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ON THE DIOPHANTINE EQUATION $x^p + 2^{2m} = py^2$

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ABSTRACT. Let p be an odd prime. In this paper, using some theorems of Adachi and the author, we prove that if $p \equiv 1 \pmod 4$ and $p \nmid B_{(p-1)/2}$, then the equation $x^p + 1 = py^2$, $y \neq 0$, and the equation $x^p + 2^{2m} = py^2$, $m \in \mathbb{N}$, $\gcd(x, y) = 1$, $p \mid y$, have no integral solutions respectively. Here $B_{(p-1)/2}$ is $(p-1)/2$th Bernoulli number.

§1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively, and let p be an odd prime. In [1], Nagell proved that if $D \in \mathbb{N}$ with $D > 2$ square-free, and $p \nmid h(-D)$, where $h(-D)$ is the class number of $\mathbb{Q}(\sqrt{-D})$, then the equation

$$x^p - 1 = Dy^2,\ x, y \in \mathbb{Z},\ xy \neq 0,$$

has no solutions with $2 \nmid x$. In [2], we proved that if $p > 3$, $D \in \mathbb{N}$ is a square-free integer which is not divisible by primes of the form $2mp + 1$, then the equation

$$x^p + C = Dy^2,\ C \in \{-1, 1\},\ x, y \in \mathbb{Z},\ xy \neq 0,$$

has no solutions with $2p \nmid y$, except $1^p + 1 = 2 \cdot (\pm 1)^2$ and $3^5 - 1 = 2 \cdot (\pm 11)^2$. In 1995, Le [3] proved that if $p > 3$, then the equation

$$x^p - 2^n = py^2,\ x, y, n \in \mathbb{N},\ \gcd(x, y) = 1,$$

has no solutions. Rabinowitz [4] found all solutions of the equations

$$x^3 \pm 2^n = 3y^2,\ x, y, n \in \mathbb{Z}.$$

Now, from [1] and [2] we have that the equation

$$x^p - 1 = py^2,\ x, y \in \mathbb{Z},$$

has no solutions since the class number $h(-p)$ of $\mathbb{Q}(\sqrt{-p})$ is less than p.

In this paper, we will discuss the solvability of the equations

$$x^p + 1 = py^2,\ x, y \in \mathbb{Z},\ y \neq 0, \tag{1}$$

and

$$x^p + 2^{2m} = py^2,\ x, y \in \mathbb{Z},\ m \in \mathbb{N},\ \gcd(x, y) = 1, \tag{2}$$

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by using a theorem of Adachi [5] and a result of the author [6], [7] on Pell's equation. Let B_n denote nth Bernoulli number. We prove the following:

Theorem. *If $p \equiv 1 \pmod 4$ and $p \nmid B_{(p-1)/2}$, then we have that*
(A) *equation (1) has no solutions;*
(B) *equation (2) has no solutions with $p \mid y$.*

§2. Preliminaries

We use the following lemmas to prove our theorem.

Lemma 1 (Adachi [5]). *If $p \equiv 1 \pmod 4$, $p \nmid B_{(p-1)/2}$ and $p \mid y$, then all solutions of the equation*

$$x^2 - py^2 = z^p,\ x, y, z \in \mathbb{Z},\ \gcd(x, y) = 1,\ 2 \nmid x + y,$$

are given by $z = a^2 - pb^2$,

$$x + y\sqrt{p} = (a + b\sqrt{p})^p,$$

where $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$.

