

GROWTH PROPERTIES OF SUPERHARMONIC FUNCTIONS ALONG RAYS

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ABSTRACT. This paper gives a precise topological description of the set of rays along which a superharmonic function on \mathbb{R}^n may grow quickly. The corollary that arbitrary growth cannot occur along all rays answers a question posed by Armitage.

1. INTRODUCTION

The purpose of this paper is to show that a result of Armitage (Theorem A below) about the growth along rays of superharmonic functions on \mathbb{R}^2 does indeed, contrary to what is suggested by the observations in [1], have a natural (but rather non-obvious) generalization to higher dimensions. As a corollary we solve a problem posed in [1] by showing that a superharmonic function on \mathbb{R}^n cannot grow too quickly on a topologically large set of rays. We begin by recording the following [1, Theorem 2].

Theorem A. *Let $E \subseteq [0, 2\pi)$ and let u be a superharmonic function on \mathbb{R}^2 . If*

$$\liminf_{r \rightarrow +\infty} u(re^{i\theta}) > -\infty \quad (\theta \in E)$$

and, for each positive number μ , the closure of the set

$$\{\theta \in [0, 2\pi) : r^{-\mu}u(re^{i\theta}) \rightarrow +\infty \text{ as } r \rightarrow +\infty\}$$

contains E , then E is of first category.

However, it was remarked in [1] that the natural analogue of Theorem A in higher dimensions fails because of the following fact. We denote by S the unit sphere in \mathbb{R}^n .

Theorem B. *Let $n \geq 3$ and let $M : [0, +\infty) \rightarrow (0, +\infty)$ be increasing. There exists a subset E of S and a superharmonic function u on \mathbb{R}^n such that*

$$u(ry)/M(r) \rightarrow +\infty \quad (r \rightarrow +\infty; y \in E)$$

and $S \setminus E$ is of first category in S .

It was even shown in [1, Proposition 2] that the set E in Theorem B can have full surface area measure, and this led to the question [1, p. 251] of whether it is possible to have $E = S$. We will show below that Theorem A does have a natural

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generalization to higher dimensions, and will observe as an immediate consequence that E cannot equal S in Theorem B. First we make some definitions.

The fine topology on \mathbb{R}^n ($n \geq 1$) is the coarsest topology for which all superharmonic functions are continuous. (See Doob [3, 1.XI] for an account of this topology and the associated notion of thinness of a set.) It has a natural extension to compactified space $\mathbb{R}^n \cup \{\infty\}$ (see [3, 1.XI.5]). From now on we will assume that $n \geq 2$. By the *fine topology on S* we mean the topology induced on S by the fine topology on $\mathbb{R}^{n-1} \cup \{\infty\}$ under the stereographic projection $f : \mathbb{R}^{n-1} \cup \{\infty\} \rightarrow S$ given by

$$f(x') = \begin{cases} (1 + |x'|^2)^{-1}(2x_1, \dots, 2x_{n-1}, |x'|^2 - 1) & (x' = (x_1, \dots, x_{n-1})), \\ (0, \dots, 0, 1) & (x' = \infty), \end{cases}$$

where $|x'| = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$. When $n = 2$ the fine topology on S coincides with the Euclidean one since the superharmonic functions on \mathbb{R} are precisely the concave functions, and hence are already continuous. The fine topology on \mathbb{R}^n has the Baire property, so the same is true of the fine topology on S . When $n \geq 3$, a subset A of S will be called *polar in S* if $A \setminus \{(0, \dots, 0, 1)\}$ is the stereographic image of a polar set in \mathbb{R}^{n-1} . This is equivalent (cf. [10, Chapter II, §§3, 4]) to A having zero capacity with respect to the kernel $|x - y|^{3-n}$ ($n \geq 4$) or $\log 1/|x - y|$ ($n = 3$). When $n = 2$, only the empty set will be called polar in S . Our main result is as follows.

