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HIGHER ORDER SYMMETRIC SPACES AND THE ROOTS OF THE IDENTITY IN A LIE GROUP

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ABSTRACT. Let $r_k(G)$ denote the set of all k-roots of the identity in a Lie group G. We show that $r_k(G)$ is always an embedded submanifold of G, having the conjugacy classes of its elements as open submanifolds. These conjugacy classes are examples of k-symmetric spaces and we show, more generally, that every k-symmetric space of a Lie group G is a covering manifold of an embedded submanifold Orb of G. We compute also the Hessian of the inclusions of $r_k(G)$ and Orb into G, relative to the natural connection on the domain and to the symmetric connection on G.

1. INTRODUCTION

There are several examples of homogeneous manifolds of a Lie group G that can be realized equivariantly as connected components of the set $r_k(G)$ of all k-roots of the identity e of G, with G acting by conjugation, in particular as conjugation classes of elements of G: If E is a Hermitian vector space, the Grassmann manifold $Gr_p(E)$, whose elements are the p-dimensional vector subspaces, can be realized as a connected component of $r_2(U(E))$, as observed by Uhlenbeck [6], and, in an analogous way, if E is a Euclidean space, $Gr_p(E)$ admits a connected component of $r_2(O(E))$ as an equivariant model; More generally, if E is a Hermitian space, the connected components of $r_k(U(E))$ are models of the flag manifolds $\mathcal{G}_{p_1,\dots,p_k}(E)$, whose elements are the systems $(F_1,\dots,F_k)$ of mutually orthogonal subspaces, with dimensions $p_1,\dots,p_k$, whose direct sum is E [3]; If E is a Euclidean space, the connected components of $r_3(O(E))$ are models of the manifolds $\mathcal{F}_p(E)$, whose elements are the p-dimensional partially complex structures, i.e. the couples (F,J), where J is a compatible complex structure on the $2p$-dimensional real subspace $F\subset E$ [4], with the exception of the extreme case $\dim(E)=2p$, where $\mathcal{F}_p(E)$ is the union of two connected components.

With the previous examples in mind, we prove that, for a general Lie group G, $r_k(G)$ is always a submanifold of G, in general with variable dimension, having the conjugation classes of its elements as open submanifolds (we will use always the word "submanifold" with the meaning "embedded submanifold").

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The conjugation classes of elements of $r_k(G)$ are examples of k-symmetric manifolds; they are of the form G/G^τ, where G^τ is the fixed point subgroup of a smooth automorphism $\tau\colon G \to G$, satisfying $\tau^k = Id_G$. The structure and classification of k-symmetric manifolds have been extensively studied in [5] and it has been known for a long time (cf. [2]) that every 2-symmetric manifold of the form G/G^τ can be one-to-one immersed into the Lie group G, by associating $g\tau(g^{-1})$ with the class of an element $g \in G$, a fact that can be generalized trivially to k-symmetric spaces. We will prove, more precisely, that this one-to-one immersion is always an embedding, and hence that the k-symmetric space G/G^τ admits a model Orb that is a submanifold of G. This fact was established in [1], for the special case where G is compact, where the name "Cartan embedding" is used (in fact, Burstall uses the inverse $\tau(g)g^{-1}$ instead of $g\tau(g^{-1})$ but this makes no essential difference). Of course, for general k-symmetric spaces, those of the form G/H, with H an open subgroup of G^τ, all we can guarantee is that they are covering manifolds of the manifold Orb.

Every k-symmetric space is reductive in a canonical way and, as such, it has a canonical connection, and it is known [2] that, for $k = 2$, its embedding into the Lie group G, considered with its symmetric connection, is totally geodesic. This fact led us to compute, for general k, the Hessian of the inclusions of $r_k(G)$ and Orb into G.

In the next section we will prove a basic lemma that guarantees that, under certain conditions, the image of a smooth map is a smooth manifold. This lemma will be used in section 3 in order to prove that Orb is a submanifold but it is possible that it may present some independent interest.