Lemma 2 (Cao [6], [7]). *Let $a, b \in \mathbb{N}$ with ab not a square. If $c \in \{1, 2, 4\}$, $1 < a \neq c$ and there exist $x, y \in \mathbb{N}$ such that*

$$ax^2 - by^2 = c,\ x \mid^* a \text{ or } y \mid^* b,$$

where symbol $x \mid^* a$ means that each prime factor of x divides a, then

$$\frac{ax^2 + by^2}{2} + xy\sqrt{ab} = \begin{cases} \frac{1}{2}\varepsilon \quad \text{or} \quad \frac{1}{2}\varepsilon^3, & \text{for } c = 1, \\ \varepsilon \quad \text{or} \quad \varepsilon^3, & \text{for } c = 2, \\ \Omega \quad \text{or} \quad \frac{1}{4}\Omega^3, & \text{for } c = 4, \end{cases}$$

except $(a, b, c, x, y) = (5, 1, 4, 5, 11)$. Here $\varepsilon = u_0 + v_0\sqrt{ab}$ and $\Omega = U_0 + V_0\sqrt{ab}$ is the least positive integer solution of Pell's equation $u^2 - abv^2 = 1$ and $U^2 - abV^2 = 4$, respectively.

Lemma 3. *Let $p \equiv 1 \pmod 4$, and $D_1, D_2 \in \mathbb{Z}$. If $m \in \mathbb{N}$ and $2 \nmid D_1x^2 - D_2y^2$, then the equation*

$$2^{m+1}\sqrt{D_1} = (x\sqrt{D_1} + y\sqrt{D_2})^p + (x\sqrt{D_1} - y\sqrt{D_2})^p,\ x, y \in \mathbb{Z}, \tag{3}$$

has no solutions.

Proof. From the binomial formula and (3), we obtain that

$$\begin{aligned} 2^{m+1}\sqrt{D_1} &= \sum_{j=0}^{p} \binom{p}{j} (x\sqrt{D_1})^{p-j}(y\sqrt{D_2})^j + \sum_{j=0}^{p} \binom{p}{j} (x\sqrt{D_1})^{p-j}(-y\sqrt{D_2})^j \\ &= 2\sum_{j=0}^{(p-1)/2} \binom{p}{2j} (x\sqrt{D_1})^{p-2j}(y\sqrt{D_2})^{2j} \\ &= 2x\sqrt{D_1}\sum_{j=0}^{(p-1)/2} \binom{p}{2j} (D_1x^2)^{(p-1)/2-j}(D_2y^2)^j. \end{aligned}$$

Hence, we have $x = \pm 2^m$, $2 \nmid D_2y^2$, and

$$\pm 1 = \sum_{j=0}^{(p-1)/2} \binom{p}{2j} (2^{2m}D_1)^{(p-1)/2-j} (D_2y^2)^j \tag{4}$$
$$= \sum_{j=0}^{(p-1)/2} \binom{p}{p-1-2j} (2^{2m}D_1)^{j} (D_2y^2)^{(p-1)/2-j}$$
$$= \sum_{j=0}^{(p-1)/2} \binom{p}{2j+1} (2^{2m}D_1)^{j} (D_2y^2)^{(p-1)/2-j}.$$

Let $2^a \| p-1$. Then $a \geq 2$ since $p \equiv 1 (\mathrm{mod}\ 4)$. Since $2 \nmid D_2y^2$, we have

$$(D_2y^2)^{(p-1)/2} = ((D_2y^2)^{(p-1)/2^a})^{2^{a-1}} \equiv 1 \ (\mathrm{mod}\ 2^{a+1}). \tag{5}$$

Also, let $2^b \| 2j$, $b \geq 1$. Then we have $2j \geq b+1$ for $j \geq 1$. So if $j \geq 1$, then

$$\binom{p}{2j+1} 2^{2mj} = \frac{p(p-1)}{(2j+1)(2j)} \binom{p-2}{2j-1} 2^{2mj} \tag{6}$$
$$\equiv 0 \ (\mathrm{mod}\ 2^{a+1}).$$

Thus, from (4), (5) and (6) we have

$$\pm 1 = p(D_2y^2)^{(p-1)/2}$$
$$+ \sum_{j=1}^{(p-1)/2} \binom{p}{2j+1} (2^{2mj}D_1^j)(D_2y^2)^{(p-1)/2-j}$$
$$\equiv p \equiv 2^a + 1 \ (\mathrm{mod}\, 2^{a+1}),$$

which is impossible since $a \geq 2$.