Theorem 1. *Let $E \subseteq S$ and $n \geq 2$. The following are equivalent:*

(a) *there is a superharmonic function u on \mathbb{R}^n such that*

$$(1) \quad \liminf_{r \rightarrow +\infty} u(ry) > -\infty \quad (y \in E)$$

and such that, for each positive number μ and each finely open subset U of S for which $U \cap E$ is non-polar in S , the set

$$\{y \in U : r^{-\mu}u(ry) \rightarrow +\infty \text{ as } r \rightarrow +\infty\}$$

is non-polar in S ;

(b) *there is a superharmonic function u on \mathbb{R}^n such that, for each positive number μ ,*

$$r^{-\mu}u(ry) \rightarrow +\infty \quad (r \rightarrow +\infty; y \in E);$$

(c) *for each increasing function $M : [0, +\infty) \rightarrow (0, +\infty)$ there is a superharmonic function u on \mathbb{R}^n such that*

$$u(ry)/M(r) \rightarrow +\infty \quad (r \rightarrow +\infty; y \in E);$$

(d) *E is of first category with respect to the fine topology on S .*

Corollary 1. *There is no superharmonic function u on \mathbb{R}^n such that $r^{-\mu}u(ry) \rightarrow +\infty$ as $r \rightarrow +\infty$ for each y in S and each positive number μ .*

When $n = 2$, the implication (a) \Rightarrow (d) of Theorem 1 corresponds to Theorem A since we can delete “finely” and “fine”, and replace “non-polar in S ” by “non-empty”. In higher dimensions this implication is more delicate and most of the proof of Theorem 1 is devoted to it. Corollary 1 answers negatively the question of Armitage mentioned above.

Following some preliminary lemmas in §2, Theorem 1 is proved in §3. The arguments involve modifications of ideas from [7] and [4] together with some new ingredients. We note from [4, Proposition 1] that there is no implication in either

direction between the notions of first category with respect to the Euclidean and fine topologies on S .

2. PRELIMINARY LEMMAS

2.1. We refer to [7, Lemma 2] or [8] for a proof of the following lemma.

Lemma A. *Let $n \geq 3$ and $A' \subseteq \mathbb{R}^{n-1}$, let $x' \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. Then A' is thin at x' in \mathbb{R}^{n-1} if and only if $A' \times \mathbb{R}$ is thin at (x', t) in \mathbb{R}^n .*

A subset A of S will be called S -thin at a point z of S if z is not a limit point of A with respect to the fine topology on S .

Lemma 1. *Let $n \geq 3$, let $A \subseteq S$ and $C = \{ry : y \in A \text{ and } r > 0\}$, let $z \in S$ and $\rho > 0$. Then C is thin at ρz if and only if A is S -thin at z . Also, C is polar in \mathbb{R}^n if and only if A is polar in S .*

In proving Lemma 1 we may assume that $\rho = 1$, since thinness is invariant under dilation. Let A' (resp. z') denote the pre-image of A (resp. z) under the stereographic projection. Then, by inverting A' in the unit sphere of \mathbb{R}^{n-1} , we see that A is S -thin at $(0, \dots, 0, 1)$ if and only if the reflection of A in $\mathbb{R}^{n-1} \times (0, +\infty)$ is S -thin at $(0, \dots, 0, -1)$. Since thinness in \mathbb{R}^n is preserved by reflection, we may assume from now on that $z \neq (0, \dots, 0, 1)$. Further, since thinness is a local property, we may assume that $(0, \dots, 0, 1) \notin \bar{A}$.

By definition A is S -thin at z if and only if A' is thin at z' in \mathbb{R}^{n-1} , which is equivalent, by Lemma 1, to the condition that $A' \times (-1, 1)$ is thin at $(z', 0)$ in \mathbb{R}^n . Since the mapping

$$(x', x_n) \mapsto \exp(x_n)f(x'),$$

from \mathbb{R}^n to $\mathbb{R}^n \setminus \{0\}$, is smooth and has a smooth inverse, it is routine to check that $A' \times (-1, 1)$ is thin at $(z', 0)$ if and only if the set

$$\{ry : y \in A \text{ and } e^{-1} < r < e\}$$

is thin at z . (See [9, Theorem 10.6] for an argument of this type.) This latter condition is equivalent to the thinness of C at z , again because thinness is a local property.

The final assertion of Lemma 1 follows from the characterization of polar sets in \mathbb{R}^n as sets which are everywhere thin.

2.2. If $E \subseteq \mathbb{R}^n$ and $r > 0$, then we define $rE = \{ry : y \in E\}$.

Lemma 2. *Let $n \geq 3$, let $U \subseteq S$ be open with respect to the fine topology on S and let $C = \{ry : y \in U \text{ and } r > 0\}$. Then the fine components of C are of the form $D = \{ry : y \in V \text{ and } r > 0\}$, where V is open and connected with respect to the fine topology on S .*

To prove this, we first observe that, if Ω is a finely open set in \mathbb{R}^n , then there exists $E \subseteq S$, polar as a subset of \mathbb{R}^n , such that the set $\{r > 0 : ry \in \Omega\}$ is open in \mathbb{R} whenever $y \in S \setminus E$. This can be established using an argument similar to that employed to prove the analogous statement with parallel lines in place of rays (see [4, Corollary 1(i)]).

