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A CRITERION FOR SPLITTING C^*-ALGEBRAS IN TENSOR PRODUCTS

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ABSTRACT. The goal of the paper is to prove the following theorem: if A,D are unital C^*-algebras, A simple and nuclear, then any C^*-subalgebra of the C^*-tensor product of A and D, which contains the tensor product of A with the scalar multiples of the unit of D, splits in the C^*-tensor product of A with some C^*-subalgebra of D.

Using essentially a result of J.B. Conway on numerical range and certain sets considered by J. Dixmier in type III factors ([C]), L. Ge and R.V. Kadison proved in [G-K] : for R, Q, M W^*-algebras, R factor, satisfying

$$R\bar{\otimes}1_Q \subset M \subset R\bar{\otimes}Q,$$

we have $M = R\bar{\otimes}P$ with P some W^*-subalgebra of Q.

Making use of the generalization of Conway's result for global W^*-algebras, due independently to H. Halpern ([Hlp]) and Ş. Strătilă and L. Zsidó ([S-Z1]), as well as of an extension of Tomita's Commutation Theorem to tensor products over commutative von Neumann subalgebras, it was subsequently proved in [S-Z2]: for R, Q, M W^*-algebras with

$$R\bar{\otimes}1_Q \subset M \subset R\bar{\otimes}Q,$$

M is generated by $R\bar{\otimes}1_Q$ and $M \cap (Z(R)\bar{\otimes}Q)$, where $Z(R)$ stands for the centre of R.

Using a result from [H-Z], the C^*-algebraic counterpart of the above cited results from [Hlp] and [S-Z1], as well as a slice map theorem for nuclear C^*-algebras due to S. Wassermann ([W], Prop. 10), we shall prove here:

Theorem. *Let A, D, C be unital C^*-algebras, A simple and nuclear, such that*

$$A \otimes 1_D \subset C \subset A \otimes_{\min} D.$$

Then

$$C = A \otimes_{\min} B$$

for some C^*-subalgebra $B \subset D$.

The next result is essentially contained in [H-Z], Corollary of Theorem 4:

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Approximation Lemma. *Let A be a simple unital C^*-algebra without tracial state. Denoting by $\mathcal{I}_A$ the closure of the convex hull*

$$\mathcal{F}_A = \text{ conv } \{ A \ni a \mapsto uau^* \in A ; u \in A \text{ unitary } \}$$

with respect to the topology of the pointwise norm convergence in the Banach algebra of all bounded linear operators on A, and by $S(A)$ the state space of A, we have

$$S(A)1_A = \{ \Phi \in \mathcal{I}_A ; \Phi(A) \subset \mathbb{C}1_A \}.$$

Proof. Clearly, $\{ \Phi \in \mathcal{I}_A ; \Phi(A) \subset \mathbb{C}1_A \} = K \cdot 1_A$ with $K \subset S(A)$ convex. K is also weak* closed. Indeed, if $K \ni \psi_\iota \to \varphi \in S(A)$ in the weak* topology, then $\mathcal{I}_A \ni \psi_\iota \cdot 1_A \to \varphi \cdot 1_A$ in the topology of pointwise norm convergence; hence $\varphi \cdot 1_A \in \mathcal{I}_A$, i.e. $\varphi \in K$.

Let us denote for any $a \in A$

$$C_A(a) = \overline{\mathcal{F}_A(a)} \cap \mathbb{C}1_A.$$

By [H-Z], Corollary of Theorem 4,

$$C_A(a) \neq \emptyset \text{ for all } a \in A$$

and, plainly,

$$C_A(\Phi(a)) \subset C_A(a), \ \Phi \in \mathcal{F}_A, a \in A.$$

Using the idea of the proof of Lemma 4 in [D], Ch. III, §5, it is easy to see that, for every

$$a \in A, \ \lambda \cdot 1_A \in C_A(a), \ a_1, \dots, a_n \in A, \ \varepsilon > 0$$

there exists $\Phi \in \mathcal{F}_A$ with

$$\begin{aligned} &\| \Phi(a) - \lambda \cdot 1_A \| < \varepsilon, \\ &\| \Phi(a_j) - \lambda_j \cdot 1_A \| < \varepsilon \text{ for some } \lambda_j \in \mathbb{C}, \quad 1 \leq j \leq n. \end{aligned}$$

