquad b_0 = \sup_{t\in[0,1]} \int_0^1 (-1)^{n-p}\, G(t,s)\, \phi(s)\, g\left(s^p\right)\, ds$$

where $G(t,s)$ *is the Green's function for* (1.5). *Then* (2.25) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y > 0$ *on* $(0,1]$ *and* $|y|_0 < r$.

Proof. The result follows from Theorem 1.3. □

Theorem 2.5. *Assume* (2.2)–(2.4), (2.6), (2.26) *and* (2.27) *hold. In addition suppose*

(2.29)

$$\text{there exists } R > r \text{ with } \frac{R\, g\left(\frac{R}{4^p}\right)}{g(R)\, g\left(\frac{R}{4^p}\right) + g(R)\, h\left(\frac{R}{4^p}\right)} \le \int_{\frac{1}{4}}^1 (-1)^{n-p}\, G(\sigma,s)\, \phi(s)\, ds;$$

here $0 \le \sigma \le 1$ *is such that*

(2.30)

$$\int_{\frac{1}{4}}^1 (-1)^{n-p}\, G(\sigma,s)\, \phi(s)\, ds = \sup_{t\in[0,1]} \int_{\frac{1}{4}}^1 (-1)^{n-p}\, G(t,s)\, \phi(s)\, ds.$$

Then (2.25) *has a solution* $y \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y > 0$ *on* $(0,1]$ *and* $r < |y|_0 \le R$.

Proof. The proof is similar to that in Theorem 2.2. In this case

$$K = \{u \in C[0,1] : \; u(t) \geq 0 \;\text{ for }\; t \in [0,1] \;\text{ and }\; u(t) \geq t^p \, |u|_0 \;\text{ for }\; t \in [0,1]\} . \qquad \square$$

Theorem 2.6. *Assume* (2.2)–(2.4)*,* (2.6)*,* (2.26)*,* (2.27) *and* (2.29) *hold. Then* (2.25) *has two solutions* $y_1, y_2 \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $y_1 > 0$*,* $y_2 > 0$ *on* $(0,1]$ *and* $|y_1|_0 < r < |y_2|_0 \leq R$*.*

Remark 2.6. By imposing other conditions on the nonlinearity $h + g$ it is easy to establish the existence of more than two solutions to (2.1) and (2.25).

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Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260

E-mail address: matravip@nus.edu.sg

Department of Mathematics, National University of Ireland, Galway, Ireland

E-mail address: donal.oregan@nuigalway.ie