This completes the proof of Lemma 3.

§3. Proof of the Theorem

Proof of (A). Suppose (x, y, p) is a solution of equation (1). Then

$$1 - py^2 = (-x)^p, \ x, y \in \mathbb{Z}, \ y \neq 0. \tag{7}$$

From [2] and (1), we see that $2p \mid y$. Hence, by Lemma 1, we get from (7) that

$$1 + y\sqrt{p} = (a + b\sqrt{p})^p, \ -x = a^2 - pb^2, \tag{8}$$

where $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$ and $2 \nmid a^2 - pb^2$. Clearly, (8) gives $a|1$ and so $a = \pm 1$. So $-x = 1 - pb^2$, and from (7) we have

$$py^2 - (pb^2 - 1)\left((pb^2-1)^{(p-1)/2}\right)^2 = 1. \tag{9}$$

If $b = 0$, then $y = 0$ by (9). If $b \neq 0$, by Lemma 2, then we have from (9) that

(10)
$$py^2 + (pb^2 - 1)\left((pb^2-1)^{(p-1)/2}\right)^2 + 2y(pb^2-1)^{(p-1)/2}\sqrt{p(pb^2-1)} = \varepsilon \text{ or } \varepsilon^3,$$

where $\varepsilon = u_0 + v_0\sqrt{p(pb^2-1)}$ is the least positive integer solution of Pell's equation

$$u^2 - p(pb^2-1)v^2 = 1. \tag{11}$$

Clearly, $(u,v)=(2pb^2-1,2|b|)$ is a positive integer solution of Pell's equation (11). It is well known [8] that if (u_1,v_1) is a positive integer solution of equation (11) and $u_1>\frac{1}{2}v_1^2-1$, then $u_1+v_1\sqrt{p(pb^2-1)}$ is the least positive integer solution of equation (11). So $\varepsilon=2pb^2-1+2|b|\sqrt{p(pb^2-1)}$. Since

$$\begin{aligned}\varepsilon^3=&(2pb^2-1)\left((2pb^2-1)^2+12pb^2(pb^2-1)\right)\\&+2|b|\left(3(2pb^2-1)^2+4pb^2(pb^2-1)\right)\sqrt{p(pb^2-1)},\end{aligned}$$

we get from (10) that

$$\begin{aligned}py^2+(pb^2-1)\left((pb^2-1)^{(p-1)/2}\right)^2&=2pb^2-1,\\ y(pb^2-1)^{(p-1)/2}&=|b|,\end{aligned}\tag{12}$$

or

(13)

$$\begin{aligned}py^2+(pb^2-1)\left((pb^2-1)^{(p-1)/2}\right)^2&=(2pb^2-1)\left((2pb^2-1)^2+12pb^2(pb^2-1)\right),\\ y(pb^2-1)^{(p-1)/2}&=|b|\left(3(2pb^2-1)^2+4pb^2(pb^2-1)\right).\end{aligned}$$

Clearly, (12) is impossible. For (13), by (9) we have

$$2(pb^2-1)\left((pb^2-1)^{(p-1)/2}\right)^2+1=(2pb^2-1)\left(16pb^2(pb^2-1)+1\right),$$

i.e.

$$(pb^2-1)^{(p-1)/2}=4pb^2-1,$$

which also is impossible. (A) of the Theorem is proved.

Proof of (B). Suppose (x,y,p,m) is a solution of equation (2) with $p\mid y$. By Lemma 1, we get from (2) that

$$2^m+y\sqrt{p}=(a+b\sqrt{p})^p,\ -x=a^2-pb^2,\tag{14}$$

where $a,b\in\mathbb{Z},\ \gcd(a,b)=1$ and $2\nmid a^2-pb^2$. By Lemma 3, (14) is impossible since (14) implies that

$$2^{m+1}=(a+b\sqrt{p})^p+(a-b\sqrt{p})^p.$$

Hence, (B) of the Theorem is proved.

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