2. A basic lemma

Lemma 1. *Let X, Y and Z be manifolds and $f\colon X \to Y$ and $\varphi, \psi\colon Y \to Z$ be smooth maps such that $\varphi \circ f = \psi \circ f$. Let $x_0 \in X$ be such that every vector $v \in T_{f(x_0)}(Y)$ verifying the condition $D\varphi_{f(x_0)}(v) = D\psi_{f(x_0)}(v)$ is in the image of $Df_{x_0}\colon T_{x_0}(X) \to T_{f(x_0)}(Y)$. Then the set $B = \{y \in Y \mid \varphi(y) = \psi(y)\}$ is a submanifold of Y at $f(x_0)$, with the image of Df_{x_0} as tangent space, and $f(X)$ is a neighborhood of $f(x_0)$ in B; in particular $f(X)$ is also a submanifold of Y at $f(x_0)$, with the same tangent space.*

Proof. The question being local, we may assume that X, Y and Z are open sets in finite dimensional spaces E, F and G and that $x_0 = 0$ and $f(x_0) = 0$. Let us fix norms in these vector spaces, let $H \subset F$ be a direct sum complement of the vector subspace $Df_0(E)$ and let $g = \psi - \varphi\colon Y \to G$. The fact that the linear map $Dg_0\colon F \to G$ is one-to-one in H allows us to consider $\delta > 0$ such that, for each $w \in H$, $\|Dg_0(w)\| \geq \delta\|w\|$ (if H is not trivial, let δ be the minimum of $\|Dg_0(w)\|$, for $w \in H$ with $\|w\| = 1$). Let $\varepsilon > 0$ be such that, for each $y \in F$ with $\|y\| < \varepsilon$, we have $\|Dg_y - Dg_0\| \leq \delta/2$. By the mean value theorem, if $\|y'\| < \varepsilon$ and $\|y''\| < \varepsilon$, then $\|g(y') - g(y'') - Dg_0(y' - y'')\| \leq \delta/2\|y' - y''\|$. Let $U \subset X$ and $W \subset H$ be open sets, with $0 \in U$ and $0 \in W$, such that, for each $x \in U$ and $w \in W$, $\|f(x) + w\| < \varepsilon$ and let us remark that, for $x \in U$ and $w \in W$, $f(x) + w \in B$ if and only if $w = 0$. In fact, one of the implications is trivial and, for the other, if $f(x) + w \in B$, then

$$\delta\|w\| \leq \|Dg_0(w)\| = \|g(f(x) + w) - g(f(x)) - Dg_0(w)\| \leq \frac{\delta}{2}\|w\|;$$

hence $w = 0$. The derivative at $(0,0)$ of the map $X \times H \to F$, $(x,w) \mapsto f(x)+w$, maps (u,v) onto $Df_0(u)+v$ and is hence onto, so that a standard result about submersions guarantees the existence of an open set V, with $0 \in V \subset Y$, and of smooth maps $\sigma_1 \colon V \to U \subset E$ and $\sigma_2 \colon V \to W \subset H$, satisfying $\sigma_1(0) = 0$, $\sigma_2(0) = 0$ and $f(\sigma_1(y)) + \sigma_2(y) = y$, for each $y \in V$. By derivation, we have

$$Df_0(D\sigma_{1_0}(v)) + D\sigma_{2_0}(v) = v; \tag{1}$$

hence $D\sigma_{2_0}$ is the projection from F onto H associated to the direct sum, in particular is onto. As we proved above, for $y \in V$, we have $y \in B$ if, and only if, $\sigma_2(y) = 0$; hence B is a submanifold of Y at 0 and $T_0(B)$ is the kernel of the linear map $D\sigma_{2_0} \colon F \to H$, so that, by (1), $Df_0 \colon E \to T_0(B)$ is onto. This implies that $f(X)$ is indeed a neighborhood of 0 in B. □

Although we will not apply it, we cannot resist stating the following trivial consequence of the previous lemma:

Corollary 1. *Let X be a manifold and $f \colon X \to X$ be a smooth map such that $f \circ f = f$. Then $f(X) = \{y \in X \mid f(y) = y\}$ is a submanifold of X and*

$$T_y(f(X)) = Df_y(T_y(X)) = \{u \in T_y(X) \mid Df_y(u) = u\}.$$

3. Embedding a k-symmetric space

In this section we will fix an integer $k \geq 2$, a Lie group G and a smooth automorphism $\tau \colon G \to G$, such that $\tau^k = Id_G$, and we will consider the corresponding k-symmetric space G/G^τ, where $G^\tau = \{g \in G \mid \tau(g) = g\}$ is the fixed point subgroup. For each $g \in G$, we will denote by $[h]$ the corresponding class in G/G^τ.