Now let U and C be as in the statement of Lemma 2. It follows from Lemma 1 that C is finely open in \mathbb{R}^n . Let D be a fine component of C . Then D is also

finely open by the local connectedness of the fine topology (see [5, p. 92]). By the observation in the preceding paragraph there exists $\kappa > 1$ such that

$$(rD) \cap D \neq \emptyset \quad (\kappa^{-1} < r < \kappa).$$

Since rD is also a fine component of the dilation-invariant set C , it follows that $rD = D$ whenever $\kappa^{-1} < r < \kappa$. Repeated application of this fact leads to the conclusion that $rD = D$ for all $r > 0$. Hence D is of the form $\{ry : y \in V \text{ and } r > 0\}$, and it follows from Lemma 1 again that V is finely open in S . Finally, it is clear from the fine connectedness of D that V must be finely connected in S .

3. PROOF OF THEOREM 1

3.1. We begin by proving that (d) implies (c). For this we need the following (see [6, Theorem 3.19 and Corollary 3.21] or [11] for more general results of this type, and [6, §3.2] for a discussion of local connectedness in this context).

Theorem C. *Let F be a (Euclidean) closed set in \mathbb{R}^n such that $(\mathbb{R}^n \cup \{\infty\}) \setminus F$ is connected and locally connected and such that the fine interior of F is empty. Then, for any continuous function u on F and any positive number ε , there exists a harmonic function v on \mathbb{R}^n such that $|v - u| < \varepsilon$ on F . (If $n = 2$, then the word “fine” can be omitted.)*

Now suppose that condition (d) holds and that $n \geq 3$. Then there is an increasing sequence (K_k) of (Euclidean) closed subsets of S which have empty interiors with respect to the fine topology on S , and a set Z which is polar in S , such that $E \subseteq (\bigcup_k K_k) \cup Z$ (see [4, Lemma 2]). The set

$$Y = \{rz : z \in Z \text{ and } r > 0\}$$

is polar in \mathbb{R}^n , by Lemma 1, so there is a positive superharmonic function s on \mathbb{R}^n such that $s = +\infty$ on Y . Given an increasing function $M : [0, +\infty) \rightarrow (0, +\infty)$, let $g : [0, +\infty) \rightarrow (0, +\infty)$ be a continuous function such that

$$g(r) \geq 1 + rM(r) \quad (r \geq 0),$$

and let

$$F = \bigcup_{k=1}^{\infty} \{ry : y \in K_k \text{ and } r \geq k\}.$$

Then F is a (Euclidean) closed set and, since the sets K_k are nowhere (Euclidean) dense in S , the set $(\mathbb{R}^n \cup \{\infty\}) \setminus F$ is connected and locally connected. In fact, the sets K_k are nowhere dense with respect to the fine topology on S and so, by Lemma 1, the fine interior of F is empty. We can now apply Theorem C to see that there is a harmonic function v on \mathbb{R}^n such that

$$|v(x) - g(|x|)| < 1 \quad (x \in F).$$

The function $u = v + s$ is thus a superharmonic function on \mathbb{R}^n which satisfies

$$u(x) > g(|x|) - 1 \geq |x|M(|x|) \quad (x \in F)$$

and $u = +\infty$ on Y . Hence condition (c) of Theorem 1 holds.

If $n = 2$ and condition (d) holds, then there is an increasing sequence (K_k) of closed nowhere dense subsets of S such that $E \subseteq \bigcup_k K_k$, and we can again use Theorem C to see that condition (c) follows.

3.2. It is clear that (c) \Rightarrow (b) \Rightarrow (a), so it remains to show that (a) \Rightarrow (d). The case where $n = 2$ is covered by Theorem A so we will assume for the remainder of this section that $n \geq 3$.