Fix some $a \in A$ and $\lambda 1_A \in C_A(a)$. Since the closure of

$$\mathcal{F}_{a_1,\dots,a_n;\varepsilon} = \left\{ \Phi^{**} ; \begin{matrix} \Phi \in \mathcal{F}_A, \| \Phi(a) - \lambda 1_A \| < \varepsilon, \\ d(\Phi(a_j), \mathbb{C}1_A) < \varepsilon \text{ for } 1 \leq j \leq n \end{matrix} \right\} \neq \emptyset$$

with respect to the topology of pointwise $\sigma(A^{**}, A^*)$-convergence is compact,

$$\bigcap_{\substack{a_1,\dots,a_n \in A \\ \varepsilon > 0}} \overline{\mathcal{F}_{a_1,\dots,a_n;\varepsilon}}^{\text{ pointwise } \sigma(A^{**},A^*)}$$

contains some Ψ. Clearly,

$$\Psi(a) = \lambda 1_A.$$

Fix now also some $b \in A$. For every $\varphi_1, \dots, \varphi_n \in A^*$ and $\varepsilon > 0$ there is some

$$\Theta \in \mathcal{F}_{b;\varepsilon}$$

such that

$$| \varphi_j (\Psi(b) - \Theta(b)) | < \varepsilon \| \varphi_j \|, \ 1 \leq j \leq n,$$

and then some

$$\mu \in \mathbb{C}$$

with

$$\|\Theta(b) - \mu \cdot 1_A\| < \varepsilon.$$

Then

$$|\mu| < \|b\| + \varepsilon \text{ and } |\varphi_j(\Psi(b) - \mu \cdot 1_A)| < 2\varepsilon\|\varphi_j\|,\ 1 \leq j \leq n.$$

It follows that the downward directed compact sets

$$\left\{\mu \in \mathbb{C}; \begin{matrix} |\mu| \leq \|b\| + \varepsilon, \\ |\varphi_j(\Psi(b) - \mu \cdot 1_A)| \leq 2\varepsilon\|\varphi_j\| \text{ for } 1 \leq j \leq n \end{matrix}\right\},$$

$$\varphi_i, \ldots, \varphi_n \in A^*,\ \varepsilon > 0,$$

are not empty; hence their intersection contains some $\mu(b) \in \mathbb{C}$. Then

$$\Psi(b) = \mu(b) \cdot 1_A \in \mathbb{C}1_A.$$

By the above,

$$\Psi(A) \subset \mathbb{C}1_A.$$

Moreover, since $\Psi|A$ takes values in $\mathbb{C}1_A \subset A$, it belongs to the pointwise $\sigma(A, A^*)$-closure of the convex set

$$\{\Phi^{**}|A; \Phi \in \mathcal{F}_A\} = \mathcal{F}_A,$$

which is the pointwise norm closure $\mathcal{I}_A$ of $\mathcal{F}_A$.

We conclude: for any $a \in A$ and any $\lambda \cdot 1_A \in C_A(a)$ there exists $\Phi \in \mathcal{I}_A$ with $\Phi(A) \subset \mathbb{C}1_A$ and $\Phi(a) = \lambda \cdot 1_A$. In other words,

$$\lambda = \psi(a) \text{ for some } \psi \in K.$$

By the Hahn-Banach theorem $K = S(A)$ follows if we show that for every $a^* = a \in A$ with

$$(*) \qquad \psi(a) \leq \lambda_0 \text{ for all } \psi \in K$$

we have

$$\varphi(a) \leq \lambda_0 \text{ for all } \varphi \in S(A).$$

But, according to [H-Z], Corollary of Theorem 4, we have for every $\varphi \in S(A)$

$$\varphi(a) \cdot 1_A \in C_A(a),$$

so, by the above,

$$\varphi(a) = \psi(a) \text{ for some } \psi \in K$$

and (*) yields

$$\varphi(a) \leq \lambda_0.$$

□

Now we prove the main ingredient for the proof of the announced theorem:

Invariance Lemma. *Let A, D, C be unital C^*-algebras, A simple, such that*

$$A \otimes 1_D \subset C \subset A \otimes_{\min} D.$$

Then, for any state φ on A,

$$(\varphi \cdot 1_A \otimes \ id_D)(C) \subset C;$$

hence

$$(\varphi \cdot 1_A \otimes \ id_D)(C) = C \cap (1_A \otimes D).$$

Proof. First we reduce the proof to the case in which A has no tracial state.

Let A_0 be any simple unital C^*-algebra without tracial state (e.g. the Calkin algebra or a type III factor of countable type), and φ_0 any state on A_0. Then $A_0 \otimes_{\min} A$ is a simple unital C^*-algebra without tracial state and

$$(A_0 \otimes_{\min} A) \otimes 1_D \subset A_0 \otimes_{\min} C \subset (A_0 \otimes_{\min} A) \otimes_{\min} D.$$

If we assume that in this case

$$(\varphi_0 \cdot 1_{A_0} \otimes \varphi \cdot 1_A \otimes \ \mathrm{id}_D)(A_0 \otimes_{\min} C) \subset A_0 \otimes_{\min} C,$$

then

$$1_{A_0} \otimes (\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(C) = (\varphi_0 \cdot 1_{A_0} \otimes \varphi \cdot 1_A \otimes \ \mathrm{id}_D)(1_{A_0} \otimes C) \subset A_0 \otimes_{\min} C,$$

$$(\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(C) \subset C.$$

Now let us assume that A has no tracial state. According to the Approximation Lemma there exists a net $(\Phi_\iota)_\iota$ in $\mathcal{F}_A$ such that

$$\|\Phi_\iota(a) - \varphi(a) \cdot 1_A\| \to 0 \text{ for all } a \in A;$$

hence

$$\|(\Phi_\iota \otimes \ \mathrm{id}_D)(a \otimes d) - (\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(a \otimes d)\| \to 0 \text{ for all } a \in A \text{ and } d \in D.$$

Since every $\Phi_\iota \otimes \ \mathrm{id}_D$ is contractive, it follows that

$$\|(\Phi_\iota \otimes \ \mathrm{id}_D)(x) - (\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(x)\| \to 0 \quad \text{ for all } x \in A \otimes_{\min} D.$$

But every $\Phi_\iota \otimes \ \mathrm{id}_D$ is convex combination of operators of the form $Ad(u \otimes 1_D)$ with $u \otimes 1_D \in A \otimes 1_D \subset C$, so it leaves C invariant. Consequently also their pointwise norm limit $\varphi 1_A \otimes \ \mathrm{id}_D$ leaves C invariant. □

Proof of the theorem. $C \cap (1_A \otimes D)$ is of the form $1_A \otimes B$ for some C^*-subalgebra $B \subset D$ and we have to prove that the C^*-subalgebras $A \otimes_{\min} B \subset C$ of $A \otimes_{\min} D$ coincide.

By the Invariance Lemma

$$C \cap (1_A \otimes D) = \{(\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(x); \varphi \in S(A), x \in C\},$$

so

$$B = \{E_\varphi(x); \varphi \in S(A), x \in C\},$$

where $E_\varphi : A \otimes_{\min} D \to D$ is the slice map defined by

$$1_A \otimes E_\varphi(x) = (\varphi \cdot 1_A \otimes \ \mathrm{id}_D)(x), \ x \in A \otimes_{\min} D.$$

In other words,

$$E_\varphi(x) \in B \text{ for all } x \in C \text{ and } \varphi \in S(A).$$

Since A is nuclear, we can apply S. Wassermann's slice map theorem ([W], Prop. 10) and conclude that $C \subset A \otimes_{\min} B$. □

We notice that in the above proof the nuclearity of A was used only by applying Wassermann's slice map theorem to A. It is an open question whether this slice map theorem holds assuming A only exact, that is, if

$$A \text{ exact } C^*\text{-algebra, } B \subset D \ C^*\text{-algebras,}$$

$$x \in A \otimes_{\min} D,\ E_\varphi(x) \in B \text{ for all } \varphi \in S(A)$$

imply $x \in A \otimes_{\min} B$ (see [H-K], Remark 2.3 and [Ki], Section 9). In the case of positive answer it would be enough to assume in our theorem A simple and exact.