We will denote $\mathcal{G} = T_e(G)$ the Lie algebra of G and $\theta = D\tau_e \colon \mathcal{G} \to \mathcal{G}$ the corresponding Lie algebra automorphism, that satisfies $\theta^k = Id_{\mathcal{G}}$. Of course, the Lie algebra of the subgroup G^τ is $T_e(G^\tau) = \mathcal{G}^\theta = \{u \in \mathcal{G} \mid \theta(u) = u\}$. The equality

$$(Id - \theta) \circ (Id + \theta + \cdots + \theta^{k-1}) = 0,$$

with commuting factors having trivial intersection kernels, implies that $\mathcal{G} = \mathcal{H}_{[e]} \oplus \mathcal{M}_{[e]}$, where

$$\mathcal{H}_{[e]} = \mathcal{G}^\theta = \ker(Id - \theta) = \{u + \theta(u) + \cdots + \theta^{k-1}(u)\}_{u \in \mathcal{G}},$$
$$\mathcal{M}_{[e]} = \ker(Id + \theta + \cdots + \theta^{k-1}) = \{u - \theta(u)\}_{u \in \mathcal{G}}.$$

If $g \in G^\tau$, the fact that the conjugation automorphism c_g commutes with τ implies that the Lie algebra automorphism Ad_g commutes with θ and hence that the direct sum $\mathcal{G} = \mathcal{H}_{[e]} \oplus \mathcal{M}_{[e]}$ is Ad_g-invariant. We have hence a well-defined structure of reductive homogeneous space on G/G^τ, the one that will be considered implicitly. We remark that this is the same reductive structure in G/G^τ that has been defined in [1], using the eigenspaces of the complexification of θ; however the direct approach will be useful later.

Proposition 1. *Let us consider the smooth action of G in G defined by $g \cdot h = gh\tau(g^{-1})$ and let $Orb = \{g\tau(g^{-1})\}_{g \in G}$ be the orbit of e for this action. Let $B \subset G$ be the set*

$$B = \{h \in G \mid h\tau(h) \cdots \tau^{k-1}(h) = e\}.$$

Then Orb is a submanifold of G, open in B, and there is an equivariant diffeomorphism $f\colon G/G^\tau \to Orb$ defined by $f([g]) = g\tau(g^{-1})$. Moreover, $T_e(Orb) = \mathcal{M}_{[e]} = \mathcal{M}_e$.

Proof. It is straightforward to verify that we have a well defined one-to-one smooth equivariant map $f\colon G/G^\tau \to G$, $[g] \mapsto g\tau(g^{-1})$, whose image is Orb, so that all we have to prove is that Orb is a submanifold of G, open in B. The fact that B, like Orb, is invariant by the action of G reduces us to proving that Orb is a neighborhood of e in B and that B is a submanifold of G at the point e. To simplify notations, let us denote by $\widehat{f}\colon G \to G$ the smooth map $g \mapsto g\tau(g^{-1})$, whose image is Orb. Let $\varphi\colon G \to G$ be the smooth map defined by $\varphi(h) = h\tau(h) \cdots \tau^{k-1}(h)$. We have $\varphi(\widehat{f}(g)) = e$, for each $g \in G$; in particular $Orb \subset B$. By differentiating at e, we obtain $D\widehat{f}_e(u) = u - \theta(u)$ and $D\varphi_e(v) = v + \theta(v) + \cdots + \theta^{k-1}(v)$, so that by what was discussed above, both the image of $D\widehat{f}_e$ and the kernel of $D\varphi_e$ are equal to $\mathcal{M}_{[e]}$. Applying Lemma 1, with a constant map as ψ, ends the proof. □

The fact that Orb is a reductive homogeneous manifold of the Lie group G gives it a natural G-invariant connection. One method of characterizing this connection is to compute the Hessian of the inclusion of Orb into G, when we consider in G its natural symmetric connection. That is what we do now, limiting our computation to what happens at $e \in Orb$, because the general formula can be obtained through the left and right invariance of the connection of G. Following the formalism of [2], we compute first the Maurer-Cartan form $\beta_e\colon T_e(Orb) \to \mathcal{M}_e \subset \mathcal{G}$. We recall that β_e is the inverse of the restriction to $\mathcal{M}_e$ of the derivative at e, $\rho_e\colon \mathcal{G} \to T_e(Orb)$, of the map $G \to Orb$, $g \mapsto g \cdot e = g\tau(g^{-1})$, a linear map that is hence defined by $\rho_e(u) = u - \theta(u)$.