Suppose that (a) holds but (d) fails, and let

$$E_k = \{y \in E : u(ry) > 1 - k \text{ for all } r > 0\} \quad (k \in \mathbb{N}).$$

Then $\bigcup_k E_k = E$ by (1), so there exists k_0 in \mathbb{N} such that the closure of E_{k_0} with respect to the fine topology on S has non-empty fine interior U . By Lemma 2 there is a non-empty finely open subset V of U such that the set

$$D = \{ry : y \in V \text{ and } r > 0\}$$

is finely connected in \mathbb{R}^n . From Lemma 1 and the fine continuity of u it follows that

$$(2) \quad u \geq 1 - k_0 > -k_0 \quad \text{on } D.$$

The set $V \cap E$ is non-polar in S , so the same is true, by hypothesis, of the sets

$$V_m = \{y \in V : r^{-m}u(ry) \rightarrow +\infty \text{ as } r \rightarrow +\infty\} \quad (m \in \mathbb{N}).$$

Let

$$(3) \quad V_{m,l} = \{y \in V : r^{-m}u(ry) \geq 1 \text{ whenever } r \geq 2^l\} \quad (m, l \in \mathbb{N}).$$

Then $V_m \subseteq \bigcup_l V_{m,l}$. For each m in \mathbb{N} we choose $l(m)$ such that $V_{m,l(m)}$ is non-polar in S . Let

$$(4) \quad D_m = \{ry : y \in V_{m,l(m)} \text{ and } r \geq 1\}$$

and let v_m denote the fine balayage of the constant function 1 relative to the set D_m in D . (We refer to Fuglede [5] for concepts related to finely superharmonic functions.) By Lemma 1 the set D_m is non-polar in \mathbb{R}^n , so $v_m > 0$ in D .

Let

$$(5) \quad F_{m,j} = \{x \in V : v_m(x) > 1/j\} \quad (m, j \in \mathbb{N}).$$

Then $\bigcup_j F_{m,j} = V$ for each m . Let $0 < \varepsilon < \mathcal{C}(V)$ where $\mathcal{C}(\cdot)$ denotes Newtonian capacity in \mathbb{R}^n . We choose $j(1)$ such that

$$\mathcal{C}(F_{1,j(1)}) > \varepsilon$$

and then a compact subset K_1 of $F_{1,j(1)}$ such that $\mathcal{C}(K_1) > \varepsilon$. We proceed inductively: given a compact subset K_m of V such that $\mathcal{C}(K_m) > \varepsilon$, we choose $j(m+1)$ such that $\mathcal{C}(K_m \cap F_{m+1,j(m+1)}) > \varepsilon$, and then a compact subset K_{m+1} of $K_m \cap F_{m+1,j(m+1)}$ such that $\mathcal{C}(K_{m+1}) > \varepsilon$. We define the compact set $K = \bigcap_m K_m$. Then K is non-polar in \mathbb{R}^n and $K \subseteq F_{m,j(m)}$ for all m . Let v denote the fine balayage of the constant function 1 relative to K in D and let

$$(6) \quad F_j = \{x \in V : v(x) > j^{-1}\} \quad (j \in \mathbb{N}).$$

Then $v > 0$ and $\bigcup_j F_j = V$, so there exists j_0 such that the set F_{j_0} , which is finely open in S , is non-empty.

If $A \subseteq \mathbb{R}^n$ and w is a positive superharmonic function on \mathbb{R}^n , then let \widehat{R}_w^A denote the balayage of w relative to A with respect to superharmonic functions on \mathbb{R}^n . We

define

$$g(x, y) = |x - y|^{2-n} - \widehat{R}_{|\cdot - y|^{2-n}}^{\mathbb{R}^n \setminus D}(x) \quad (x, y \in \mathbb{R}^n; x \neq y),$$

and note that $g(x, \cdot)$ is subharmonic on $\mathbb{R}^n \setminus \{x\}$ for each x , and $g(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$ for each y . (See [3, 1.X.3] for the symmetry of $g(\cdot, \cdot)$.) Since $g(\cdot, \cdot)$ is locally bounded above on

$$\{ry : y \in S \text{ and } r > 3/2\} \times \{ry : y \in S \text{ and } 0 < r < 3/2\},$$

a result of Avanissian [2, Théorème 9] now asserts that $g(\cdot, \cdot)$ is subharmonic on this subset of \mathbb{R}^{2n} . In particular, g is upper semicontinuous on the set $(2S) \times S$ (recall the definition of rE in §2.2). Let F be a compact subset of F_{j_0} with non-empty fine interior and let

$$L_i = \{(y, z) \in F^2 : g(2y, z) \geq i^{-1} \text{ and } g(y, 2z) \geq i^{-1}\} \quad (i \in \mathbb{N}).$$