Since the closure of the union of every upward directed family of nuclear C^*-subalgebras is still a nuclear C^*-subalgebra (see e.g. [M], Th. 6.3.10), the Zorn lemma implies that any nuclear C^*-subalgebra is contained in a maximal nuclear C^*-subalgebra.

Corollary. *Let D be a unital C^*-algebra, $1_D \in B \subset D$ a maximal nuclear C^*-subalgebra, and A a nuclear, simple, unital C^*-algebra. Then $A \otimes_{\min} B$ is a maximal nuclear C^*-subalgebra of $A \otimes_{\min} D$.*

Proof. The C^*-algebra $A \otimes_{\min} B (= A \otimes_{\max} B)$ is clearly nuclear.

Now let $A \otimes_{\min} B \subset C \subset A \otimes_{\min} D$ be an arbitrary nuclear C^*-subalgebra. By the above theorem

$$C = A \otimes_{\min} B_0$$

for some C^*-subalgebra $B \subset B_0 \subset D$. The nuclearity of C implies the nuclearity of B_0, and then the maximal nuclearity of B in D yields $B_0 = B$. Thus

$$C = A \otimes_{\min} B.$$

□

We notice also that in the case of positive answer to the above slice map question for exact C^*-algebras, when our theorem would follow for A only simple and exact, a counterpart of the above corollary would hold for maximal exact C^*-subalgebras.

Note added in proof. After this work was completed, we learned that our theorem was independently obtained also by Joachim Zacharias in his preprint: “A note on a result of Ge and Kadison and its C^*-algebra version”, 1998. He uses a different way to approximate states on simple unital C^*-algebras by elementary mappings.

References

[C] J. B. Conway, *The numerical range and a certain convex set in an infinite factor*, J. Funct. Anal. **5** (1970), 428–435. MR **41:**7444

[D] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier Villars, 1957. MR **20:**1234

[G-K] L. Ge and R. V. Kadison, *On tensor products of von Neumann algebras*, Invent. Math. **123** (1996), 453-466. MR **97c:**46074

[H-K] U. Haagerup - J. Kraus, *Approximation properties for group C^*-algebras and group von Neumann algebras*, Trans. Amer. Math. Soc. **344** (1994), 667-699. MR **94k:**22008

[H-Z] U. Haagerup - L. Zsidó, *Sur la proprieté de Dixmier pour C^*-algèbres*, C.R. Acad. Sci. Paris, Série I, **298** (1984), 173-175. MR **85i:**46077

[Hlp] H. Halpern, *Essential central spectrum and range for elements of a von Neumann algebra*, Pacific J. Math. **43** (1972), 349-380. MR **48:**2787

[Ki] E. Kirchberg, *Exact C^*-algebras, Tensor Products and the Classification of Purely Infinite Algebras,* Proc. Internat. Congress of Mathematicians (Zürich, 1994), Birkhäuser, 1995, pp. 943-959. MR **97g:**46074

[M] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990. MR **90m:**46084

[S-Z1] Ş. Strătilă - L. Zsidó, *An algebraic reduction theory for W^*-algebras II*, Revue Roum. Math. Pures Appl. **18** (1973), 407-460. MR **50:**1012

[S-Z2] Ş. Strătilă - L. Zsidó, *The commutation theorem for tensor products over von Neumann algebras*, to appear.

[W] S. Wassermann, *A pathology in the maximal ideal space of $L(H) \otimes L(H)$*, Indiana Univ. Math. J. **27** (1978), 1011-1020. MR **80d:**46113

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