Lemma 2. *The Maurer-Cartan form $\beta_e\colon T_e(Orb) = \mathcal{M}_e \to \mathcal{M}_e$ is defined by*

$$\beta_e(v) = -\frac{1}{k}\sum_{j=1}^{k-1} j\,\theta^j(v).$$

Proof. All we have to prove is that the linear map β_e, defined above, maps $\mathcal{M}_e$ into $\mathcal{M}_e$ and verifies $\rho_e \circ \beta_e = Id_{\mathcal{M}_e}$ and this is a straightforward calculation if we recall that $\theta^k = Id_{\mathcal{G}}$ and the characterization of $\mathcal{M}_e$ as a kernel. □

Proposition 2. *The Hessian $h_e\colon \mathcal{M}_e \times \mathcal{M}_e \to \mathcal{G}$, of the inclusion $Orb \to G$ at e, is given by*

$$h_e(u,v) = [\beta_e(u) - \frac{1}{2}u, v - \theta(v)].$$

Proof. Let us denote by ∇ and ∇^G the covariant derivatives associated to the connections we are considering in Orb and in G. Let $u, v \in \mathcal{M}_e$ and let Y be the vector field on Orb associated to v and to the action of G on Orb, that is defined by $Y_g = DR_{g_e}(v) - DL_{g_e}(\theta(v))$, where R_g and L_g denote the right and left translations by g. Let us denote also by Y the vector field on G defined by the same formula. Then $\nabla^G Y_e(u) = -\frac{1}{2}[u, v + \theta(v)]$ and

$$\nabla Y_e(u) = -\rho_e([\beta_e(u), v]) = -[u, \theta(v)] - [\beta_e(u), v - \theta(v)],$$

and the result is now a consequence of the formula $h_e(u,v) = \nabla^G Y_e(u) - \nabla Y_e(u)$. □

For $k = 2$, we have $\beta_e(u) = \frac{1}{2}u$; hence $h_e(u, v) = 0$, and we retrieve the conclusion that Orb is a totally geodesic submanifold.

Remark 1. There is another equivariant embedding of the k-symmetric space G/G^τ that, for $k = 2$, coincides with the previous one: The product group G^k acts transitively on the manifold G^{k-1} by

$$(g_1, \ldots, g_k) \cdot (h_1, \ldots, h_{k-1}) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_3^{-1}, \ldots, g_{k-1} h_{k-1} g_k^{-1})$$

and G^{k-1} is then a k-symmetric manifold, with the permutation automorphism $\tau\colon G^k \to G^k$, $\tau(g_1, \ldots, g_k) = (g_2, \ldots, g_k, g_1)$ associated to the base point $(e, \ldots, e) \in G^{k-1}$. The isotropy subgroup is the diagonal $\{(g_1, \ldots, g_k) \in G^k \mid g_1 = \cdots = g_k\}$, the corresponding Lie algebra is $\mathcal{H}_{(e,\ldots,e)} = \{(u_1, \ldots, u_k) \in \mathcal{G}^k \mid u_1 = \cdots = u_k\}$ and the corresponding direct complement is $\mathcal{M}_{(e,\ldots,e)} = \{(u_1, \ldots, u_k) \mid u_1 + \cdots + u_k = 0\}$. We have hence an associated connection on G^{k-1}. We can consider the smooth morphism $\psi\colon G \to G^k$, $\psi(g) = (g, \tau(g), \ldots, \tau^{k-1}(g))$ and we have then a ψ-equivariant and ψ-reductive map $\Psi\colon G/G^\tau \to G^{k-1}$,

$$\Psi([g]) = (g\tau(g^{-1}), \tau(g)\tau^2(g^{-1}), \ldots, \tau^{k-2}(g)\tau^{k-1}(g^{-1})).$$

This map is hence totally geodesic and, by looking to the first coordinate, we conclude that it is an embedding of G/G^τ into a submanifold of G^{k-1}.

4. The manifold $r_k(G)$

In this section we will fix an integer $k \geq 2$ and a Lie group G and we will denote by $r_k(G)$ the set of k-roots of the identity in G,

$$r_k(G) = \{g \in G \mid g^k = e\}.$$

The group G acts on $r_k(G)$ by conjugation: $h \cdot g = c_h(g) = hgh^{-1}$. For each $g \in r_k(G)$, we will denote by $Orb_g = \{hgh^{-1}\}_{h\in G}$ the orbit of g for this action. We will denote by R_g and L_g the right and left translations by g.