Then $\bigcup_i L_i = F^2$, since when $y, z \in D$, the non-negative functions $g(y, \cdot)$ and $g(\cdot, z)$ are finely superharmonic and not identically zero on the fine domain D , and hence are strictly positive there. The product topology on S^2 , generated from the fine topology on S , also has the Baire property since it has a neighbourhood base of Euclidean compact sets (cf. [3, pp. 167, 168]). Hence there exists i_0 in \mathbb{N} such that the compact set L_{i_0} has non-empty interior with respect to this product topology. In particular, there exist compact subsets M_1 and M_2 of F_{j_0} which have non-empty interiors with respect to the fine topology on S and which satisfy

$$(7) \quad g(2y, z) \geq i_0^{-1} \quad \text{and} \quad g(y, 2z) \geq i_0^{-1} \quad (y \in M_1, z \in M_2).$$

The sets M_1 and M_2 have positive $(n - 1)$ -dimensional measure and hence are non-polar in \mathbb{R}^n . We define $c = \min\{\mathcal{C}(M_1), \mathcal{C}(M_2)\}$ and choose m_1 such that $2^{m_1} > i_0/c$.

Now let $G_\Omega(\cdot, \cdot)$ denote the Green function of the open set

$$\Omega = \{x \in \mathbb{R}^n : u(x) + k_0 > 0\}$$

and let $l \geq l(m_1)$. (For the definition of $l(m_1)$ see the line following (3).) Then $D \subseteq \Omega$ by (2), and since $u(2^l x) \geq 2^{lm_1}$ when $x \in D_{m_1}$ (see (3) and (4)), it follows that

$$u(2^l x) + k_0 \geq 2^{lm_1} v_{m_1}(x) \quad (x \in D)$$

(see the line following (4) for the definition of v_{m_1}). Thus

$$u(2^l x) + k_0 \geq \frac{2^{lm_1}}{j(m_1)} \quad (x \in F_{m_1, j(m_1)})$$

in view of (5), so

$$u(2^l x) + k_0 \geq \frac{2^{lm_1}}{j(m_1)} v(x) \quad (x \in D),$$

since $K \subseteq F_{m_1, j(m_1)}$. Hence

$$u(2^l x) + k_0 \geq \frac{2^{lm_1}}{j_0 j(m_1)} \quad (x \in F_{j_0})$$

in view of (6). In particular, the above inequality holds for all x in $M_1 \cup M_2$.

Thus

$$(8) \quad u(x) + k_0 \geq \frac{2^{lm_1}}{j_0 j(m_1)} w_{i,l}(x) \quad (x \in \Omega; i = 1, 2),$$

where $w_{i,l}$ denotes the balayage of the constant function 1 relative to the set $2^l M_i$ with respect to positive superharmonic functions on Ω . Each function $w_{i,l}$ is the potential on Ω of a measure $\mu_{i,l}$ with support in the compact set $2^l M_i$ and

$$(9) \quad \mu_{i,l}(2^l M_i) \geq 2^{l(n-2)} c,$$

using the fact that $G_\Omega(x, y) \leq |x - y|^{2-n}$ and the standard dilation property of Newtonian capacity. Similarly

$$(10) \quad g(rx, ry) = r^{2-n} g(x, y) \quad (x, y \in \mathbb{R}^n; r > 0).$$

Since

$$g(x, y) \leq |x - y|^{2-n} - \widehat{R}_{|\cdot - y|^{2-n}}^{\mathbb{R}^n \setminus \Omega}(x) = G_\Omega(x, y) \quad (x, y \in \mathbb{R}^n; x \neq y),$$

we see from (7), (9) and (10) that

$$\begin{aligned} w_{i,l}(x) &= \int_{2^l M_i} G_\Omega(x, y) d\mu_{i,l}(y) \\ &\geq \int_{2^l M_i} g(x, y) d\mu_{i,l}(y) \\ &\geq 2^{(l-1)(2-n)} i_0^{-1} \mu_{i,l}(2^l M_i) \\ &\geq c/i_0 \quad (x \in 2^{l-1} M_{3-i}; i = 1, 2). \end{aligned}$$

Thus, by (8),

$$u(x) + k_0 \geq (c/i_0) 2^{lm_1} \{j_0 j(m_1)\}^{-1} \quad (x \in 2^{l-1}(M_1 \cup M_2)),$$

and repeated application of this argument yields

$$u(x) + k_0 \geq (c/i_0)^l 2^{lm_1} \{j_0 j(m_1)\}^{-1} \quad (x \in M_1 \cup M_2).$$

Since m_1 was chosen such that $2^{m_1} > i_0/c$, and l can be arbitrarily large, we obtain the contradictory conclusion that $u = +\infty$ on the non-polar set $M_1 \cup M_2$. Thus condition (d) of Theorem 1 must hold and the proof is complete.

Corollary 1 follows immediately.

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