Proposition 3. *The set $r_k(G)$ is a closed submanifold of G and the orbits Orb_g, with $g \in r_k(G)$, are open in $r_k(G)$. Moreover, for each $g \in r_k(G)$,*

$$T_g(r_k(G)) = DR_{g_e}(\mathcal{M}_g) = DL_{g_e}(\mathcal{M}_g) = \{DR_{g_e}(u) - DL_{g_e}(u)\}_{u\in\mathcal{G}},$$

where $\mathcal{M}_g \subset \mathcal{G}$ is defined by

$$\mathcal{M}_g = \ker(Id + Ad_g + \cdots + {Ad_g}^{k-1}) = \{u - Ad_g(u)\}_{u\in\mathcal{G}}.$$

Proof. Let $g \in r_k(G)$. Then $c_g\colon G \to G$ is a smooth automorphism, such that ${c_g}^k = Id_G$ and the corresponding Lie algebra automorphism is $Ad_g\colon \mathcal{G} \to \mathcal{G}$. By Proposition 1, we conclude that $Orb_{(g)} = \{hc_g(h^{-1})\}_{h\in G}$ is a submanifold of G, open in

$$B_{(g)} = \{h \in G \mid hc_g(h) \cdots c_g^{k-1}(h) = e\}$$

and having $\mathcal{M}_g$ as tangent space at e. Considering the diffeomorphism $R_g\colon G \to G$, we conclude that $R_g(Orb_{(g)}) = Orb_g$ is a submanifold of G open in $R_g(B_{(g)}) = r_k(G)$ and that $T_g(r_k(G)) = T_g(Orb_g) = DR_{g_e}(\mathcal{M}_g)$. The other characterizations of $T_g(r_k(G))$ follow from the equality $DL_{g_e} = DR_{g_e} \circ Ad_g$ and from the Ad_g-invariance of $\mathcal{M}_g$. $\square$

Each orbit $Orb_g = \{hgh^{-1}\}_{h\in G}$ is a homogeneous manifold of G and the restriction of the right translation to the homogeneous manifold $Orb_{(g)} = \{hc_g(h^{-1})\}_{h\in G}$, associated to the automorphism $c_g\colon G \to G$, is an equivariant diffeomorphism $R_g\colon Orb_{(g)} \to Orb_g$. By transport through this equivariant diffeomorphism, Orb_g becomes a reductive homogeneous space, such that, for each $g' = hgh^{-1} \in Orb_g$, the horizontal space $\mathcal{M}_{g'} \subset \mathcal{G}$ is equal to the horizontal space of $Orb_{(g)}$ at $hc_g(h^{-1})$; hence

$$\mathcal{M}_{g'} = Ad_h(\mathcal{M}_e) = \{Ad_h(u) - Ad_h(Ad_g(u))\}_{u\in\mathcal{G}} = \{v - Ad_{g'}(v)\}_{v\in\mathcal{G}},$$

which is compatible with the notation used in the previous proposition and proves, in particular, that the reductive structure in Orb_g does not depend of the choice of g in the orbit.

The reductive homogeneous structure of the submanifolds Orb_g entitles them to, and hence $r_k(G)$, with an associated connection. The fact that $R_g\colon Orb_{(g)} \to Orb_g$ and $R_g\colon G \to G$ are totally geodesic maps allows us to deduce the following proposition from Proposition 2:

Proposition 4. *For each $g \in r_k(G)$, the Hessian $h_g\colon T_g(r_k(G)) \times T_g(r_k(G)) \to T_g(G)$, of the inclusion $r_k(G) \to G$ at g, is defined by*

$$h_g(DR_{g_e}(u), DR_{g_e}(v)) = DR_{g_e}([\beta_{(g)}(u) - \frac{1}{2}u, v - Ad_g(v)]),$$

for $u, v \in \mathcal{M}_g$, where $\beta_{(g)}(u) = -\frac{1}{k}\sum_{j=1}^{k-1} j\, {Ad_g}^j(u)$.

Again, for $k = 2$, we have $\beta_{(g)}(u) = \frac{1}{2}u$, hence $h_g = 0$, and we retrieve the conclusion that $r_2(G)$ is a totally geodesic submanifold of G.

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